Stable assignment

1. Gale and Shapley (1962) College admissions and stability of marriage
2. Becker (1973) A theory of marriage, part 1

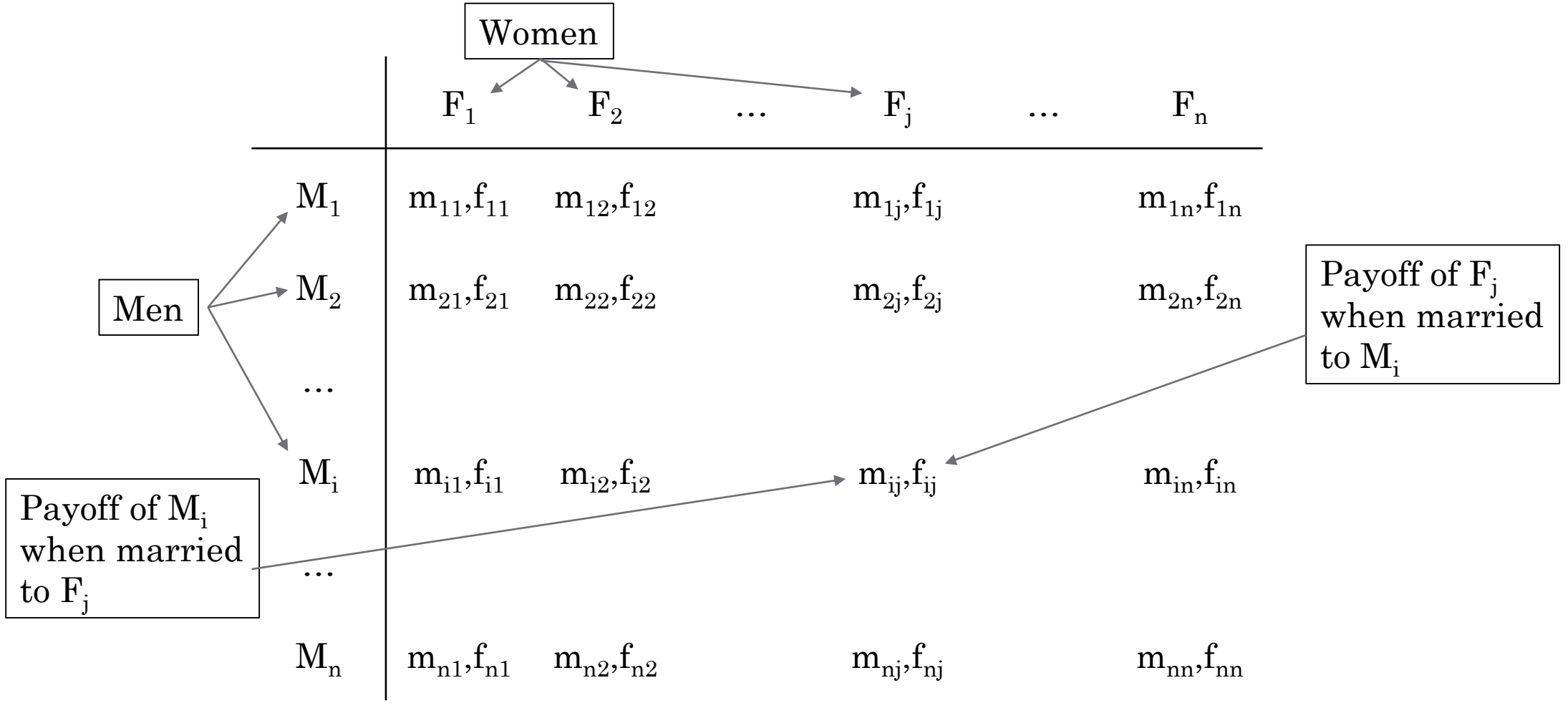
Matching under perfect information

- The premise of perfect information implies that knowledge about the quality of a connection is immediately and costlessly available.
- Key concept: Stability of matching (pairing assignment)

Definition: We call a matching stable if:
1. There is no person who is married and would prefer to be single;
2. There are no two individuals (single or married) who would prefer to form a new relationship together

- Lack of stability means that a given pairing assignment either will not occur at all or will not endure.

Stable matching when transfers between partners are not allowed: payoff matrix



2x2 example

	Alice	Betty
Adam	4,4	2,2
Bob	3,6	6,1

Is the following assignment stable?

- Alice-Adam and Betty-Bob?

Yes. Even though both women would prefer to be with different men, no man wants to switch.

What about the following assignment?

- Alice-Bob and Betty-Adam?

Also yes. Even though both men would prefer to be with different women, no woman wants to switch.

4x4 example

	Alice	Betty	Cindy	Doris
Adam	8,8	5,8	2,8	3,6
Bob	9,5	6,6	7,9	2,2
Chad	5,9	6,3	4,1	2,4
Derek	6,4	2,1	3,3	1,3

- The number of possible pairings is as many as 24.
- Which ones of them are stable?
- Are we sure that at least one pairing is stable?
- How to search for stable pairings? Is there any algorithm for that?

„Man proposes" algorithm

- The algorithm finds a solution in a series of rounds.
- Round 1.
  - Each man proposes to his preferred woman
  - Each woman who has been proposed to...
    - by only one man accepts the proposal;
    - by more than one man rejects all but the one proposed by her most preferred man; she accepts his proposal.
- Round 2.
  - Men who were rejected in Round 1 propose to their second most preferred women.
  - Each woman repeats the pattern from Round 1. Some women break off previous engagements and accept proposals from new, more preferred suitors.
- The algorithm repeats until no man is rejected. Then, all women marry the men they are currently engaged to

4x4 example cont. – from the payoff matrix to the ranking matrix

- In practice, it is more convenient to use a ranking matrix corresponding to the initial payoff matrix when searching for a stable solution
- It is created by ranking potential partners according to preferences derived from the payoff matrix (1 - most preferred, 2 - second in preference order, etc.)

8,8	5,8	2,8	3,6
9,5	6,6	7,9	2,2
5,9	6,3	4,1	2,4
6,4	2,1	3,3	1,3

1,2	2,1	4,2	3,1
1,3	3,2	2,1	4,4
2,1	1,3	4,4	3,2
1,4	3,4	2,3	4,3

Searching for stable assignments

Man proposes algorithm

		Alice	Betty	Cindy	Doris
Rounds	1	A,B,D	C		
	2	A	C	B,D	
	3	A	C,D	B	
	4	A	C	B	D
	5				
	6				

	Ali	Bet	Cin	Dor
Ada	1,2	2,1	4,2	3,1
Bob	1,3	3,2	2,1	4,4
Cha	2,1	1,3	4,4	3,2
Der	1,4	3,4	2,3	4,3

Would the "woman proposes" algorithm find the same or a different stable solution?

Searching for stable assignments

		Adam	Bob	Chad	Derek
Rounds	1				
	2				
	3				
	4				
	5				
	6				

	Ali	Bet	Cin	Dor
Ada	1,2	2,1	4,2	3,1
Bob	1,3	3,2	2,1	4,4
Cha	2,1	1,3	4,4	3,2
Der	1,4	3,4	2,3	4,3

Searching for stable assignments

Man proposes algorithm

		Alice	Betty	Cindy	Doris
Rounds	1	A,B,D	C		
	2	A	C	B,D	
	3	A	C,D	B	
	4	A	C	B	D

Woman proposes algorithm

		Adam	Bob	Chad	Derek
Rounds	1	B, D	C	A	
	2	B	C	A, D	
	3	B	C	A	D

	Alice	Betty	Cindy	Doris
Adam	8,8	5,8	2,8	3,6
Bob	9,5	6,6	7,9	2,2
Chad	5,9	6,3	4,1	2,4
Derek	6,4	2,1	3,3	1,3

Properties of the „man proposes” algorithm

- The proposed solution is at least as good as any other for each man and at least as bad as any other for each woman.
- If it were women proposing to men, the situation would be reversed.
- Increasing the number of women in the population will help some men and not harm any man, but it will not help any woman and will harm some women.

Note! If comparing solutions obtained using both algorithms indicates that some women (men) prefer the solution obtained by the "man proposes (woman)" algorithm, it means that an error was made during the search for a solution using one or both algorithms.

Exercise

	Alice	Betty	Cindy	Doris
Adam	1,3	2,3	3,2	4,3
Bob	1,4	4,1	3,3	2,2
Chad	2,2	1,4	3,4	4,1
Derek	4,1	2,2	3,1	1,4

1. Find stable solutions using the "man proposes" algorithm.
2. Find stable solutions using the "woman proposes" algorithm.
3. Identify which men are indifferent to which algorithm is applied.

Stable assignment with transfers: payoff matrix

	F_1	F_2	…	F_j	…	F_n
M_1	z_{11}	z_{12}		z_{1j}		z_{1n}
M_2	z_{21}	z_{22}		z_{2j}		z_{2n}
…						
M_i	z_{i1}	z_{i2}		z_{ij}		z_{in}
…						
M_n	z_{n1}	z_{n2}		z_{nj}		z_{nn}

Joint payoff for couple F_j i M_i

Stable assignment with transfers: what has changed?

- We are searching for a stable solution as before
- This time, men and women can negotiate the division of the total marriage payoff: m_{ij} and f_{ij} are variables such that $m_{ij} + f_{ij} = z_{ij}$
- If a man wants to maximize m_{ij} and a woman wants to maximize f_{ij}, how do they reach an agreement?
  - Competition forces the restraint of one's own expectations
  - Analogy: Companies want to pay their employees as little as possible, while employees want their pay to be as high as possible. Despite these conflicting expectations, companies and employees reach an agreement.

This time, assuming that z_{ij} and z_{rs} represent any pair of payoffs in a stable solution, the division of the payoff between husband and wife must meet the criterion

$$m_{ij} + f_{rs} \geq z_{is}$$

2x2 example

	Alice	Betty
Adam	8	4
Bob	9	7

Is the assignment Alice-Adam and Betty-Bob stable?

Yes, because we can find such m_{11}, m_{22}, f_{11} and f_{22} that:

- $m_{11} + f_{22} \geq z_{12}$ and
- $m_{22} + f_{11} \geq z_{21}$

What about the Alice-Bob and Adam-Betty assignment?

No, because there are no such m_{12}, m_{21}, f_{12} and f_{21} that:

- $m_{12} + f_{21} \geq z_{11}$ and
- $m_{21} + f_{12} \geq z_{22}$

2x2 example cont.

	Alice	Betty
Adam	8	4
Bob	9	7

- We know that:
  - $m_{11} + f_{11} = z_{11}$
  - $m_{22} + f_{22} = z_{22}$
- For the stability criterion to hold, we must have:
  - $m_{11} + f_{22} \geq z_{12}$
  - $m_{22} + f_{11} \geq z_{21}$
- Substituting from equations into inequalities, we get:
  - $z_{11} - f_{11} + f_{22} \geq z_{12}$
  - $z_{22} + f_{11} - f_{22} \geq z_{21}$
- Adding sides, we get:

$$z_{11} + z_{22} \geq z_{12} + z_{21}$$

Conclusion: Assignment with transfers is stable when it generates the highest sum of payoffs (when it maximizes social welfare)

The algorithm of finding stable assignment with transfers

1. We are looking for pairings where the sum of payouts (across all marriages) will be the largest → this is **the only stable assignment**
2. We divide the payouts in each marriage between the woman and the man in such a way that for any pair z_{ij} and z_{rs} that forms the stable solution we get $m_{ij} + f_{rs} \geq z_{is}$

Attention! If it is not possible to divide the payoffs in such a way that for any pair z_{ij} and z_{rs} the inequality $m_{ij} + f_{rs} \geq z_{is}$ holds, t is likely that the assignment of women to men does not maximize the sum of payouts. Therefore, the assignment is not stable, and it is necessary to return to its search.

4x4 example

	Alice	Betty	Cindy	Doris
Adam	16	(13)	10	9
Bob	14	12	(16)	4
Chad	(14)	9	5	6
Derek	10	3	6	(4)

	Alice	Betty	Cindy	Doris
Adam	(8,8)	(5,8)	2,8	3,6
Bob	9,5	6,6	(7,9)	2,2
Chad	(5,9)	(6,3)	4,1	2,4
Derek	6,4	2,1	3,3	(1,3)

4x4 example: is the assignment under the division of payoffs as per the ranking matrix stable?

	Alice	Betty	Cindy	Doris
Adam	16	**5,8**	10	9
Bob	14	12	**7,9**	4
Chad	**5,9**	9	5	6
Derek	10	3	6	**1,3**

	Alice	Betty	Cindy	Doris
Adam	8,8	5,8	2,8	3,6
Bob	9,5	6,6	7,9	2,2
Chad	5,9	6,3	4,1	2,4
Derek	6,4	2,1	3,3	1,3

Answer: NO!

4x4 example: How about now?

	Alice	Betty	Cindy	Doris
Adam	16	**8,5**	10	9
Bob	14	12	**8,8**	4
Chad	**5,9**	9	5	6
Derek	10	3	6	**2,2**

	Alice	Betty	Cindy	Doris
Adam	8,8	5,8	2,8	3,6
Bob	9,5	6,6	7,9	2,2
Chad	5,9	6,3	4,1	2,4
Derek	6,4	2,1	3,3	1,3

This time it is stable.