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Should economics students be taught mathematics?

Abstract

The paper obviously points at a positive answer to this question, but it raises several doubts with respect to conventional approaches to teaching mathematics. Based on the author's experience in teaching microeconomics, environmental economics and several mathematics courses (such as convex programming and differential equations), two problems are discussed. The first one is whether economics students should be familiar with methods and techniques only, or should they learn proofs too. The second one is the list of mathematical concepts and theorems that seem to be most useful in economic analyses.

It can be observed that few economics students appreciate the value of rigorous reasoning in their analyses. Most try to grasp concepts quickly, relate them to everyday experience, and learn how to apply formulae or procedures in order to obtain quantitative results. They think that the task of proving things should be left to mathematicians. As a result, quite often the conclusions they arrive at are shallow and based on politically correct statements rather than established facts. Therefore teaching mathematics is more than simply providing routine solutions. If properly implemented, it can augment the analytical skills of students, raise the level of healthy scepticism, and nourish innovativeness.

Mathematicians and economists may have diverse views on what theorems are really necessary for the students of BA, MA, and PhD levels, respectively. It is fairly obvious what measure theory or probability concepts are applied in modern econometric techniques. However, when it comes to economic analysis, things are much more difficult to sort out. An easy distinction would be to stick to Lagrangians and the Kuhn-Tucker theorem for the BA level and to switch to Hamiltonians and the Pontryagin principle for the PhD level. This is an oversimplification, as most advanced microeconomic models base on the "simple" Kuhn-Tucker theorem whereas some basic and intermediate models – such as e.g. the Hotelling rule – call for the optimal control framework. Compiling lists of mathematical concepts and theorems for all levels is still a challenge.

The paper concludes with some hints on how a successful curriculum can be arrived at. In the absence of Europe-wide standards coupled with an increased mobility, it is unlikely that any student holding an undergraduate degree from elsewhere will be ready to fully benefit from a graduate course offered at another university. Nevertheless by carefully identifying the mathematics necessary to enrol in a course and publicising this knowledge, universities may establish a Europe-wide awareness of what is really worth studying.

1. Is mathematics useful for economic theory?

For many generations economists raised truly difficult questions, but tried to answer them by referring either to common sense or to philosophical assertions. It was as late as in the 19th century that economics started to mimic physics and other natural sciences in their attempt to apply mathematics in order to derive verifiable corollaries from simple assumptions. A number of useful results were obtained by assuming optimising behaviour, such as e.g. utility maximisation for consumers and profit maximisation for firms. In the 20th century second thoughts came with the recognition of the fact that consumers may actually take irrational decisions and firms may not maximise profits. Several "brooks" diverged from the mainstream economics questioning its scientific adequacy and policy relevance. While it is true that some important economic problems are not understood enough to be quantified, not all the critique of the mainstream is justified.

It is easy to demonstrate that one can obtain ridiculous results by applying a rigorous argument with respect to a wrong initial assumption. Nevertheless critics of the mainstream economics go too far by rejecting its general approach and moving to a purely descriptive method. They return to where economics was before the 19th century. All too often a politically correct statement is used as a substitute for a logical argument. All too often economics students are convinced that the reality is complex and no firm conclusions can be arrived at. Obviously a bit of scepticism is healthy, but the core of the scientific method is to simplify things in order to achieve simplified results. The difference between "simplified" and "oversimplified" is in practical relevance. Simplified statements can serve as a basis for an effective action; an oversimplified statement leads to inadequate actions.

Box 1

I taught Microeconomics in an MBA class. The problem of public goods was discussed extensively. Typical examples – such as lighthouse, air defence, nice landscapes etc. – were identified and explained. First order conditions for an optimum were derived. On the exam I asked whether the provision of air defence can be privatised. Very few students answered properly that it could, but it would be economically inefficient. Most of them gave irrelevant answers such as e.g. "we should avoid yet another war" or "NATO strikes against Belgrade were a barbaric act". Apparently the students think that a politically correct statement can serve as a good answer.

University studies provide an opportunity to develop skills and virtues. Of course, both are important. Virtues can be nurtured by being exposed to honest teachers, by helping the poor or by learning how good struggles with bad in the world. However, the world needs not only the sensitive, but also the clever. Therefore it is of equal importance that young people can tell the difference between what is politically correct and what is scientifically justified. Mathematics is an indispensable tool allowing to arrive at unambiguous conclusions. If one does not like the conclusion, he or she has to challenge the assumptions. Otherwise it is all too easy to complain that things go not as we would like them to go and to look for someone to blame even though the real causes stay undetected.

Mathematics has served as a method to make scientific predictions, as well as a method to verify empirical data. For instance, by applying a textbook model of production one can predict that the ratio of marginal factor productivities is equal to the factor price ratio. This requires the knowledge of simple calculus. Then one can check this prediction against real market data. This time a scholar must apply statistical tests in order to judge whether the data are consistent with the optimising model behind the entire argument. If the tests suggest that the prediction is incorrect, one has to look at initial assumptions. Perhaps firms do not maximise their profits; or perhaps they buy factors at prices that are different from those statistically recorded; or maybe they face some technological constraints the scholar is not aware of? Mathematical models structure our thinking, suggest explanations of phenomena observed and identify data requirements. Students who do not appreciate mathematics tend to accept half-truths and can be easily misled by poor explanations.

Thus mathematics is essential for university instruction not so much as a source of practical knowledge, but as a way to train young people to think critically. Consequently what is taught is of lesser importance than how it is taught. Building a conceptual framework, training to prove even intuitively obvious things, and demonstrating that not all apparently obvious things are true are perhaps the most important outcomes of a mathematical study.

There is a tradition in teaching non-mathematics students to concentrate on methods and formulae and to omit all the proofs. The rationale behind is that non-mathematicians are not supposed to push the frontiers of mathematics; they are expected to use mathematics in order to obtain results which belong to their disciplines like economics, sociology or history. This approach, however, has a serious drawback. By omitting proofs students are used to accepting someone else's competence and to being freed from an ardent task of seeking the truth. Consequently one expects that studying is easy and absolves himself or herself of thinking whenever a hardship is confronted.

Box 2

In 1988 I published a text of *Differential and difference equations* based on my one-semester lecture delivered to economics students in 1986-1988. The lecture was "self-contained" in a sense that almost all the material – including the Cauchy-Piccard theorem and Volterra model – was defined and proved. There was only one exception: the Liapunov stability theorem (the necessity of stability criteria) whose proof would have required too many lecture hours without giving the students any additional insight into the area. Nevertheless the students seemed to understand that almost everything was accessible with the power of their brains and they did not have to take for granted whatever their teacher said. They were expected to know the techniques, but the knowledge was not a witchcraft; everything could have been traced to the mathematics they should have known from earlier stages of education.

Having claimed that the specific material studied is of minor importance, a serious qualification must be added. Even though the main purpose of teaching mathematics is training young minds, one cannot escape from the fact that a typical student expects to see a direct use of what he or she is learning. Hence mathematics courses have to take into account specific requirements and traditions of a given discipline. Otherwise the students will revolt.

2. What kind of mathematics is called for in economics curricula?

The 19th century breakthrough in economic theory was achieved by means of simple calculus. In order to let students understand the logic of microeconomics, theorems on unconstrained minima and maxima need to be introduced. This is also a good opportunity to demonstrate the usefulness of convexity and concavity concepts. Sooner or later, however, an annoying student will observe that in some circumstances the so-called first order conditions are not necessary for the solution to an optimisation problem. This is where the 20th century mathematics starts to prove its usefulness. The Kuhn-Tucker theorem is perhaps the most important piece of mathematical analysis not included in a standard course which nevertheless should be taught in economics departments. In fact, it can be derived in less than two hours from the classical Lagrange method using the separation theorem i.e. a fairly standard algebraic technique. In other words, in order to understand basic consumer and producer theory, one needs to be familiar with standard calculus and linear algebra supplemented with the Kuhn-Tucker theorem.

For several generations economists believed that equilibria were optimal (by the way, this demonstrates that relying on common sense may lead to grave mistakes). The shock came in the mid-20th century with the concept of Nash equilibrium. It turned out that economic agents can be stuck in positions where nobody has a motivation to unilaterally move away even though no optimum has been reached yet. Game theory is a branch of mathematics that helped not only to reinterpret some of the 19th century theories (e.g. models of Cournot and Bertrand), but also to develop completely new concepts (such as e.g. self-enforcing agreements and incentive compatible auctions). Game theory is therefore a key 20th century mathematical discipline that is worth teaching economics students. If lectured properly, it will combine rigour with the sense of usefulness, both necessary in shaping young economic minds.

There is a growing body of economic literature based on optimal control theory concepts. It would be hard to find an issue of a mainstream economic journal without a paper referring to the Pontryagin principle. Control theory concepts (including the classical 19th century Hamiltonian) permeate modern microeconomics, but they are applied in management science as well. Also national accounting interprets sustainable domestic product in terms of a Hamiltonian (the present value of the future welfare flow). If one's model is based on discrete time variables, then optimal control is replaced with dynamic programming and integrals with infinite time series. In either case the mathematics becomes quite sophisticated.

A competent course on control theory is definitely too difficult for non-mathematics undergraduate students. Consequently it seems obvious that it fits graduate – especially PhD – level. Nevertheless there are some simple intermediate microeconomic models that call for the Pontryagin principle. The Hotelling model of extraction of an exhaustible resource makes a good example. The so-called Hotelling rule ("the expected rate of rent growth is equal to the discount rate") can be derived almost without any mathematics, but explaining a trajectory of extraction cannot be achieved without optimal control theorems. Thus lecturers of undergraduate courses have to choose between a superficial treatment of an important topic and applying sophisticated mathematics – the Pontryagin theorem in this case – without an adequate preparation. Nobody claimed that teaching economics is a trivial job!

For two generations economists failed to prove the existence of a Walrasian equilibrium. They were simply comforted by the fact that the Walras model has the same number of variables and equations. Even a beginner in mathematics knows that this equality means nothing. It is thus surprising that the first acceptable treatment of the model came as late as in the middle of the 20th century. It seems that economists simply did not pay sufficient attention to the existence and uniqueness of the mathematical objects they studied. The tide seems to have changed now and modern economics is aware of several fixed-point theorems. Especially the Brouwer theorem has found many applications in economic problems and hence it is worth proving somewhere in a curriculum, perhaps at the graduate level. At the undergraduate level a simple geometrical interpretation on a plane (supported with an analytical argument) is sufficient. Even a simplified proof is better than a statement or a line of reasoning referring to the common sense.

A large part of modern economics owes to econometric techniques and ultimately to mathematical statistics. Fluency in statistics requires confidence in calculus in the first place; otherwise one cannot compute frequency distributions, likelihood functions, moments and so on. It also requires understanding of the Lebesgue measure theory and Borel set families, but here a fundamental problem arises. A good lecture on measure theory requires at least one semester and it would be considered prohibitively difficult by an average economics student. To complicate things further, one should stress that the Lebesgue integral turns out to be superior to the Riemann integral especially in the context of limit theorems that are particularly useful in statistical analyses. Thus one faces a dilemma: to stick to the principle of avoiding superficial treatment, prove everything whatever is needed and lose contact with a majority of the class, or to review all the concepts necessary for analyses and stay satisfied with superficial understanding of theorems. There is no good solution to the dilemma. Perhaps the best way out is to make a compromise with the general rule outlined in the first section of this paper and to accept that only few students (i.e. those who wish to make original contributions to econometric theory) will learn the measure theory comprehensively. Others will be expected to simply acknowledge that statistical analysis they are supposed to use rests on a theory too difficult to be a part of a standard curriculum.

Differential equations (especially partial ones) make a substantial part of applied mathematics. Also in economics one can easily identify numerous applications of differential and difference equations. Hence there are good reasons to include such a course in economics curricula, and some students even demand this. Even though it is true that economists will benefit from learning differential equations, given time constraints, it should not be a prerequisite for everybody. Only those who are interested in mathematical modelling should take such a course.

Calculus and basic linear algebra are at the core of mathematics to be required in economic classrooms. One should make sure, however, that the Kuhn-Tucker theorem is included in the package, because otherwise a large part of microeconomics rests on waving hands rather than firm foundations. All students should have an opportunity to appreciate basic game theory including the concept of Nash equilibrium. Other mathematical courses should be addressed to selected groups of students. Differential equations, control theory, fixed point theorems and Lebesgue measure have been widely applied in economic research, but they are not prerequisites for an economic diploma.

3. How to teach mathematics?

Mathematics can be taught in two ways: in separate courses or as part of non-mathematic courses. Every topic deserves to be taught separately, but perhaps only calculus and linear algebra should be offered as mandatory courses to economics students. Other subjects can be left as electives or they can be included in economics courses.

There are some topics that are suitable for being included in intermediate microeconomics. For instance, game theory can be naturally introduced as a chapter in a microeconomics lecture. Both undergraduate and PhD level courses allow such extensions. At the undergraduate level several lecture hours are sufficient for students to get acquainted with the definition of Nash equilibrium, mixed strategies and prisoner's dilemma. At the PhD level a more elaborate exposition is needed in order to introduce the concept of Subgame Perfect Nash Equilibrium, repeated games, "Folk Theorem", Nash reversion strategy and the like. Obviously, all these topics can fill a separate course, but a more realistic solution is to offer students packages.

Optimal control theory is a special case. At the undergraduate level it can be referred to in the context of the Hotelling model. Even though the latter makes use of Hamiltonians, proving the Pontryagin principle would perhaps be an excessive effort; leaving unproved theorems from time to time is not a grave sin. In contrast, at the PhD level a separate full blown control theory lecture could be a good solution. PhD students should read and understand the economics literature which makes extensive use of the Pontryagin principle. Its applications are so diverse – ranging from microeconomics, to portfolio selection, to growth theory – that it deserves a separate course.

Box 3

There are two styles in teaching mathematics. One approach is to weaken assumptions of theorems as much as possible. The best possible outcome is to be able to prove reverse implications demonstrating that the assumptions cannot be weakened any further. An alternative is to identify a final result that the course is supposed to arrive at, and to prove the result with the least conceptual effort. While many mathematicians favour the former, the latter approach is perhaps more suitable for non-mathematics students. In the 1980s I taught a course on *Convex programming*. An inequality between the solutions of respective primal and dual problems was identified as the final result. The entire sequence of definitions and theorems was then designed so as to reach the result at the least effort using the conjugate duality theory. As a rule, assumptions of theorems were not weakened. They were adopted at the level of generality that is just necessary to reach the final conclusion. In order to please mathematics students sitting in the class I indicated occasionally how a given theorem can be refined by weakening its assumptions without, however, offering a proof.

A separate question is who is supposed to teach mathematics. There are obvious arguments both for leaving the job to mathematicians and to economists. When a mathematician delivers a lecture there is a better chance that theorems are proved properly and difficult technical questions are taken. Nevertheless economists may have a better feeling of why one needs a concept or a theorem – something that non-mathematics students appreciate very much. There is, however, a gray area between pure theory and applications where neither mathematicians nor economists guarantee a success. This is the art of formulating a real-life problem in the language of mathematics. Sometimes it requires simply acknowledging that a sample average is a random variable with a known distribution. Sometimes one needs to make an extra effort to get rid of non-substantial facts and to distil from the problem only that information which can be translated into the framework of a suitable mathematical theory. A choice of a theory and a selection of substantial information is by far not an easy task.

There is no ready method to teach economics students how to apply mathematics competently. Mathematicians may not appreciate such a need, and economists may not understand all the intricacies involved. In some curricula there are separate workshops or laboratories aimed at framing economic questions as mathematical problems. Instructors should have experience in both disciplines.

Progress in the availability of computer programmes has changed the role of applied mathematics dramatically. A generation ago students were expected to solve linear programming problems using the simplex method with the help of hand calculators. Today only computer scientists need to know the simplex method. Everybody else needs to know how to apply linear programming modules offered by dozens of Windows-based programmes. Likewise, in the 1970s students enrolled in econometrics courses were supposed to manually perform operations required by the Ordinary Least Squares (OLS) method. Today students can choose from a range of packages each of which has a built-in OLS module; estimating regression coefficients requires no more effort than clicking on an icon from a computer screen. Only mathematicians need to understand now what operations are necessary in order to invert large matrices. Analytical tractability of microeconomic models – e.g. an advanced duopoly model – depends on how fast one can differentiate functions and solve equations. The availability of programmes that perform such operations instantly and without errors has accelerated the entire process enormously. As a result, models that used to be too complex to be investigated, can now be solved by a computer-literate scholar.

Have computers changed the role of mathematics in applied sciences? Of course, they have. Nevertheless the role of mathematics as a way to develop an argument and to go beyond what seems to be consistent with common sense and/or political correctness has not changed. The availability of software helping scholars to apply mathematical formulae and routines does not reduce the role of mathematics as a mind-training practice. Hence all comments from section 1 will remain valid in the future too.

Summary and conclusions

There is no doubt that mathematics ought to be studied by economics students. Yet mathematics plays a double role. The first advantage of studying mathematics is to appreciate the power of the mind. Economists without such a training are more likely to accept half-truths and to overlook logical holes in their arguments. An economist with a strong mathematical background is more likely to develop a consistent economic argument even if no mathematical methods are applied. The second advantage of studying mathematics is to learn specific techniques that help to see inter-relationships and draw conclusions. At times mathematics has been abused by applying sophisticated techniques to poorly identified problems. This resulted in lowering the appreciation of mathematical methods and the acceptance of the mainstream economics that relies on them.

The first advantage can be accomplished by teaching mathematics at an adequate pace, without omitting proofs, and by paying sufficient attention to the consistency of the entire theory. The second one requires that students understand what problems can be tackled by a given technique, and sufficient attention is paid to phrasing economic questions in a formal language. Perhaps separate courses may be required to train students in this skill which does not have to be appreciated by an average economist or mathematician.

Modern economics relies on a number of mathematical theories some of which permeate it so deeply that they deserve separate courses. Calculus and basic linear algebra belong to this group. There are several generations of experience in teaching both subjects which means that students can rely on good texts, exercises and programmes. However, many such programmes ignore the Kuhn-Tucker theorem which turns out to be very useful in a number of applications, and hence it should be taught.

Other mathematical theories can be taught either separately or as topics in economics courses. Moreover their relevance may vary across education profiles and levels so that specific curricula can differ with respect to such theories. All in all teaching mathematics is a serious matter and detailed recommendations should be arrived at as a result of a cautious deliberation.