The Black-Scholes vs. the Merton jump-diffusion model applied to selected WIG20 companies in the year 2011

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Abstract
Two classical models - the Black-Scholes model and the Merton jump-diffusion model of the evolution of stock prices are considered. The models are applied to 2011 data on stocks of nine major WIG20 companies, representing different branches of the economy. The findings of this article show that the simpler Black-Scholes model outperforms the Merton model in the period considered.

1 Introduction

In the year 2011, especially in its second half, financial markets used to react strongly to the signs of the deepening Eurozone crisis. In this turbulent time stock prices behaved in very volatile way reacting to the inflow of the informations on the debt problems of the different Eurozone members and emergency solutions applied. To capture such a rapid, sometimes even discontinuous movements of stock prices, the Merton jump-diffusion model of the evolution of stock prices seems to be adequate. This model, in opposite to the classical Black-Scholes model, does not infer the normality of logarithmic returns and it usually better fits the distribution of the logarithmic returns in shorter periods (e.g. daily logarithmic returns) than the Black-Scholes model.

In this article we compared two models mentioned. We calibrated and tested this models using real data about stock prices in 2011 of nine major companies quoted at the Warsaw
Stock Exchange. To obtain better comparison we chose companies from different branches of economy: ASSECO (IT), GTC (building), KERNEL (agriculture), KGHM (mining), ORLEN (oil refining and retail), PKO BP (banking), TAURON (energy), TP S.A. (telecommunication) and TVN (television). All chosen companies were major, WIG20 companies, thus one may assume that the actions of the individual investors had limited impact on their prices. On the Figure 1 we present the graphs of these prices in the year 2011.

**Figure 1** Graphs of the evolution of the prices in the year 2011 of nine companies considered

One may observe that the prices declined rapidly after July 2011, and only prices of ASSECO and TP S.A. managed to rise to their previous levels. Thus, it seems that the Merton model would be a better tool for modelling the evolution of stock prices in this period. However, the findings of this article suggests that the Black-Scholes model still
outperforms the Merton model, though the normality of the daily logarithmic returns in the period considered is rejected for all nine companies. This will become evident from the calculations presented in the following sections.

Let us comment on the organization of the paper. In the next section we shortly present the Black-Scholes model and test the normality of daily logarithmic returns of the companies considered. In the third section we present shortly the Merton jump-diffusion model and estimate coefficients of two models considered. Next, we present the comparison of the goodness of fit tests of the models obtained, tested on the new, independent sample. We compare also the values of $\text{VaR}_{95\%}$ obtained from two models. In the last section we present the concluding remarks.

In all calculations we used data from the bossa.pl web page and R software for statistical computing and graphics.

2 The Black-Scholes model and its consequences. Normality tests for returns

2.1 The Black-Scholes model

The classical model of the evolution of stock prices $S_t$ in continuous time $t \geq 0$ is the Black-Scholes (cf. [BS]) model. In this model the evolution of $S_t$ is described by the following stochastic differential equation

\begin{equation}
\text{(BS)} \quad dS_t = vS_t \, dt + \sigma S_t \, dB_t,
\end{equation}

where $B_t$ is a standard Brownian motion and $v, \sigma$ are constants, called drift and volatility respectively. Equation (BS) has unique strong solution, which is the geometric Brownian motion process given by the formula

\begin{equation}
S_t = S_0 e^{(v - \sigma^2/2)t + \sigma B_t} = S_0 e^{\mu t + \sigma B_t},
\end{equation}

where $\mu = v - \frac{\sigma^2}{2}$.

The immediate consequence of the Black-Scholes model is the lognormal distribution of the stock price at a fixed time in the future as well as the normality of logarithmic returns.
We have

\[
\ln \frac{S_{t+\Delta t}}{S_t} = \ln \frac{S_0 e^{\mu(t+\Delta t)+\sigma B_{t+\Delta t}}}{S_0 e^{\mu t+\sigma B_t}} \\
= \mu(1+\Delta t) + \sigma B_{t+\Delta t} - \mu t - \sigma B_t \\
= \mu \Delta t + \sigma (B_{t+\Delta t} - B_t) \sim N(\mu \Delta t, \sigma^2 \Delta t),
\]

i.e. \(\ln(S_{t+\Delta t}/S_t)\) has normal distribution with mean \(\mu \Delta t\) and variance \(\sigma^2 \Delta t\). However, the empirical data suggests that the distribution of logarithmic returns for shorter time lags \(\Delta t\) fails to be normal. In the next subsection we present tests of normality for daily logarithmic returns, corresponding to \(\Delta t = 1\).

2.2 Normality tests for daily returns

In Table 1 we present results of two tests for normality of daily logarithmic stock returns in the analyzed period 1st of January 2011 - 31st of July 2011. The first test was the Shapiro-Wilk test and the second was the Jarque-Bera test. The null hypothesis in both tests assumes the normality of a sample. Both test reject the hypothesis about the normality of the returns at 0.01 significance level for five analyzed assets: GTC, KERNEL, ORLEN, TP S.A. and TVN.

Table 1 Normality tests for daily logarithmic returns in the period 1st January – 31st July 2011

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shapiro-Wilk test statistic's p-value</th>
<th>Jarque-Bera test statistic's p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>0.3179</td>
<td>0.7429</td>
</tr>
<tr>
<td>GTC</td>
<td>0.0007</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>KERNEL</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>KGHM</td>
<td>0.0751</td>
<td>0.0774</td>
</tr>
<tr>
<td>ORLEN</td>
<td>0.0048</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>PKO BP</td>
<td>0.2749</td>
<td>0.1381</td>
</tr>
<tr>
<td>TAURON</td>
<td>0.2460</td>
<td>0.1746</td>
</tr>
<tr>
<td>TP SA</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>TVN</td>
<td>0.0095</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

When we analyze the whole year 2011 (more precisely - the period 1st of January 2011 - 20th of December 2011) the hypothesis about the normality of the returns is rejected at
Thus we see that empirical tests show that the normality of returns does not hold for daily returns in the analyzed periods for majority of the assets. This assumption holds approximately for longer time periods $\Delta t$ (e.g. $\Delta t \geq 3$ months), but on the other hand, the assumption about constant value of volatility and drift in longer periods is unrealistic. The rejection of the hypothesis of normality of daily logarithmic returns, when the sample from whole year 2011 is considered, has its roots in the high market volatility in the second half of the year 2011. In fact, in June and the following months of 2011, financial markets reacted strongly to the signs of the deepening Eurozone crisis. For example, in June 2011 Standard and Poor's downgraded Greece's debt rating to the lowest in the world (CCC), following the findings of a bilateral EU-IMF audit. On 6 July 2011 the ratings agency Moody's had cut Portugal's credit rating to junk status, Moody's also launched speculation that Portugal may follow Greece in requesting a second bailout.

### 3. Merton's jump diffusion model vs. the Black-Scholes model

#### 3.1 The Merton's jump diffusion model

Due to the limitations of the Black-Scholes model in modelling distribution of

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shapiro-Wilk test statistic's p-value</th>
<th>Jarque-Bera test statistic's p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>GTC</td>
<td>0.0003</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>KERNEL</td>
<td>0.0002</td>
<td>0.0048</td>
</tr>
<tr>
<td>KGHM</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>ORLEN</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>PKO BP</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>TAURON</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>TP SA</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>TVN</td>
<td>&lt;0.0001</td>
<td>&lt;0.0001</td>
</tr>
</tbody>
</table>

0.01 significance level for all analyzed assets (cf. Table 2).
logarithmic stock returns in shorter periods, Merton (cf. [M]) enriched the Black-Scholes model with the possibility of jumps, which occur according to a Poisson process, independent from the Brownian motion appearing in (BS). Dynamics of the stock prices in the Merton model may be described by the following equation

\[ S_t = S_0 e^{(\mu + \sigma \beta_t) t + X_t}, \]

where \( X_t \) is a compound Poisson process,

\[ X_t = \sum_{i=1}^{N_t} Y_i, \]

\( N_t \) is a Poisson process and \( Y_i, i = 1, 2, \ldots, \) are i.i.d. normal random variables. Processes \( B_t, N_t \) and variables \( Y_i, i = 1, 2, \ldots \) are independent.

**Remark** Notice, that the Merton model is not a solution of the stochastic differential equation

\((M)\)

\[ dS_t = S_{t-} \left( (\mu dt + \sigma dB_t + dX_t) \right), \]

where \( S_{t-} = \lim_{s \uparrow t} S_s \). Explicit solution to \((M)\) is given by the Doléans-Dade exponential (cf. [A]),

\[ S_t = S_0 e^{(\mu + \sigma \beta_t) t} \prod_{0 \leq s \leq t} \left( 1 + \Delta X_s \right) e^{-\Delta X_s}, \]

where \( \mu = \nu - \frac{\sigma^2}{2} \) and \( \Delta X_s = X_s - X_{s-}, \) \( 0 \leq s \leq t, \) denote jumps of the process \( X_s \) which occur according to the Poisson process \( N_t \) and for \( s \) such that \( N_s - N_{s-} = 1 \) one has \( \Delta X_s = Y_i, \). Thus, to obtain the Merton model one has to consider rather the following equation

\[ dS_i = S_{i-} \left( (\mu dt + \sigma dB_t + d\tilde{X}_t) \right), \]

where

\[ \tilde{X}_t = \sum_{i=1}^{N_t} \tilde{Y}_i, \]

and \( \tilde{Y}_i \) is such that \( \left( 1 + \tilde{Y}_i \right) e^{-\tilde{Y}_i} = e^\nu \) for \( i = 1, 2, \ldots \).
Now we calculate the logarithmic returns in the Merton model

\[
\ln \frac{S_{i+\Delta t}^{N}}{S_{i}^{N}} = \ln \frac{S_{0}^{N}e^{(\mu(t + \Delta t) + \sigma B_{i+\Delta t} + X_{i+\Delta t})}}{S_{0}^{N}e^{\mu t + \sigma B_{i} + X_{i}}} \\
= \mu(t + \Delta t) + \sigma B_{i+\Delta t} + X_{i+\Delta t} - \mu t - \sigma B_{i} - X_{i} \\
= \mu \Delta t + \sigma(B_{i+\Delta t} - B_{i}) + \sum_{i=N_{t}+1}^{N_{t+\Delta t}} Y_{t}.
\]

Let \( \lambda \) be the intensity of the Poisson process and \( m, s^2 \) be the (common) mean and the variance of the variables \( Y_{t}, i = 1, 2, \ldots \). We see that the distribution of \( \ln(S_{i+\Delta t}^{N}/S_{i}^{N}) \) is a mixture of the following normal distributions

- \( N(\mu \Delta t, \sigma^2 \Delta t) \) with the probability \( p_{0} = P(N_{t+\Delta t}^{N} - N_{t}^{N} = 0) = e^{-\lambda \Delta t}; \)
- \( N(\mu \Delta t + m, \sigma^2 \Delta t + s^2) \) with the probability \( p_{1} = P(N_{t+\Delta t}^{N} - N_{t}^{N} = 1) = e^{-\lambda \Delta t} \lambda \Delta t; \)
- \( N(\mu \Delta t + km, \sigma^2 \Delta t + ks^2) \) with the probability

\[
p_{k} = P(N_{t+\Delta t}^{N} - N_{t}^{N} = k) = e^{-\lambda \Delta t} \left( \frac{\lambda \Delta t}{k!} \right)^{k}, k = 2, 3, \ldots
\]

For shorter time periods \( \Delta t \) one may assume that \( \lambda \Delta t \approx 0 \), hence \( p_{2}, p_{3}, p_{6}, \ldots \) are negligible and we may set \( p_{3} = 1 - p_{0} - p_{1} - p_{2}, p_{k} = 0 \) for \( k \geq 4 \).

### 3.2 Calibration of the Merton and Black-Scholes models from daily returns data

To estimate the parameters \( \mu, \sigma, m, s \) and \( p_{0} \) for \( \Delta t = 1 \) day we will utilize maximum likelihood approach. Let \( r_{1}, r_{2}, \ldots, r_{n} \) be consecutive observations of daily logarithmic returns of an asset and \( \hat{\mu}_{M}, \hat{\sigma}_{M}, \hat{m}_{M}, \hat{s}_{M}, \hat{p}_{0,M} \) be maximum likelihood estimators of \( \mu, \sigma, m, s \) and \( p_{0} \) for \( \Delta t = 1 \) in the Merton model. Using the approximation

\[
p_{3} = 1 - p_{0} - p_{1} - p_{2}, p_{k} = 0 \text{ for } k \geq 4,
\]

we have

\[
(\hat{\mu}_{M}, \hat{\sigma}_{M}, \hat{m}_{M}, \hat{s}_{M}, \hat{p}_{0,M}) = \arg \max_{\mu, \sigma, m, s, p_{0}} \prod_{i=1}^{n} \left( \sum_{j=0}^{3} p_{j} \exp \left( \frac{-(r_{i}-\mu-\sigma j)^{2}}{2(\sigma^{2} + j \cdot s^{2})} \right) \right).
\]

In Table 3 we present the obtained maximum likelihood estimates \( (\hat{\mu}_{M}, \hat{\sigma}_{M}, \hat{m}_{M}, \hat{s}_{M}, \hat{p}_{0,M}) \)
for nine assets considered. The estimation was based on the data from the period 1st of January 2011 - 20th of September 2011. In the maximum likelihood estimation the optim procedure of the R environment was utilized.

Table 3 Maximum likelihood estimators for the Merton model parameters based on the data from the period 1st of January 2011 - 20th of September 2011

<table>
<thead>
<tr>
<th>Asset</th>
<th>(\hat{\mu}_M)</th>
<th>(\hat{\sigma}_M)</th>
<th>(\hat{m}_M)</th>
<th>(\hat{\delta}_M)</th>
<th>(\hat{\theta}_{0,M})</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>-0.0014</td>
<td>0.0138</td>
<td>-0.0023</td>
<td>0.0351</td>
<td>0.8494</td>
</tr>
<tr>
<td>GTC</td>
<td>-0.0009</td>
<td>0.0213</td>
<td>-0.0361</td>
<td>0.0329</td>
<td>0.8984</td>
</tr>
<tr>
<td>KERNEL</td>
<td>-0.0026</td>
<td>0.0125</td>
<td>0.0064</td>
<td>0.0332</td>
<td>0.7389</td>
</tr>
<tr>
<td>KGHM</td>
<td>0.0003</td>
<td>0.0182</td>
<td>-0.0060</td>
<td>0.0327</td>
<td>0.8922</td>
</tr>
<tr>
<td>Orlen</td>
<td>-0.0006</td>
<td>0.0150</td>
<td>-0.0034</td>
<td>0.0387</td>
<td>0.8208</td>
</tr>
<tr>
<td>PKO BP</td>
<td>0.0002</td>
<td>0.0126</td>
<td>-0.0085</td>
<td>0.0283</td>
<td>0.8189</td>
</tr>
<tr>
<td>TAURON</td>
<td>-0.0008</td>
<td>0.0122</td>
<td>-0.0060</td>
<td>0.0357</td>
<td>0.9017</td>
</tr>
<tr>
<td>TP SA</td>
<td>0.0023</td>
<td>0.0153</td>
<td>-0.0264</td>
<td>0.0398</td>
<td>0.9085</td>
</tr>
<tr>
<td>TVN</td>
<td>-0.0002</td>
<td>0.0167</td>
<td>-0.0088</td>
<td>0.0503</td>
<td>0.9055</td>
</tr>
</tbody>
</table>

The reason for considering the period 1st of January 2011 - 20th of September 2011 instead of the period 1st of January 2011 - 31st of July 2011 was such that the models obtained from the shorter period fail to fit the distribution of the daily returns during the remaining period of the year 2011. This was tested with the use of the Kolmogorov-Smirnov goodness of fit tests.

**Remark** The obtained estimates for \(p_0\) justify the simplification that \(p_k = 0\) for \(k \geq 4\).

Indeed, since for \(\Delta t = 1\), \(p_0 = e^{-\lambda \Delta t} = e^{-\lambda}\), then \(\lambda = \ln p_0^{-1}\) and \(p_k = p_0(\ln p_0^{-1})^k / k!\). Now, since for \(p_0 > e^{-k}\) the function \(p_0 \mapsto p_k = p_0(\ln p_0^{-1})^k / k!\) is decreasing, for all assets we have

\[
\hat{p}_k \leq \left| p_0(\ln p_0^{-1})^k / k! \right|_{p_0=0.7389} < 0.0003 \quad \text{for } k \geq 4.
\]

Similarly, we considered the maximum likelihood estimators of \(\mu, \sigma\) in the Black-Scholes model. In the Black-Scholes model case it is possible to give explicit formulas for these estimators, denoted respectively by \(\hat{\mu}_{BS}, \hat{\sigma}_{BS}\).
The formulas read as

\[
\hat{\mu}_{BS} = \bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i, \quad \hat{\sigma}_{BS} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (r_i - \bar{r})^2}
\]

and, using the data from the period 1st of January 2011 - 20th of September 2011, we obtain estimators presented in Table 4.

<table>
<thead>
<tr>
<th>Asset</th>
<th>(\hat{\mu}_{BS})</th>
<th>(\hat{\sigma}_{BS})</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
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<td>0.0200</td>
</tr>
<tr>
<td>GTC</td>
<td>-0.0049</td>
<td>0.0266</td>
</tr>
<tr>
<td>KERNEL</td>
<td>-0.0006</td>
<td>0.0229</td>
</tr>
<tr>
<td>KGHM</td>
<td>-0.0004</td>
<td>0.0221</td>
</tr>
<tr>
<td>ORLEN</td>
<td>-0.0014</td>
<td>0.0231</td>
</tr>
<tr>
<td>PKO BP</td>
<td>-0.0017</td>
<td>0.0186</td>
</tr>
<tr>
<td>TAURON</td>
<td>-0.0015</td>
<td>0.0169</td>
</tr>
<tr>
<td>TP SA</td>
<td>-0.0002</td>
<td>0.0210</td>
</tr>
<tr>
<td>TVN</td>
<td>-0.0011</td>
<td>0.0234</td>
</tr>
</tbody>
</table>

Due to bigger number of parameters, the obtained Merton model fits better than the Black-Scholes model the distribution of the daily returns in the period 1st of January 2011 - 20th of September 2011. This may be measured e.g. with the Kolmogorov-Smirnov statistics

\[
D_M = \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} I_{(x < r_i)} - \sum_{j=0}^{3} \hat{p}_{j,M} \frac{x - \hat{\mu}_M - j \cdot \hat{\sigma}_M}{\sqrt{\hat{\sigma}_M^2 + \hat{\sigma}_M}} \right|
\]

\[
D_{BS} = \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} I_{(x < r_i)} - \Phi \left( \frac{x - \hat{\mu}_{BS}}{\hat{\sigma}_{BS}} \right) \right|
\]

Here

\[
I_{(x < r_i)} = \begin{cases} 
1 & \text{if } x \geq r_i, \\
0 & \text{if } x < r_i
\end{cases}
\]

and \(\frac{1}{n} \sum_{i=1}^{n} I_{(x < r_i)}\) is an empirical cumulative distribution function obtained from the sample

while \(\sum_{j=0}^{3} \hat{p}_{j,M} \Phi \left( \frac{x - \hat{\mu}_M - j \cdot \hat{\sigma}_M}{\sqrt{\hat{\sigma}_M^2 + \hat{\sigma}_M}} \right)\), \(\Phi \left( \frac{x - \hat{\mu}_{BS}}{\hat{\sigma}_{BS}} \right)\) are cumulative distribution functions in the
Merton and Black-Scholes models respectively (by \( \Phi(x) \) we denote cumulative
distribution function of a standard normal distribution, \( \Phi(x) = \left(2\pi\right)^{-1/2} \int_{-\infty}^{x} \exp\left(-t^2/2\right) dt \).
For all assets considered, \( D_M < D_{BS} \), which corresponds to better fit of the calibrated and
observed distribution in the Merton model.

3.3 Goodness of fit tests for daily returns

What is of the utmost importance for risk managers is the prediction of the future market
movements and assessment of the risk of the positions assumed. Additionally, the
goodness of fit of the distribution obtained shall be performed on the independent sample,
not on the same sample from which the parameters of the distribution were obtained. This
is why we have abandoned checking the goodness of fit of the distribution obtained on
the same sample from which the parameters of the distribution were obtained (i.e. on the
data from the period 1st of January 2011 - 20th of September 2011).

In Table 5 we present the results of the Kolmogorov-Smirnov goodness of fit test of the
models obtained, applied to the daily returns from the remaining period of the year 2011,
i.e. to the period 20th of September 2011 - 20th of December 2011.

Table 5 Results of the Kolmogorov-Smirnov goodness of fit test of the models obtained applied to the
data from the period 20th of September 2011 - 20th of December 2011

<table>
<thead>
<tr>
<th>Asset</th>
<th>Merton model, K-S statistic's p-value</th>
<th>B-S model, K-S statistic's p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>0.0725</td>
<td>0.1436</td>
</tr>
<tr>
<td>GTC</td>
<td>0.2516</td>
<td>0.4917</td>
</tr>
<tr>
<td>KERNEL</td>
<td>0.0318</td>
<td>0.5734</td>
</tr>
<tr>
<td>KGHM</td>
<td>0.0210</td>
<td>0.0567</td>
</tr>
<tr>
<td>ORLEN</td>
<td>0.0583</td>
<td>0.5222</td>
</tr>
<tr>
<td>PKO BP</td>
<td>0.0620</td>
<td>0.3111</td>
</tr>
<tr>
<td>TAURON</td>
<td>0.7129</td>
<td>0.4695</td>
</tr>
<tr>
<td>TP SA</td>
<td>0.1395</td>
<td>0.3624</td>
</tr>
<tr>
<td>TVN</td>
<td>0.0185</td>
<td>0.2122</td>
</tr>
</tbody>
</table>

We see that the Kolmogorov-Smirnov tests does not reject the hypothesis about the
accuracy of the Black-Scholes neither Merton model at the 0.01 significance level for all
assets, but the first model significantly outperforms the Merton model in most cases, and K-S statistics for Merton model has bigger \( p \)-value (which corresponds to \( D_M < D_{BS} \)) only for TAURON. It may be argued that the K-S test gives better results for Black-Scholes model since the K-S statistics equally treats all quantiles of the distribution, while Merton model improves Black-Scholes' model ability in modelling very low and very high quantiles. Due to the fact that we consider three month period, it is not possible to check this. However, calculating low (5\%) and high (95\%) empirical quantiles of the daily returns and comparing them with quantiles obtained in two models considered, we still observe that the Black-Scholes model and the Merton model give similar results.

For the sample of daily returns \( r_{n+1}, r_{n+2}, \ldots, r_{n+N} \) from the period 20th of September 2011 - 20th of December 2011 we have \( N = 62 \) and thus the empirical 5\% quantile reads as

\[
\hat{x}_{5\%} = r_{(n+3): (n+59)}
\]

and the empirical 95\% quantile reads as

\[
\hat{x}_{95\%} = r_{(n+56): (n+59)}
\]

where \( r_{(n+1): (n+59)} \leq r_{(n+2): (n+59)} \leq \ldots \leq r_{(n+59): (n+59)} \) is an increasing ordering of the sample \( r_{n+1}, r_{n+2}, \ldots, r_{n+59} \). The estimated 5\% and 95\% quantiles in the Merton model read as

\[
\hat{x}_{5\%,M} = \inf \left\{ x : \sum_{j=0}^{3} \hat{p}_{j,M} \Phi \left( \frac{x - \hat{\mu}_M - j \cdot \hat{\sigma}_M}{\sqrt{\hat{\sigma}_M^2 + j \cdot \hat{\sigma}_M^2}} \right) \geq 5\% \right\}
\]

and

\[
\hat{x}_{95\%,M} = \inf \left\{ x : \sum_{j=0}^{3} \hat{p}_{j,M} \Phi \left( \frac{x - \hat{\mu}_M - j \cdot \hat{\sigma}_M}{\sqrt{\hat{\sigma}_M^2 + j \cdot \hat{\sigma}_M^2}} \right) \geq 95\% \right\}
\]

respectively. Similarly, the estimated 5\% and 95\% quantiles in the Black-Scholes model read as

\[
\hat{x}_{5\%,BS} = \inf \left\{ x : \Phi \left( \frac{x - \hat{\mu}_{BS}}{\hat{\sigma}_M} \right) \geq 5\% \right\}
\]
\[
\hat{x}_{95\%} = \inf \left\{ x : \Phi \left( \frac{x - \hat{\mu}_{BS}}{\hat{\sigma}_{M}} \right) \geq 95\% \right\}.
\]

In Tables 6 and 7 we present comparison of the empirical quantiles of daily returns with quantiles obtained in two models considered.

**Table 6** Comparison of the 5% empirical quantiles of daily returns with quantiles obtained in two models considered

<table>
<thead>
<tr>
<th>Asset</th>
<th>Merton model 5% quantile</th>
<th>B-S model 5% quantile</th>
<th>Empirical 5% quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>-0.031</td>
<td>-0.035</td>
<td>-0.044</td>
</tr>
<tr>
<td>GTC</td>
<td>-0.050</td>
<td>-0.049</td>
<td>-0.051</td>
</tr>
<tr>
<td>KERNEL</td>
<td>-0.032</td>
<td>-0.038</td>
<td>-0.053</td>
</tr>
<tr>
<td>KGHM</td>
<td>-0.036</td>
<td>-0.037</td>
<td>-0.106</td>
</tr>
<tr>
<td>ORLEN</td>
<td>-0.036</td>
<td>-0.039</td>
<td>-0.039</td>
</tr>
<tr>
<td>PKO BP</td>
<td>-0.032</td>
<td>-0.032</td>
<td>-0.042</td>
</tr>
<tr>
<td>TAURON</td>
<td>-0.025</td>
<td>-0.029</td>
<td>-0.029</td>
</tr>
<tr>
<td>TP SA</td>
<td>-0.033</td>
<td>-0.035</td>
<td>-0.022</td>
</tr>
<tr>
<td>TVN</td>
<td>-0.035</td>
<td>-0.040</td>
<td>-0.061</td>
</tr>
</tbody>
</table>

**Table 7** Comparison of the 95% empirical quantiles of daily returns with quantiles obtained in two models considered

<table>
<thead>
<tr>
<th>Asset</th>
<th>Merton model 95% quantile</th>
<th>B-S model 95% quantile</th>
<th>Empirical 95% quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>0.027</td>
<td>0.031</td>
<td>0.046</td>
</tr>
<tr>
<td>GTC</td>
<td>0.034</td>
<td>0.039</td>
<td>0.050</td>
</tr>
<tr>
<td>KERNEL</td>
<td>0.038</td>
<td>0.037</td>
<td>0.046</td>
</tr>
<tr>
<td>KGHM</td>
<td>0.035</td>
<td>0.036</td>
<td>0.056</td>
</tr>
<tr>
<td>ORLEN</td>
<td>0.032</td>
<td>0.037</td>
<td>0.050</td>
</tr>
<tr>
<td>PKO BP</td>
<td>0.025</td>
<td>0.029</td>
<td>0.049</td>
</tr>
<tr>
<td>TAURON</td>
<td>0.023</td>
<td>0.026</td>
<td>0.025</td>
</tr>
<tr>
<td>TP SA</td>
<td>0.029</td>
<td>0.034</td>
<td>0.023</td>
</tr>
<tr>
<td>TVN</td>
<td>0.032</td>
<td>0.037</td>
<td>0.035</td>
</tr>
</tbody>
</table>
3.4 Three month VaR calculation

The main application of the modelling dynamics of financial assets’ prices is the
calculation of the risk measures associated with this dynamics. The big advantage of the
models considered is the independence of the returns on the non-overlapping periods.
This, together with the time-homogeneity gives the following formulas for the
distribution of three month logarithmic returns.

- For the Merton model the distribution of three month logarithmic return is a mixture
  of normal distributions \( N(\mu \Delta t + km, \sigma^2 \Delta t + ks^2) \) with the weights

\[
p_{3m,k} = P(N_{t+\Delta t} - N_t = k) = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^k}{k!}, \quad k = 0, 1, 2, 3, \ldots,
\]

where \( \lambda \) is the same as for the daily returns, but now \( \Delta t = 62 \), since in three month
period there are 63 trading days (on average). This applies to the considered period,
i.e. 20th of September 2011 - 20th of December 2011.

- Similarly, in the Black-Scholes model the distribution of three month logarithmic
  return has normal distributions \( N(\mu \Delta t, \sigma^2 \Delta t) \).

Notice that now, since \( \Delta t = 62 >> 1 \), the probabilities \( p_{3m,k} \) are no longer negligible for
\( k > 3 \). For example, for KERNEL HOLDING, setting \( \lambda = \ln \hat{p}_0 = \ln 0.7389^{-1} = 0.3026 \)
we have

\[
p_{3m,20} = e^{-0.30259\times62} \frac{(0.30259\times62)^{20}}{20!} = 0.0853,
\]
\[
p_{3m,30} = e^{-0.30259\times62} \frac{(0.30259\times62)^{10}}{10!} = 0.0042,
\]

and

\[
p_{3m,35} = e^{-0.30259\times62} \frac{(0.30259\times62)^{35}}{35!} = 0.0003.
\]

Thus we may assume that the probabilities \( p_{3m,k} \) are negligible only for \( k \geq 35 \).

The most common risk measure is \( VaR \) and if the three month loss is defined as

\[
L = S_0 - S_{\Delta t},
\]
then \( \text{VaR}_{95\%}(L) \) is defined with the following formula

\[
\text{VaR}_{95\%}(L) = \inf \{ l : P(L > l) \leq 5\% \}.
\]

Denoting \( r_{3m} = \ln \frac{S_t}{S_0} \) we obtain \( L = S_0(1 - e^{r_{3m}}) \) and

\[
\text{VaR}_{95\%}(L) = \inf \{ l : P(S_0(1 - e^{r_{3m}}) > l) \leq 5\% \}
= \inf \{ l : P(r_{3m} < \ln(1 - l / S_0)) \leq 5\% \}
= \inf \{ (1 - e^{l})S_0 : P(r_{3m} < x) \leq 5\% \}
= (1 - \exp(\sup\{x : P(r_{3m} < x) \leq 5\%\}))S_0.
\]

In both - the Merton and the Black-Scholes model, from the continuity of the return’s distribution we get that \( \sup\{x : P(r_{3m} < x) \leq 5\%\} \) corresponds to 5% quantile of the three month logarithmic return:

\[
\sup\{x : P(r_{3m} < x) \leq 5\%\} = \inf\{x : P(r_{3m} < x) \geq 5\%\} = x_{3m, 5\%}.
\]

Hence

\[
\text{VaR}_{95\%}(L) = (1 - e^{x_{3m, 5\%}})S_0.
\]

In Table 8 we present calculations of \( \text{VaR}_{95\%}(L) \) in both models as well as the realized losses i.e. the difference of the prices as of 20th of December and as of 20th of September 2011.

**Table 8 Calculations of \( \text{VaR}_{95\%}(L) \) in both models and the realized losses**

<table>
<thead>
<tr>
<th>Asset</th>
<th>Price as of 20th of Sept. 2011</th>
<th>( \text{VaR}_{95%}(L) ) - Merton model</th>
<th>( \text{VaR}_{95%}(L) ) - B-S model</th>
<th>realized loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSECO</td>
<td>39.42</td>
<td>30.87135</td>
<td>34.85691</td>
<td>-10.37</td>
</tr>
<tr>
<td>GTC</td>
<td>10.1</td>
<td>9.254616</td>
<td>9.606616</td>
<td>1.35</td>
</tr>
<tr>
<td>KERNEL</td>
<td>65.65</td>
<td>48.55412</td>
<td>59.5688</td>
<td>0.4</td>
</tr>
<tr>
<td>KGHM</td>
<td>159.2</td>
<td>136.8537</td>
<td>142.9109</td>
<td>52.6</td>
</tr>
<tr>
<td>ORLEN</td>
<td>36.9</td>
<td>29.65935</td>
<td>33.68745</td>
<td>2.62</td>
</tr>
<tr>
<td>PKO BP</td>
<td>32.05</td>
<td>24.08707</td>
<td>27.69789</td>
<td>-0.5</td>
</tr>
<tr>
<td>TAURON</td>
<td>5.11</td>
<td>3.76996</td>
<td>4.27768</td>
<td>-0.01</td>
</tr>
<tr>
<td>TP SA</td>
<td>17.4</td>
<td>13.83569</td>
<td>15.38344</td>
<td>0.45</td>
</tr>
<tr>
<td>TVN</td>
<td>13.94</td>
<td>11.64837</td>
<td>12.74169</td>
<td>4.09</td>
</tr>
</tbody>
</table>
4 Concluding remarks

Comparison of the Merton and the Black-Scholes models of the evolution of stock prices, applied to the stocks of nine major companies quoted on the Warsaw Stock Exchange, which belong to WIG20 stock market index and represent different branches of the economy, shows that the Black-Scholes model gives better fit of the distribution of returns in the period 20th of September 2011 - 20th of December 2011. The models were calibrated to the data from the preceding period of the year 2011 - January 2011 - 20th of September 2011. Although the Merton model is more complicated and allows to model rare, large deviations from the "usual" market behavior, the estimation based on this model of low (5%) and high (95%) quantiles, gives similar results as the simpler Black-Scholes model. Two models give also similar values of the popular risk measure - VaR_{95%} and, surprisingly, the Black-Scholes model gives in all cases slightly higher values of VaR_{95%} than the Merton model.

References

