Tomasz Żylicz
University of Warsaw

Microeconomics for PhD students

There are two PhD curricula offered by the Faculty of Economic Sciences at the University of Warsaw: a 'regular' one addressed to students intrested in economics, and 'Quantitative Psychology and Economics' (QPE) addressed to students interested in both disciplines. Microeconomics is an element of either of these curricula.

There are two major parts of economics: microeconomics and macroeconomics. The names are somewhat misleading, since the former suggests small objects, while the latter – big research objects. Despite the terminology, both microeconomics and macroeconomics can address problems of the entire economy. The difference between the two is in a perspective adopted. Microeconomics is about economy as seen from the point of view of individual decision-makers. Macroeconomics is about economy as seen from the point of view of relationships between such aggregates as: GDP, exchange rate, unemployment, growth rate etc. Microeconomics tries to explain economic phenomena by asking questions on motivations decision-makers are likely to have when they do what they do. In contrast, macroeconomics is not necessarily interested in individual motivations. It tries to find relationships between various phenomena as they are observed.

Microeconomics is often considered more credible, because it offers more accurate predictions. Yet, despite frequently erroneous predictions, macroeconomics addresses very important questions. For instance, what will be the exchange rate of euro to dollar next month? Macroeconomics looks at what happens in European and American economies, and tries to predict future rates. Mistakes it makes are spectacular sometimes. Nevertheless the question is extremely interesting, and it is quite obvious that many people would like to see a specific answer rather than an analysis of hypothetical motives of various decision-makers. The distinction between microeconomics and macroeconomics is in a perspective adopted – not in an area studied.

The subject matter of economics is difficult, since it addresses human choices in routine voluntary transactions. Many people think of economics as something which gives answers to questions: "how to become rich?", "why is the price of a given stock sky-rocketing?", "what to do in order to eradicate poverty?" and so on. These are important questions, but they do not define economics. For at least 100 years, most economists have understood their discipline as the science of making choices. Of particular interest are choices when people cannot have everything they would like to have, and if they wish to have something, they need to give up something else.

It is important to emphasise that economics is about choices made by ordinary people – not necessarily by the clever and the virtuous ones. We often say: "she must be an idiot if she spends so much on something" or "if he were ethical, he would not have done this". As citizens, neighbours, teachers, parents we can say so, but economists are not expected to judge choices made by other people. They are supposed to analyse people's decisions as they are, and they should not call them right or wrong.

Transactions establish typical proportions certain goods are exchanged for each other. For instance, two apples go for one orange. We say that one orange is worth two apples, or one apple is worth a half of an orange. Often one good becomes so common that people refer to it as a *numeraire*. They say: one orange is worth 1 €. The good referred to as a *numeraire* is called money. We are used to consider some currencies – like euro – as money. Yet many other goods served in this capacity for a long time: cattle, gold, vodka, man-hour, or what have you.

It is important to emphasise that economics is about routine transactions. Economists have little to say about transactions that are unique. If a criminal kills somebody "in exchange" for 100 € it does not mean that the victim's life was worth 100 €. Or if somebody spends 1 million € to cure herself it does not mean that the sickness was worth 1 million €. Only routine transactions are analysed in economics. Unique choices can be studied in psychology (or in other disciplines), but not in economics.

The last adjective in the description of economics reads "voluntary". Economics does not analyse transactions which are involuntary (that are forced). For instance, the behaviour of slaves in ancient Egypt, or prices charged in Soviet stores cannot be explained by economic methods. Voluntariness makes an important assumption, since many economic analyses are based on it. For example, economists argue that a purchase makes the buyer better off. Unless it is voluntary, the argument cannot be applied.

Thus economics is about how people make choices. They make choices every day, whenever they take decisions about what to do, what to buy, what to sell, whom to support, etc. *Homo oeconomicus*, 'economic man' (an often criticised concept) makes a convenient abstraction. Economists assume that people make choices that let them achieve well-being; people contemplate alternative decisions and choose what is most profitable given the circumstances. This does not preclude spontaneity, altruism, etc.

Microeconomics makes a 'backbone' of economics. It would be difficult to address any economic problem ignoring motivations people have. Whenever we make choices, we act as consumers, producers, members of organisations, civil servants, etc. Any role we play can be crucial, but consumers are of particular interest; whatever happens in economies is ultimately driven by what we do as consumers. That is why a typical course of microeconomics starts with the consumer.

The aim of the course is to introduce students to concepts and analytical methods used in modern economics. Some level of mathematical literacy is necessary in order to read the literature and to contribute to it. Each of the 10 lectures contains a number of definitions and several theorems that apply them. Their choice is motivated by two things. First of all, they are referred to by economists routinely. Secondly, efforts were put to avoid theorems too difficult to be proved in the class. Nevertheless some theorems without proofs are quoted. The famous Arrow's Impossibility Theorem provides an example. On the one hand this is one of the most important pieces of microeconomics. On the other hand, its proof requires mathematical methods that many PhD students are not familiar with.

Microeconomics is one the standard classes that all PhD candidates who specialise in economics have to go through. COVID pandemics forced the University of Warsaw to switch to the online regime. This lecture was delivered online twice. I did not like this system, since I prefer contact with my students. Looking at them, asking questions or responding to their queries make me feel that they understand the topic. These two years I lacked direct contact with them resulted in preparing written outlines of my lectures. I have always used powerpoint presentations. In the online system they were controlled by me, and displayed on students' tablet or laptop screens. Both the vision and my narrative were recorded and made available for students. Nevertheless I felt that a written (perhaps simplified) version of my talks can be useful too.

The entire cycle consists of ten topics. They are accompanied by sets of open-ended questions and answers where I contain additional information important for better understanding of the problem. Initially each unit covered 180-minute lectures. Once the University of Warsaw decided to devote only 1350 minutes to microeconomics, each unit corresponded to 135 minutes (three 45-minute lectures). The list of topics remained the same, but my exposition had to be less specific.

List of topics

1. Consumer theory
2. Individual and aggregate demand
3. Production theory
4. Game theory
5. Competitive equilibrium
6. External effects and public goods
7. Imperfect competition
8. Asymmetric information
9. General equilibrium
10. Public choice theory

In this material there are only questions to the ten topics. In the case of mutiple choice questions the correct answer is underlined, and it does not require additional explanation. In the case of epen-ended questions I outline such explanations.

1. Consumer theory

1. If $\geq$ is a lexicographic preference relation on $\Re^2$ as in D.24 then for each pair $(x^0,y^0)\in\Re^2$
[a] <u>the utility function representing $\geq$ is not continuous</u>
[b] $\forall\ (x,y)\in\Re^2\ [(x,y)\sim(x^0,y^0) \Rightarrow \neg(x,y)\sim(x^0,y^0)]$
[c] $\geq$ is not transitive
[d] $\geq$ is not convex
[e] none of the above

2. A risk avert consumer choosing among lotteries L_1=(1/3,1/3,1/3) and L_2=(1/2,0,1/2) with payoffs x_1=-1, x_2=0, x_3=1, in order to maximize the expected vNM utility
[a] <u>will choose L_1</u>
[b] will choose L_2
[c] will consider both lotteries as equally attractive
[d] the choice does not depend on risk aversion
[e] none of the above

3. How can a budgetary set in $\Re_+^2$ be interpreted in a non-competitive case? I.e. let us assume that a consumer's increased demand for a good results in its higher relative price (with respect to another good).

4. Please demonstrate that if x(**p**,w) is a Walrasian demand function satisfying the weak axiom of revealed preferences then it must be homogeneous of degree 0.

5. Please demonstrate that if a Walrasian demand function x(**p**,w) is homogeneous of degree 1 with respect to w (i.e. $\forall\ \mathbf{p}\in\Re_+^L\ \forall\ w>0\ \forall\ \alpha>0\ [x(\mathbf{p},\alpha w)=\alpha x(\mathbf{p},w)]$) then $\forall \ell=1,...,L$ $[\varepsilon_{\ell w}(\mathbf{p},w)=1]$.

6. Let us assume that a consumer maximizes her utility and buys goods in a competitive market. Should her Marginal Rate of Substitiution be always equal to the respective price ratio?

7. Let us assume that preferences $\geq$ are locally non-satiated and $\mathbf{x}^*$ is maximal with respect to $\geq$ in the set $\{\mathbf{x}\in X: \mathbf{p}^T\cdot\mathbf{x}\leq w\}$. Please prove that the following implication holds: $\forall\mathbf{x}\in X$ $[\mathbf{x}\geq\mathbf{x}^* \Rightarrow \mathbf{p}^T\mathbf{x}\geq\mathbf{p}^T\mathbf{x}^*]$.

8. Consider so-called Machina's Paradox. There are three outcomes of lotteries:
   - $x_1=0$,
   - $x_2=10$ USD, and
   - $x_3=10{,}000$ USD

   It is obvious that $u(x_1)<u(x_2)<u(x_3)$; hence $L_1\leq L_2\leq L_3$, where $L_1=(1,0,0)$, $L_2=(0,1,0)$, and $L_3=(0,0,1)$. Thus, by the independence axiom (D.30), the lottery $0.001L_2+0.999L_3$ should be preferred over $0.001L_1+0.999L_3$. And yet experiments show that most people choose otherwise. Please offer a psychological explanation.

Outline answers to exercices for Consumer theory (1)

3. In the competitive case (when consumers are price takers), the relative price of x_1 (with respect to the price of x_2) is constant irrespective of how many units a consumer consumes. In a non-competitive case (when a consumer can influence the relative prices) an increased demand for good x_1 results in its higher price, i.e. in the higher slope of the budget line. Thus the budget set is not a triangle, but rather a convex area with "rounded" corners.

4. We must prove the homogeneity of degree 0, i.e. $x(\alpha\mathbf{p},\alpha w)=x(\mathbf{p},w)$. Weak axiom of revealed preferences for the Walrasian demand function can be written in the following form. Let **p** and w be a vector of prices and income, respectively, such that $\mathbf{p}^T x\leq w$ and $\mathbf{p}^T y\leq w$ (both x and y are affordable under **p** and w). Let us assume further that x was chosen under **p** and w, but it is also affordable under prices **q** and income v. If y was chosen under **q** and v, then x also could have been chosen under **q** and v. This was the axiom. To prove the homogeneity, as **q** and v we can take $\mathbf{q}=\alpha\mathbf{p}$, and $v=\alpha w$ to see that they define the same budget set. Thus whichever bundle is chosen under **p** and w, i.e. belongs to x(**p**,w) it must also belong to $x(\alpha\mathbf{p},\alpha w)$. To complete the proof, please note that not only the budget sets are identical, but the consumer's indifference curves are the same in both cases too.

5. The homogeneity of degree 1 with respect to w means that $\forall\ \mathbf{p}\in\mathcal{R}_+^L\ \forall\ w>0\ \forall\ \alpha>0$ $[x(\mathbf{p},\alpha w)=\alpha x(\mathbf{p},w)]$. In order to calculate $\varepsilon_{\ell w}(\mathbf{p},w)$of x_ℓ, we need to calculate the derivative

$\partial x_\ell(\mathbf{p},w)/\partial w = \lim_{h\to 0}(x_\ell(\mathbf{p},w+h)-x_\ell(\mathbf{p},w))/h$. In particular we can take $w+h=\alpha w$, where $\alpha\to 1$ (then indeed $h=\alpha w-w\to 0$). Then $\lim_{h\to 0}(x_\ell(\mathbf{p},w+h)-x_\ell(\mathbf{p},w))/h = \lim_{\alpha\to 1}(x_\ell(\mathbf{p},\alpha w)-x_\ell(\mathbf{p},w))/(\alpha w-w) = \lim_{\alpha\to 1}(\alpha x_\ell(\mathbf{p},w)-x_\ell(\mathbf{p},w))/(\alpha w-w) = \lim_{\alpha\to 1}(\alpha-1)x_\ell(\mathbf{p},w)/((\alpha-1)w) = x_\ell(\mathbf{p},w)/w$ = constant with respect to α. Thus income elasticity $\varepsilon_{\ell w}(\mathbf{p},w) = (x_\ell(\mathbf{p},w)/w):(x_\ell(\mathbf{p},w)/w) = 1$.

6. No. First, for some preferences (e.g. Leontiev) MRS cannot be defined. Second, the equality does not to be satisfied for corner solutions (even though the Kuhn-Tucker theorem – T.18 – holds in this case too).

7. Implication $\forall \mathbf{x}\in X\ [\mathbf{x}\succeq\mathbf{x}^* \Rightarrow \mathbf{p}^T\mathbf{x}\geq\mathbf{p}^T\mathbf{x}^*]$ is equivalent (if and only if) to implication $\forall \mathbf{x}\in X\ [\mathbf{x}\succ\mathbf{x}^* \Rightarrow \mathbf{p}^T\mathbf{x}\geq\mathbf{p}^T\mathbf{x}^*]$. Both implications from the square brackets will be rewritten in their reversed forms, i.e. "implication $\forall \mathbf{x}\in X\ [\mathbf{p}^T\mathbf{x}<\mathbf{p}^T\mathbf{x}^* \Rightarrow \mathbf{x}\prec\mathbf{x}^*]$ is equivalent to implication $\forall \mathbf{x}\in X\ [\mathbf{p}^T\mathbf{x}<\mathbf{p}^T\mathbf{x}^* \Rightarrow \mathbf{x}\preceq\mathbf{x}^*]$". The "only if" part is obvious, because $\mathbf{x}\prec\mathbf{x}^*$ implies $\mathbf{x}\preceq\mathbf{x}^*$. Thus one needs to prove the "if" part. Let us assume that – on the contrary – that $\mathbf{x}\preceq\mathbf{x}^*$ but at the same time $\mathbf{x}\prec\mathbf{x}^*$ does not hold, i.e. $\mathbf{x}\sim\mathbf{x}^*$. By the local non-satiation, there is some $\mathbf{x}'$ close to $\mathbf{x}^*$ such that $\mathbf{p}^T\mathbf{x}'<\mathbf{p}^T\mathbf{x}^*$ and $\mathbf{x}'\succ\mathbf{x}^*$. This x' contradicts the implication (which was supposed to hold $\forall\mathbf{x}\in X$).

8. Outcomes x_2 and x_3. can be interpreted as a video about Venice, and a luxurious trip to Venice for two, respectively. Then lotteries $0.001L_2+0.999L_3$ and $0.001L_1+0.999L_3$. can be interpreted as going to Venice almost for sure (99.9% probability), and getting nothing (L_2) or a video (L_1). For a person who missed a luxurious trip to Venice the frustration must be so large that he/she prefers to get nothing rather than a video. In general, the Machina's Paradox predicts that missing an almost sure attractive outcome implies such as strong frustration that people may not take rational decisions.

2. Individual and aggregate demand

Counterexample (Weak axiom of revealed preferences satisfied by a Walrasian demand function does not imply strong axiom of revealed preferences). Let us consider three price systems: $\mathbf{p}^1$, $\mathbf{p}^2$, $\mathbf{p}^3$; and three corresponding bundles: $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$ selected by a consumer according to a Walrasian demand function x (if x is a demand function it means that for given prices there exists only one bundle that the consumer prefers over other ones in the budget set):

$\mathbf{p}^1=(1,2,2)$ $\quad\mathbf{p}^2=(2,2,1)$ $\quad\mathbf{p}^3=(2,1,2)$

$\mathbf{x}^1=(2,2,1)$ $\quad\mathbf{x}^2=(2,1,2)$ $\quad\mathbf{x}^3=(1,2,2)$

We assume that in each case the consumer's income is the same: $w^1=w^2=w^3=w=8$. It is easy to see that in each of the three cases budget constraint is satisfied and entire income is spent, i.e. $(\mathbf{p}^{iT}\cdot\mathbf{x}^i)=8$.

Now we will calculate expenditures needed in order to purchase bundles $\mathbf{x}^1$, $\mathbf{x}^2$ lub $\mathbf{x}^3$ at prices different than the ones that applied when the bundles were actually selected. One needs to perform 6 such computations (3·3–3): bundle 1 at prices 2 and *vice versa*, bundle 2 at prices 3 and *vice versa* and bundle 3 at prices 1 and *vice versa*.

$(\mathbf{p}^{2T}\cdot\mathbf{x}^1) = 4+4+1 = 9 > 8$
$(\mathbf{p}^{1T}\cdot\mathbf{x}^2) = 2+2+4 = 8 \leq 8$
$(\mathbf{p}^{3T}\cdot\mathbf{x}^2) = 4+1+4 = 9 > 8$
$(\mathbf{p}^{2T}\cdot\mathbf{x}^3) = 2+4+2 = 8 \leq 8$
$(\mathbf{p}^{1T}\cdot\mathbf{x}^3) = 1+4+4 = 9 > 8$
$(\mathbf{p}^{3T}\cdot\mathbf{x}^1) = 4+2+2 = 8 \leq 8$

The first pair of comparisons is consistent with the weak axiom of revealed preferences and reveals the preference of $\mathbf{x}^1$ over $\mathbf{x}^2$. The second pair of comparisons is consistent with the weak axiom of revealed preferences and reveals the preference of $\mathbf{x}^2$ over $\mathbf{x}^3$. These two comparisons – if the strong axiom was applied – should imply that the preference of $\mathbf{x}^1$ over $\mathbf{x}^3$ will be revealed too. And yet the third pair of comparisons reveals the preference of $\mathbf{x}^3$ over $\mathbf{x}^1$ (note that the weak axiom is satisfied for this pair as well).

Hence the three observations (pairwise) satisfy the weak axiom (for i,j=1,2,3)

$(\mathbf{p}^{iT}\cdot x(\mathbf{p}^j,w^j)\leq w^i \wedge x(\mathbf{p}^j,w^j)\neq x(\mathbf{p}^i,w^i)) \Rightarrow \mathbf{p}^{jT}\cdot x(\mathbf{p}^i,w^i)>w^j$.

Despite that, the chain of comparisons violates the strong axiom (here N=3):

$(x(\mathbf{p}^2,w^2\neq x(\mathbf{p}^1,w^1) \wedge \mathbf{p}^{1T}\cdot x(\mathbf{p}^2,w^2)\leq w^1 \wedge x(\mathbf{p}^3,w^3\neq x(\mathbf{p}^2,w^2) \wedge \mathbf{p}^{2T}\cdot x(\mathbf{p}^3,w^3)\leq w^2) \Rightarrow \mathbf{p}^{3T}\cdot x(\mathbf{p}^1,w^1)>w^3$.

1. A Hicksian demand function reflecting rational locally non-satiable preferences cannot explain the Giffen paradox, because
[a] the paradox results from the lack rationality
[b] the paradox emerges when preferences violate the local non-satiation principle
[c] <u>changing price of a Giffen good results in changing the level of utility</u>
[d] Hicksian demand for a Giffen good cannot be quantified
[e] none of the above

2. Preferences of a representative (in positive sense) consumer for an aggregate demand function
[a] let determine socially optimum allocation of income
[b] determine the sum of utility (calculated as a product of the utility of this consumer and the number of consumers) resulting from a given allocation
[c] are always strictly convex
[d] <u>explain why the demand for some bundles is higher than for some other ones</u>
[e] none of the above

3. Calculate functions $x(\mathbf{p},w)$, $e(\mathbf{p},u)$ and $h(\mathbf{p},u)$ for a consumer with the following Cobb-Douglas utility function $u(x_1,x_2)=x_1^{\alpha}x_2^{1-\alpha}$.

4. Verify the Slutsky equation for the example form the exercise 3.

5. Check that the function $x(\mathbf{p},w)$ from the exercise 3 satisfies the Uncompensated Law of Demand. Under these circumstances, what can be inferred about aggregate demand function for two or more consumers?

6. Assuming that every consumer has a Cobb-Douglas utility function $u(x_1,x_2)=x_1^{1/2}x_2^{1/2}$ find the preference relation for a consumer representative in normative sense for a BSP function as in T.21

7. Robert Giffen analysed the following paradox. In 19^{th} century Ireland the demand for potatoes grew despite their growing relative price. Can you explain the paradox?

Outline answers to exercices for Individual and aggregate demand (2)

3. First, it is useful to recall that maximization or minimization problems can be simplified by taking strictly monotonic transformations. In other words, finding $\max_x f(x)$ is equivalent to finding $\max_x g(f(x))$ if g is strictly increasing. To see this, one needs to check that $f(x_1) \leq f(x_2)$ if and only if $g(f(x_1)) \leq g(f(x_2))$. This, however, is obvious given the monotonicity of g. In

particular, as the function g one can take the natural logarithm. Thus maximization of a Cobb-Douglas utility function $x_1^{\alpha}x_2^{(1-\alpha)}$ is equivalent to maximizing $\alpha \mathit{ln} x_1+(1-\alpha)\mathit{ln} x_2$.

In order to calculate x(**p**,w) one needs to $\max_x\{\alpha \mathit{ln} x_1+(1-\alpha)\mathit{ln} x_2: p_1x_1+p_2x_2=w\}$. But x_2 can be calculated from the budget constraint as $x_2=(w-p_1x_1)/p_2$. Thus the function to be maximized is $\alpha \mathit{ln} x_1+(1-\alpha)\mathit{ln}((w-p_1x_1)/p_2)$. Its derivative to be equated to zero is $\alpha/x_1-(1-\alpha)p_1/(w-p_1x_1)$. Solving the equation yields $x_1=\alpha w/p_1$. After the substitution, $x_2=(1-\alpha)w/p_2$. These two formulae define x(**p**,w).

In order to calculate e(**p**,u) one needs to $\min_x\{p_1x_1+p_2x_2: x_1^{\alpha}x_2^{(1-\alpha)}=u\}$. But x_2 can be calculated from the utility constraint as $x_2=u^{1/(1-\alpha)}\ x_1^{-1/(1-\alpha)}$. Hence the expenditure minimization problem reads: $\min_x\{p_1x_1+p_2u^{1/(1-\alpha)}\ x_1^{-1/(1-\alpha)}\}$. The derivative (to be equated to zero) is $p_1-1/(1-\alpha)p_2u^{1/(1-\alpha)}\ x_1^{(-2+\alpha)/(1-\alpha)}$. The equation is $p_1=1/(1-\alpha)p_2u^{1/(1-\alpha)}\ x_1^{(-2+\alpha)/(1-\alpha)}$. Its solution is $x_1=(\alpha/(1-\alpha))^{1-\alpha}p_1^{\alpha-1}p_2^{1-\alpha}u$ and consequently $x_2=(\alpha/(1-\alpha))^{-\alpha}p_1^{\alpha}p_2^{-\alpha}u$. The minimum expenditure is thus e(**p**,u)=
$=(\alpha/(1-\alpha))^{1-\alpha}p_1^{\alpha}p_2^{1-\alpha}u+(\alpha/(1-\alpha))^{-\alpha}p_1^{\alpha}p_2^{1-\alpha}u=$
$=(((\alpha/(1-\alpha))^{1-\alpha}+(\alpha/(1-\alpha))^{-\alpha})p_1^{\alpha}p_2^{1-\alpha}u = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u$.

By the definition of Hicksian demand $h_1(\mathbf{p},u)=x_1(\mathbf{p},e(\mathbf{p},u))=$
$=x_1(\mathbf{p},\ \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u)=\alpha\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u/p_1=\alpha^{1-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha-1}p_2^{1-\alpha}u$;
and $h_2(\mathbf{p},u)=x_2(\mathbf{p},e(\mathbf{p},u))=$
$=x_2(\mathbf{p},\ \alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u)=(1-\alpha)\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u/p_2=\alpha^{-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha}p_2^{-\alpha}u$.

4. For the two-dimensional case, Slutsky equation reads:

$$\forall \mathbf{p}\in\Re^2\forall w>0\forall \ell,k\in\{1,2\}[\partial h_\ell(\mathbf{p},u)/\partial p_k=\partial x_\ell(\mathbf{p},w)/\partial p_k+x_k(\mathbf{p},w)\partial x_\ell(\mathbf{p},w)/\partial w].$$

(In fact, there are four scalar equations.)
$\ell=1$, k=1: $L_{11}= -\alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha-2}p_2^{1-\alpha}u$;
$\ell=1$, k=2: $L_{12}= \alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha-1}p_2^{-\alpha}u$;
$\ell=2$, k=1: $L_{21}= \alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha-1}p_2^{-\alpha}u$;
$\ell=2$, k=2: $L_{22}= -\alpha^{1-\alpha}(1-\alpha)^{\alpha}p_1^{\alpha}p_2^{-\alpha-1}u$.
At the same time
$\ell=1$, k=1: $R_{11}= -\alpha w/p_1^2+(\alpha w/p_1)\alpha/p_1=\alpha(\alpha-1)w/p_1^2$;
$\ell=1$, k=2: $R_{12}= 0+(1-\alpha)(w/p_2)\alpha/p_1=(1-\alpha)\alpha w/(p_1p_2)$;
$\ell=2$, k=1: $R_{21}= 0+(\alpha w/p_1)(1-\alpha)/p_2=(1-\alpha)\alpha w/(p_1p_2)$;
$\ell=2$, k=2: $R_{22}= -(1-\alpha)/p_2^2+(1-\alpha)(w/p_2)(1-\alpha)/p_2)=-\alpha(1-\alpha)w/p_2^2$.
Equalities are confirmed if one substitutes $w\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}$ (i.e. the utility from consuming the bundle) for u, or $\alpha^{-\alpha}(1-\alpha)^{\alpha-1}p_1^{\alpha}p_2^{1-\alpha}u$ (i.e. the minimum expenditure required for purchasing the bundle) for w.

5. For the Cobb-Douglas utility function $u(x_1,x_2)=x_1^{\alpha}x_2^{1-\alpha}$. The demand is $x_1=\alpha w/p_1$ and $x_2=$ $=(1-\alpha)w/p_2$. Thus expenditures are, respectively, $(\alpha w/p_1)\ p_1=w\alpha$ and $x_2=(1-\alpha)(w/p_2)p_2=(1-\alpha)w$. In other words, the expenditures do not depend on prices (shares are constant and equal to, respectively, α and $1-\alpha$). Hence, if the price of a good increases, the quantity demanded – decreases (in order to keep the expenditure constant) which is exactly what the ULD states. According to T.16, as long as price changes do not change the distribution of incomes, the aggregate demand satisfies the ULD too.

6. The aggregate demand for the first good is $(1/2)w_1/p_1+(1/2)w_2/p_1+\ldots+(1/2)w_I/p_1$ (where w_1, w_2, …, w_I are incomes available to consumers 1, 2, …, I). This is $(1/2)(w_1+w_2+\ldots+w_I)/p_1$

$=(1/2)w/p_1$ (where w is the total income available to these consumers). Similarly the total demand for the second good is $(1/2)w_1/p_2+(1/2)w_2/p_2+\ldots+(1/2)w_I/p_2$ (where w_1, w_2, …, w_I are incomes available to consumers 1, 2, …, I). This is $(1/2)(w_1+w_2+\ldots+w_I)/p_2=(1/2)w/p_2$ (where w is the total income available to these consumers). These demand quantities may result from the optimization problem of a consumer characterized by the function $x_1^{1/2}x_2^{1/2}$. This consumer is thus representative in a positive sense for the aggregate problem (irrespective of whether the total income distribution is uniform or not). Hence if the assumptions of theorems 20 or 21 are satisfied, this consumer is representative in normative sense too.

7. T.12 from lecture 1 provides a proof that Giffen good must be an inferior one (e.g. potatoes), and so-called "Law of Demand" states that the demand for, and the price of, a good move in different directions (Δx is the change in demand caused by the change of prices from p^0 to p^1). This Law is based on economic analyses, and confirmed by a number of empirical observations. The analysis of a Giffen paradox (which contradicts the "Law") refers to the following Slutsky decomposition:

$\Delta x = x(p^1,m) - x(p^0,m) =$

$= x(p^1,m') - x(p^0,m) + x(p^1,m) - x(p^1,m') =$

$= \Delta x^s + \Delta x^m =$

= substitution effect + income effect, where m' is an arbitrary income; in particular, m' can be the same as m (then the entire change of demand is understood as "substitution effect", since "income effect" vanishes), or m' can be adjusted in such a way that the original demand – despite a new price – can be afforded. In other words, Slutsky decomposition allows for analysing what can be attributed to substitution (switching to a relatively cheaper substitute), and what can be attributed to changes in real income (which goes down if the price goes up, or *vice versa*).

If the price goes up, and the good is a normal one (such that – *ceteris paribus* – the demand declines with growing income), income effect is negative, and it has the same sign as the substitution effect. Potato, however, is an example of an inferior good (not a normal good). Thus, the sign of the income effect is different from the sign of the substitution effect. If the former is stronger than the latter, it prevails in the sum of the two elements. This is what happened in Ireland in 19th century: consumers were not compensated for the price increase.

3. Production Theory

1. The variable λ in Kuhn-Tucker conditions from T.4 and T.10 can be interpreted as

[a] marginal profit from changing prices
[b] marginal cost of production
[c] average profit enjoyed when the production is kept at the optimum level
[d] average cost paid when the production is kept at the optimum level
[e] <u>none of the above</u>

2. If the production set satisfies axiom D.5.3 (*no free lunch*), then

[a] it cannot satisfy axiom D.5.4
[b] it cannot satisfy axiom D.5.5
[c] it cannot satisfy axiom D.5.6
[d] it cannot satisfy axiom D.5.9
[e] <u>none of the above</u>

3. Find the profit and cost functions for the Cobb-Douglas production function $f(z_1,z_2)=z_1^{\alpha}\cdot z_2^{\beta}$

4. Verify the formula from the Shepard's Lemma (T.11.6) for the production function from exercise 3 above

5. Let us assume that J firms satisfy the D.4 conditions. The average cost of the *j*th function is $AC_j(q_j)=\alpha+\beta_j q_j$; the parameter $\alpha>0$ is identical for all firms. What will be the cost-minimising distribution of production $q_1+...+q_J = q$ (for $q\in(0,\alpha/\max_j\{|\beta_j|\})$, if
   (a) all $\beta_j>0$?
   (b) all $\beta_j<0$?
   (c) some $\beta_j>0$, and some $\beta_j<0$?

6. Demonstrate that $\partial z_\ell(\mathbf{w},q)/\partial q>0$ if and only if when for the production q the marginal cost is an increasing function of w_ℓ

7. Firms are assumed to be 'risk-neutral'. Can you explain why their economic behaviour is considered different from the behaviour of consumers?

Outline answers to exercices for Production Theory (3)

3. The reason that production function is defined as $f(z_1,z_2)=z_1^{\alpha}\cdot z_2^{\beta}$ (instead of taking the second exponent $1-\alpha$, as in the consumer case) is the fact that the sum $\alpha+\beta$ determines scale effects of production. If $\alpha+\beta=1$, then production exhibits constant returns to scale and the profit maximization problem does not have a solution. Apart from that, the derivation of profit and cost functions is very similar to the exercise 3 in lecture 2.

The derivation of the cost function is almost exactly the same as the derivation of $e(\mathbf{p},u)=(\alpha/(1-\alpha))^{1-\alpha}p_1^{\alpha}p_2^{1-\alpha}u+(\alpha/(1-\alpha))^{-\alpha}p_1^{\alpha}p_2^{1-\alpha}u$, except that β should be substituted for $1-\alpha$, z for x, w for p, and q for u (please note that u had the exponent $1/(\alpha+1-\alpha)$). Thus $c(w_1,w_2,q)=$
$=(\alpha/\beta)^{\beta/(\alpha+\beta)}w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}q^{1/(\alpha+\beta)}+(\alpha/\beta)^{-\alpha/(\alpha+\beta)}w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}q^{1/(\alpha+\beta)}=$
$=((\alpha/\beta)^{\beta/(\alpha+\beta)}+(\alpha/\beta)^{-\alpha/(\alpha+\beta)})w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}q^{1/(\alpha+\beta)}=\theta\varphi(w_1,w_2)q^{1/(\alpha+\beta)}$, where $\theta=(\alpha/\beta)^{\beta/(\alpha+\beta)}+(\alpha/\beta)^{-\alpha/(\alpha+\beta)}$ is a constant, and $\varphi(w_1,w_2)=w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}$. Likewise the factor demand functions are almost exactly the same as demand functions calculated in the exercise 3 in lecture 2: $z_1(w_1,w_2,q)=q^{1/(\alpha+\beta)}((\alpha/\beta)(w_2/w_1))^{\beta/(\alpha+\beta)}$, and $z_2(w_1,w_2,q)=q^{1/(\alpha+\beta)}((\beta/\alpha)(w_1/w_2))^{\alpha/(\alpha+\beta)}$ (these formulae will be required in exercise 4).

Given the computed cost function, the profit function is $\pi(p,w_1,w_2,q)=pq-\theta\varphi(w_1,w_2)q^{1/(\alpha+\beta)}$, where p is the price the output q is sold at. Its derivative $\partial\pi/\partial q$ is equal to $p-\theta\varphi(w_1,w_2)(1/(\alpha+\beta))q^{(1/(\alpha+\beta))-1}$. Its zero point identifies a maximum profit only when $\alpha+\beta\leq1$ since the profit function is then concave (the cost function is convex). However, if $\alpha+\beta=1$ (there are constant returns to scale), maxima cannot be determined (q vanishes from the equation) which is linked to the fact that any output yielding a positive profit can be increased causing the profit to grow. If $\alpha+\beta<1$, then for any prices, the unique profit maximizing output can be determined as $q(p,w_1,w_2)==(\alpha+\beta)(p/(\theta\varphi(w_1,w_2)))^{(\alpha+\beta)/(1-\alpha-\beta)}$. Substitution of this expression into the formulae for $c(w_1,w_2,q)$ and $\pi(p,w_1,w_2,q)$ solves the exercise.

4. Shephard's lemma states: if $z(\mathbf{w}^0,q)$ consists of a single point, then c is differentiable with respect to $\mathbf{w}$ in $\mathbf{w}^0$, and $D_w c(\mathbf{w}^0,q)=z(\mathbf{w},q)$. This is a vector equation which can be reduced to two

scalar equations: $\partial c(\mathbf{w}^0,q)/\partial w_1=z_1(\mathbf{w},q)$, and $\partial c(\mathbf{w}^0,q)/\partial w_2=z_2(\mathbf{w},q)$. Detailed calculations are left for the students.

5. Please note that all MC functions are increasing in (a) and decreasing in (b). In (c) some of them are decreasing which makes this case similar to (b). Namely, if there is at least one decreasing MC function, then the total cost is minimized when all the production is allocated to the firm with the largest negative β_j; other firms should not produce at all. This solves (b) and (c). In order to solve (a), let us recall that the minimization of the total cost requires that the marginal cost in all firms should be the same. Therefore we look for an allocation of q such that $q=q_1+...+q_J$ and $MC_j=x$ for all $j=1,\dots,J$. The allocation $q_1,...,q_J$ can be determined by a system of weights $a_1,...,a_J$ such that $a_1+...+a_J=1$and $q_j=a_jq$. The total cost of the *j*th firm is $(\alpha+\beta_ja_jq)a_jq=(\alpha a_jq+\beta_ja_j^2q^2)$. Hence its marginal cost is (please note that q is constant, and a_j is to be determined) $\alpha q+2\beta_ja_jq^2$. These costs are the same for two firms j and k if $\alpha q+2\beta_ja_jq^2=\alpha q+2\beta_ka_kq^2$. After cancellations this means that $\beta_j/\beta_k=a_k/a_j$; the contribution of the firms should be inversely proportional to their cost coefficients. In other words, $a_j=(1/\beta_j)(1/(1/\beta_1+...+1/\beta_J))$. Please also note that the condition $q\in(0,\alpha/\max_j\{|\beta_j|\}$ is necessary to avoid a situation (in (b) or (c)) such that the lowest cost firm (which gives the total production) operates at a negative average cost level; it is not necessary for (a).

6. By the Shephard's lemma, one can write $D_wc(\mathbf{w}^0,q)=z(\mathbf{w},q)$. Contemplating $\partial z_\ell(\mathbf{w},q)/\partial q$ implies that z is differentiable and consists of a single point (is a function in the first place). Thus the Shephard's lemma can be applied. By differentiating both sides of the equation (and in particular the relevant scalar equation $\partial c(\mathbf{w}^0,q)/\partial w_\ell=z_\ell(\mathbf{w},q)$) with respect to q we obtain: $\partial(\partial c(\mathbf{w}^0,q)/\partial w_\ell)/\partial q=\partial z_\ell(\mathbf{w},q)/\partial q$. In other words, $\partial^2 c(\mathbf{w}^0,q)/(\partial w_\ell\partial q)=\partial z_\ell(\mathbf{w},q)/\partial q$. Mathematical analysis lets us write $\partial(\partial c(\mathbf{w}^0,q)/\partial q)/\partial w_\ell=\partial z_\ell(\mathbf{w},q)/\partial q$. or $\partial MC/\partial w_\ell=\partial z_\ell(\mathbf{w},q)/\partial q$. The right hand side is positive (the exercise statement) if and only if the left hand side is (the marginal cost is an increasing function of w_λ, as the exercise asks).

7. In microeconomics a consumer is assumed to take decisions in order to maximize his or her utility. Whenever risk is involved, a consumer maximizes expected utility (not necessarily the expected monetary outcome). Profit plays the role of utility in production theory, and expected profit is to be maximized in risky situations. Firms (producers) do not have feelings as individuals do when they make a distinction between money and utility. Consequently it is assumed that firms look at monetary outcomes only.

4. Game theory

1. In Nash equilibrium each of the players applies strategies
[a] whose joint effect lets achieve the maximum sum of payoffs
[b] which let the first mover enjoy the maximum achievable payoff
[c] which let the last mover enjoy the maximum achievable payoff
[d] that let each of them enjoy the maximum payof that can be achieved by him or her in this game
[e] <u>none of the above</u>

2. A finite game with perfect information cannot include
[a] <u>two or more nodes in the same information set</u>
[b] two or more actions available for decisions in the initial node

[c] a node where the decision is taken by 'nature' (i.e. a player who does not act strategically) with certain probabilities
[d] any subgames except for the original game
[e] none of the above

3. Does a game defined by a symmetric payoff matrix have always a Nash equilibrium in pure strategies? Please provide an argument.

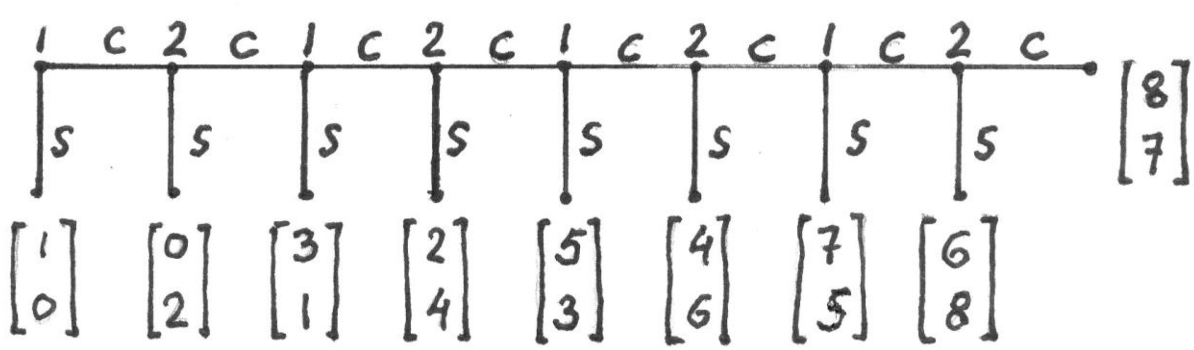
4. Prove that if the game Γ_E is its only subgame, then every Nash equilibrium is a SPNE.

5. Prove that an SPNE in Γ_E induces a SPNE in each of its subgames.

6. Assume that the market is served by the firm I, whose payoff is 2. The firm E contemplates entering the market (without entering, E's payoff is 0). If the entrance happens, the two firms will have to decide simultaneously which of the two parts of the market to serve. The market consists of two parts (interpreted as two types of buyers or two types of the product): big (B) or small (S). By choosing a segment to serve, the firms will play a game with the following payoffs: $u_E(S,S)=u_I(S,S)=-6$, $u_E(B,B)=u_I(B,B)=-3$, $u_E(S,B)=u_I(B,S)=-1$, $u_E(B,S)=u_I(S,B)=1$, where S – the choice of the small segment, B – the choice of the big segment, and the first argument of the utility function u applies to the E's decisions, while the second – the I's decisions. Let A and D denote the decisions of E to enter the market ('attack') or stay away ('do not attack'). By applying the principle of sequential rationality, please demonstrate that the entire game has two SPNE: (σ_E,σ_I) = ((A,B if A was chosen in the previous step),(S if E chose A)) and (σ_E,σ_I) = ((D,S if A was chosen in the previous step),(B if E chose A)). Please note that the second SPNE induces an irrelevant strategy for E (the entrant does not participate in the second stage at all), as well as an irrelevant strategy for I (as a sole supplier, the incumbent does not have to choose a market segment; it serves both).

7. Comment on Martin Shubik's statement: 'You need to know John Nash in order to understand the Nash Equilibrium'.

8. The so-called Centipede Game is defined as follows. This is a 2-person game that lasts 2k periods (where k=1,2,3,...). It has 2k decision nodes, and 2k+1 terminal nodes. The players move in turns, starting with the player number 1. In every node either of the two decisions can be taken: Stop (S) or Continue (C). If the game is stopped at node i=1,2,3,...,2k then the payoffs are: $P(i)=i-(1+(-1)^i)$, $D(1)=0$, and for $i>1$ $D(i)=i-(1-(-1)^i)$. If the game is not stopped by any player at any node, it terminates with the payoffs P(2k)=2k, and D(2k)=2k-1. For k=4 the game is reflected by the following tree (the name refers to the shape of the tree):



At any stage (node) each player receives a higher payoff if he/she stops the game than if the game is stopped by the other player in the next node (stage). By the sequential rationality principle, player number 2 is better off if he/she stops the game in its ultimate (*2k*th) decision node. Knowing this, player number 1 has an incentive to stop the game in its penultimate (*(2k-1)*th) decision node. Knowing this, player number 2 has an incentive to stop the game earlier, i.e. in the *(2k-2)*th node, and so on. Therefore by the sequential rationality principle (D.9), the game should be stopped in its first node. Experiments do not confirm the outcome predicted by the principle in this case. Can you interpret the fact?

Outline answers to exercices for Game Theory (4)

3. No. The following game which has a symmetric payoff matrix does not have a Nash equilibrium in pure strategies (easy to check) even though it obviously must have a Nash equilibrium in mixed strategies (by theorem T.2).

	A	B	C
a	0,0	-1,1	1,-1
b	1-,1	0,0	-1,1
c	-1,1	1,-1	0,0

4. By the definition of SPNE, it must be a Nash equilibrium which induces Nash equilibria in all subgames. If there are no subgames (other than the initial game) then any Nash equilibrium induces itself in the initial game, and hence is a SPNE.

5. One needs to observe that a subgame of a subgame requires strategies which make a fragment of a sequence listing strategies of the second subgame. Thus if this longer sequence consisted of strategies making Nash equilibria in all subgames, then the shorter one must consist of such strategies as well.

6. The game is about whether the firm E enters, and which segments of the market are served by either of the firms E and I, if the firm E entered. The game can be represented by the following payoff table (Entrant chooses rows, and Incumbent chooses columns):

	B	S
D & B	**0,2**	0,2
D & S	**0,2**	0,2
A & B	-3,-3	***1,-1***
A & S	-1,1	-6,-6

Namely, the initial market – which gives the incumbent the payoff of 2 – will give the firm which chose B the payoff of 1, but only when the other competitor chooses S (and has the payoff of -1). If both firms choose the same segment (B,B or S,S) they have negative payoffs (-3,-3 in the first case and -6,-6 in the second case).

Outcomes that are printed in bold are Nash Equilibria. However only one (out of the three), which is printed in italics, is a Nash Equilibrium in the game played in the second stage (see the payoff table made of the two bottom rows). Please note that the upper two rows corresponding to the decision 'do not attack' consist of the same payoffs (0,2), since the Entrant stays away (0), and Incumbent enjoys the monopolistic position (2); plans of the Entrant what to do if the decision 'attack' were made are purely hypothetical and they have no impact on the payoffs.

7. Among his colleagues, John Nash had a reputation of a rational and selfish individual. The concept of 'Nash Equilibrium' assumes that players behave rationally, and they do not take into account their rivals' payoffs.

8. By the sequential rationality principle, the game should be stopped immediately. Experiments, however, demonstrate that it is often continued for a long time. Apparently players trust each other when they promise not to stop it. Sometimes game theory arguments based on rivals' selfishness and narrow rationality fail to be confirmed in real-life situations.

5. Competitive equilibrium

1. The market clearing clearing condition in the definition of the competitive equilibrium can be satisfied as a strict inequality if

[a] different consumers pay different prices for the unit of the good analyzed
[b] different firms get different prices for the unit of the good analyzed
[c] <u>increase of the consumption of the analyzed good above the equilibrium level does not increase the utility</u>
[d] increase of the production of the analyzed good does not increase the cost
[e] none of the above

2. If in the Marshallian welfare model utility functions are not quasi-linear then transfers of the compound good (*ceteris paribus*)

[a] <u>can change the demand level for the analyzed good despite the unchanged price</u>
[b] can change the supply level of the analyzed good despite the unchanged price
[c] decrease the sum of the compound good to be used by the consumers
[d] increase the sum of the compound good to be used by the consumers
[e] none of the above

3. An allocation satisfying D.2 is called a strong Pareto optimum. A weak Pareto optimum is an allocation satisfying D.1 for which there does not exist another feasible allocation $(\mathbf{x}'_1,...,\mathbf{x}'_I,\mathbf{y}'_1,...,\mathbf{y}'_J)$ such that: $\forall i=1,...,I\ [u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)]$. Demostrate that every strong Pareto optimum is a weak one too. Demonstrate also that if consumers' preferences are continuous and and strictly monotonic, and consumption sets $X_i=\Re_+^L$, then every weak Pareto optimum $(\mathbf{x}^*_1,...,\mathbf{x}^*_I)$, with all $\mathbf{x}_i \gg \mathbf{0}$ is a strong Pareto optimum.

4. Let us assume that in the Marshallian model of competitive equilibrium an *ad valorem* tax was imposed with rate τ, i.e. the price paid by consumers is $p(1+\tau)$, where p – the price charged by producers. Aggregate demand function is given by $x(p)=Ap^{\varepsilon}$ ($A>0$, $\varepsilon<0$), and aggregate supply function is $q(p)=ap^{\gamma}$ ($a>0$, $\gamma>0$). Find the rate of price change caused by imposing a small tax (the rate is $p^{*\prime}(\tau)/p^*(0)$).

5. Please explain why in the exercise 4 the producers will bear the entire burden of the tax (consumers' expenditures will be unaffected) if $\gamma=0$, and consumers will bear the entire burden (producers' revenues will be unaffected) if $a=0$.

6. Please demonstrate that in the long term competitive equilibrium model (D.10), if c is strictly convex and $c(0)=0$, then $\pi(p)>0$ if and only if $p>c'(0)$

Outline answers to exercices for Competitive Equilibrium (5)

3. Let us denote a Weak Pareto Optimum WPO, and a Strong Pareto Optimum SPO. We need to prove that SPO $\Rightarrow$ WPO. This will be proved as $\neg$WPO $\Rightarrow$ $\neg$SPO. Let us therefore assume that $\mathbf{x}$ is not a WPO, i.e. there exists another feasible allocation $(\mathbf{x}'_1,...,\mathbf{x}'_I,\mathbf{y}'_1,...,\mathbf{y}'_J)$ such that: $\forall i=1,...,I\ [u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)]$. The same allocation indicates that $\mathbf{x}$ cannot be an SPO (i.e. cannot satisfy $\forall i=1,...,I\ [u_i(\mathbf{x}'_i)\geq u_i(\mathbf{x}_i)]\ \&\ \exists i=1,...,I\ [u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)]$.

Now let us assume that consumers' preferences are continuous and strictly monotonic, and $X_i=\Re_+^L$. We need to prove that WPO $\Rightarrow$ SPO. This will be proved as $\neg$SPO $\Rightarrow$ $\neg$WPO. Let us therefore assume that $\mathbf{x}$ is not a SPO, i.e. there exists another feasible allocation $(\mathbf{x}'_1,...,\mathbf{x}'_I,\mathbf{y}'_1,...,\mathbf{y}'_J)$ such that: $\forall i=1,...,I\ [u_i(\mathbf{x}'_i)\geq u_i(\mathbf{x}_i)]\ \&\ \exists i=1,...,I\ [u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)]$. Let i* be the consumer for whom $u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)$. In other words, $u_{i*}(\mathbf{x}'_{i*})>u_{i*}(\mathbf{x}_{i*})$. By the strict monotonicity at least one coordinate of $\mathbf{x}'_{i*}$ is strictly higher than a corresponding coordinate of $\mathbf{x}_{i*}$. Let us assume that k is this coordinate: $x'_{ki*}>x_{ki*}$. Given the assumption that $X_i=\Re_+^L$, $x'_{ki*}=x_{ki*}+2\varepsilon$, where $\varepsilon>0$ is some positive number. The allocation $\mathbf{x}''$, is constructed in the following way: $x''_{ki*}=x'_{ki*}-\varepsilon$, and $x''_{ki}=x'_{ki}+\varepsilon/(I-1)$ for all $i\neq i*$. Otherwise $x''_{\ell i}=x'_{\ell i}$. By the continuity and strict monotonicity, $u_i(\mathbf{x}''_i)>u_i(\mathbf{x}_i)$ for all $i=1,\ldots,I$. By the feasibility assumption, $x'_{\ell 1}+...+x'_{\ell I}\leq \omega^{\ell}+y'_{\ell 1}+...+y'_{\ell J}$ for $\ell=1,...,L$; and – by construction – the left hand side of this inequality is equal to $x''_{\ell 1}+...+x''_{\ell I}$. Thus the new allocation is feasible too. The allocation $\mathbf{x}''$ demonstrates that $\mathbf{x}$ is not a WPO.

4. T.14 refers to a unit (not to an *ad valorem* tax). Hence we cannot apply the formula derived in the theorem. In the equilibrium the supply $q(p)=ap^{\gamma}$ $(a>0,\ \gamma>0)$ is equal to the demand $x(p)=Ap^{\varepsilon}$ $(A>0,\ \varepsilon<0)$. In other words, $ap^{\gamma}=A(p(1+\tau))^{\varepsilon}$. This equation can be rewritten in order to calculate p as a function of τ. Namely: $p=(A/a)^{1/(\gamma-\varepsilon)}(1+\tau)^{\varepsilon/(\gamma-\varepsilon)}$. The rate to be calculated is defined as $p^{*\prime}(\tau)/p^*(0))$, where p^* is an equilibrium price. First, we have to observe that $p^*(0)=(A/a)^{1/(\gamma-\varepsilon)}$. By calculating the derivative with respect to τ, we obtain

$$p^{*\prime}(\tau)=(\varepsilon/(\gamma-\varepsilon))(A/a)^{1/(\gamma-\varepsilon)}(1+\tau)^{(2\varepsilon-\gamma)/(\gamma-\varepsilon)}.$$

Hence the rate is $(\varepsilon/(\gamma-\varepsilon))(1+\tau)^{(2\varepsilon-\gamma)/(\gamma-\varepsilon)}$.

5. If $\gamma=0$ or $\varepsilon=0$, then the supply or the demand functions, respectively, are horizontal ($q(p)=a=$const or $x(p)=A=$const). Therefore supply or demand curves, respectively, are vertical. Tax incidence is peculiar in either case. If the supply curve is vertical then sellers bear the entire burden of the tax. If the demand curve is vertical then buyers bear the entire burden of the tax.

6. Strict convexity (and non-negativity) of the cost function implies that $c'(q)$ is (strictly) increasing. Thus, $MC(q)=c'(q)>c'(0)$ for $q>0$. At the same time, $AC(0)=MC(0)$. To see this let us calculate $\lim_{q\to 0}c(q)/q=\lim_{h\to 0}(c(0+h)-c(0))/h=c'(0)$. The first equality is obvious, and second one is the definition of a marginal cost. But the average cost is increasing for all q greater than q_0 where $AC(q_0)=MC(q_0)$ (that is for all $q>0$). Furthermore, for such q we have: $MC(q)>AC(q)$. If firms are price-takers then for every firm active in the market $p=MC(q)$. Hence such a firm makes a positive profit $(p>AC(q))$.

6. External effects and public goods

1. A Pigouvian tax may move market equilibrium to a Pareto optimum, despite an externality, but certain condition must be satisfied:

[a] tax revenue is at least as high as the level of the external cost

[b] tax revenue is equal to the external cost

[c] <u>tax rate is equal to the marginal external cost calculated for the Pareto optimum</u>

[d] tax rate is equal to the marginal external cost calculated for the market equilibrium achieved before levying the tax

[e] none of the above.

2. In the Lindahl equilibrium model the market failure of a standard economy with a public good is corrected by

[a] not allowing consumers to cheat when they reveal their preferences with respect to the public good

[b] <u>substituting the public good by a bundle of private goods – characterized by full production complementarity – one for every consumer</u>

[c] providing that the supply of the public good is competitive – it is supplied by many independent producers

[d] charging every potential user an identical price sufficient to finance purchasing the public good

[e] none of the above

3. A student group consists of I persons. The *i*th student allocates h_i hours per week to study which subtracts from his/her utility $h_i^2/2$ (i=1,...,I). Benefits from studying depend not only on the time individually spent but also on the average group level as the following function shows: $\varphi(h_i/(h_1+...+h_I))$. The function is concave, increasing and differentiable with $\lim_{x\to 0+}\varphi'(x)=\infty$ (e.g. the function $\sqrt{}$ satisfies these conditions). Please discuss the Nash equilibrium confined to symmetric solutions, i.e. such that $h_1^*=...=h_I^*$ and compare it with the Pareto optimum (confined to symmetric solutions).

4. Consumer 1 generates a negative depletable externality affecting consumers 2 and 3. Please state and prove the Coase theorem for this case.

5. Inhabitants of a town are characterized identical utility functions $U(x_i,y)=x_i+y^{1/2}$, where x_i – consumption of the (only) private good by the *i*th consumer, and y – consumption of the (only) public good. The marginal costs of producing these goods are constant and equal to 1 for the private good and 5 for the public good. Assuming that the town has 1000 inhabitants, please calculate the optimum supply of the public good.

6. The total external cost of producing q is $EC=2q^2+1$, and the total positive externality is $EB=50q-2q^2+1$. Private benefit of this production is $PB=1000q-2q^2-1$. The knowledge of the private cost is subject to uncertainty. It can be denoted as $PC=q^2+\eta q$, where η is some positive number. Without the exact knowledge of this cost, the government would like to let the economy achieve the optimum level of production either by a direct (quantity) regulation or by imposing a Pigouvian tax. Given the uncertainty, which of the instruments is likely to cause a larger error (in terms of welfare loss with respect to a social optimum): the tax or direct regulation? The answer should be explained comprehensively.

7. Coase theorem can be restated without an assumption D.2. that utilities are quasi-linear with respect to the compound good (money). If they do depend on money in a non-linear way, then the conclusion that irrespective of the property right allocation negotiations will lead to the same Pareto optimum is not valid any more. However, what can be proved is

that negotiations will lead to a Pareto optimum (not necessarily the same one). Please provide an intuitive explanation for this fact.

Outline answers to exercices for External effects and public goods (6)

3. It is assumed in the exercise that if nobody studies then even a slight effort yields a tremendous benefit. On the other hand, if a student works quite a lot, but the others do the same, then his or her individual benefit is lowered by a high average level of competence that teachers are likely to take as a reference. Thus the strategy is to make sure that the average is not excessive. In terms of externalities, a student who studies hard creates an external cost on the rest of the group.

We can start studying the problem by maximizing the net benefit for a student (i.e. $\varphi(h_i/(h_1+...+h_I))-h_i^2/2$). By the symmetry ($h_i=h_j$ for all i and j), this boils down to maximizing $B(h)=\varphi(1/I)-h^2/2$, where $h=(h_1+...+h_I)/I=h_i$. As a sum of a constant and a concave function, B is concave. Hence it is sufficient to differentiate it and to equate its derivative to zero. The (symmetric) maximum is thus for $h_1=...=h_I=h=0$. In other words, the best students can do (provided that all of them do the same) is not to study at all. The solution (0,0,…,0) is, however, not a Nash Equilibrium. This is because of the shape of the function $\varphi(h_i/(h_1+...+h_I))-h_i^2/2$. Namely in a small neighbourhood of (0,…,0) $\partial\varphi/\partial h_i>\partial(h_i^2/2)/\partial h_i=h_i$. Hence any individual will be made better off if he or she studies slightly more than 0, provided that others do not do this. Thus the solution found is not a Nash equilibrium. The question remains whether some other symmetric solution $h_1=...=h_I=h>0$ can be a Nash equilibrium. But we do not have sufficient information on the derivative of φ in neighbourhoods of points $(h,…,h)\neq(0,…,0)$. The information $\partial\varphi/\partial h_i>\partial(h_i^2/2)/\partial h_i$ (which enabled us to conclude that (0,…0) was not a Nash equilibrium) is not available for $h_i>0$. In other words, we cannot exclude the possibility that $\partial\varphi/\partial h_i<h_i$, i.e. that students have no motivation to unilaterally move away from h_i. If the function φ changes slowly this may happen. For instance, if φ is a square root function then this inequality holds for $h_i>1/4^{1/3}\approx0.63$. Hence, if all the students agree to study 0.63 hours, then nobody has a motivation to unilaterally change this time (if somebody studies less, then he/she saves on disutility less than loses from a lower grade; if somebody studies more, then he/she suffers more disutility than gains from a higher grade).

What needs to be checked is whether there are Pareto optima (h,…,h). The (0,…,0) is not a Pareto optimum, because if one student decides to devote some (positive but short) time to study, then his gain is positive, while all the rest enjoy the same level of utility. To see if there are positive symmetric Pareto optima, one needs to check if someone's additional effort above h (resulting in his/her additional net gain) can be compensated by a sufficient time savings for others (resulting in no net losses). There is no sufficient information on the φ function to chek this easily.

4. An analogue of the Coase theorem cannot be proved for three agents affected by non-depletable externalities. However, for depletable externalities *free riding* is not possible, and hence negotiations will drive the parties towards an efficient solution. Formally this can be phrased in the following way.

Three-agent Coase theorem

Let I=3. It is assumed that utilities depend not only on the bundle purchased, but also on the activity $h_i\geq0$ undertaken by the consumer i=1: $u_i(x_{1i},...,x_{Li},h_i)$ for i=1,2,3, with $\partial u_2/\partial h_2\neq0$, $\partial u_3/\partial h_3\neq0$, and $h_2+h_3=h$. In equilibrium the consumers enjoy utilities $v_i(\mathbf{p},w_i,h_i) = \mathrm{Max}_{x_i\geq0}\{u_i(\mathbf{x}_i,h_i): \mathbf{p}^{*T}\cdot\mathbf{x}_i\leq w_i\}$. It is also assumed that utilities are quasi-linear functions with respect to the compound good, so $v_i(\mathbf{p},w_i,h_i) = \varphi_i(\mathbf{p},h_i)+w_i$. Functions $\varphi_i(\mathbf{p},.)$ are strictly concave with respect to h (i.e. $\partial^2\varphi_i/\partial h_i^2<0$). By offering compensations, the consumers can agree on the level of externality without paying transaction costs. If consumers 2 and 3 have the right to prevent the consumer 1 from generating the externality, then the consumer 1 can motivate the consumers 2 and 3 to voluntarily waive this right in exchange for money compensations. If the consumer 1 has the right to generate the externality, then consumers 2 and 3 can motivate the consumer 1 to voluntarily waive this right in exchange for a money compensation. In both cases the same Pareto optimum level of the externality will be achieved.

Proof:
First, we need to calculate the efficient level. Like in the two-agent case, the first order condition for Pareto optimality of the externality is ($h=h_1$):

$$\partial\varphi_1(\mathbf{p},h_1^0)/\partial h \leq -\partial\varphi_2(\mathbf{p},h_2^0)/\partial h-\partial\varphi_2(\mathbf{p},h_3^0)/\partial h, \text{ and if } h^0>0, \text{ then the equality holds.}$$

Now, assuming that an internal solution holds, the numbers $\partial\varphi_1(\mathbf{p},h_1^0)/\partial h$, $-\partial\varphi_2(\mathbf{p},h_2^0)/\partial h$, and $\partial\varphi_2(\mathbf{p},h_3^0)/\partial h$ (where $h=h_2+h_3$), respectively, are the compensations that agents are likely to offer or accept.

5. The utility of an inhabitant is $U(x_i,y)=x_i+y^{1/2}$. The marginal utility from consuming the private good is $\partial U(x_i,y)/\partial x_i = 1$. At the same time the marginal utility from consuming the public good by a single inhabitant is $\partial U(x_i,y)/\partial y = (1/2)y^{-1/2}$, but – by the non-rivalry principle – for the entire city it is $1000(1/2)y^{-1/2}$.The marginal costs of providing a unit of these goods are, respectively, 1 and 5. For an optimum (internal) solution, the proportion of marginal utilities should be the same as the proportion of marginal cost. In other words, $1000(1/2)y^{-1/2}:1 = 5:1$. This is equivalent to $100y^{-1/2}=1$, i.e. y=10,000.

6. T.14 (Weitzman rule) can be rephrased by separating private values (both benefits and costs) from externalities (both positive and negative). We adopt a notational convention that private values are interpreted as benefits (thus private costs are considered private negative benefits). Conversely, externalities are interpreted as costs (thus positive external effects are considered negative external costs). In other words, MNSC = MSC-MEB (Marginal Net Social Cost = Marginal Social Cost – Marginal External Benefit), MNEC = MNSC-MPC (Marginal Net External Cost = Marginal Net Social Cost – Marginal Private Cost), and MNPB = MPB-MPC (Marginal Net Private Benefit = Marginal Private Benefit – Marginal Private Cost). Then the classic Pareto optimum condition MSB=MSC is equivalent to MNEC=MNPB. The proof is simple: MSC=MSB $\Leftrightarrow$ MSC-MEB=MSB-MEB $\Leftrightarrow$ MNSC=MPB $\Leftrightarrow$ MNSC-MPC=MPB-MPC $\Leftrightarrow$ MNEC=MNPB. When MNEC is known with perfect accuracy, but only the slope of MNEC is known, then: -(MNPB)' > (MNEC)' $\Rightarrow$ price (Pigouvian tax) regulation implies a lower error than quantity regulation; and conversely: -(MNPB)' < (MNEC)' $\Rightarrow$ price (Pigouvian tax) regulation implies a greater error than quantity regulation.

Now we need to calculate marginal values. MEC=4q, MEB=50-4q, MPB=1000-4q, and MPC=$2q+\eta$. Applying the notation from the earlier paragraph, MNPB=$1000-4q-2q-\eta=1000-6q-\eta$, and MNEC=4q-50+4q=-50+8q. Hence -(MNPB)'=6, and (MNEC)'=8; quantity regulation is safer.

7. Let the property right is with the second agent (victim of the externality). Then – as the beneficiary of the compensation – he or she is wealthier as a result of negotiations. In the case the property right is with the first agent, the second one is less wealthy, since it is him or her to pay the compensation. Similar logic applies to the other agent. If the marginal utilities depend on the wealth (the ownership of the composite good) then there is no reason why the equation

$\partial v_1(\mathbf{p},w_1,h)/\partial h = -\partial v_2(\mathbf{p},w_2,h)/\partial h$ (for h>0) should lead to same solution for h. Specifically, a wealthier agent's utility can respond to the externality differently.

7. Imperfect competition

1. In a Bertrand duopoly model the equilibrium will include the competitive price, since
[a] any number of firms can compete in the market
[b] both firms are pricetakers
[c] <u>if one of the rivals sets a price higher than the competitive one then the other one can win all the clients and make a positive profit</u>
[d] the Nash equilibrium is always established at the level that is bad for the players – as illustrated by the 'prisoner's dilemma'
[e] none of the above

2. Let us assume that the competition in an oligopolistic market is modelled as a two-stage Cournot game with a linear demand function (T.11). If the equilibrium number of firms active in the market is 8, then doubling the entry cost will cause
[a] the number of firms to increase to 9
[b] the number of firms to increase to 16
[c] the number of firms to decrease to 4
[d] the number of firms to decrease to 3
[e] <u>none of the above</u>

3. A monopolistic mark-up (so-called Lerner's index of the monopolistic power) is defined as $(p^m – c'(q^m))/p^m$. Demonstrate that it is the inverse of the price elasticity at the price p^m.

4. There are three identical firms with constant returns to scale and average production cost c=2. The demand is given by the formula q=80–10p. Assuming that the firms form a Cournot oligopoly, calculate the equilibrium price, supply, and profit of each firm.

5. In a two-stage entry game, the firms comptete in the second stage as in a modified oligopolistic Bertrand model; the firms playing the game have constant returns to scale, but not necessarily the same average cost. Under these circumstances will the SPNE include the lowest cost firm? Please provide a detailed explanation of your answer.

6. Demonstrate the following theorem. If a dynamic game in an oligopolistic Bertrand model (J>1) is infinite then every consistent choice of any fixed price $p \in [c,p^m]$ enforced by the Nash reversion strategy is SPNE if and only if $\rho \leq 1/(J–1)$.

7. There is a variant of the Nash reversion strategy which assumes that a 'wrong' choice is punished only once (and not for ever). In other words, the rival who applies the Nash reversion strategy plays a Nash strategy in the stage which follows immediately the 'wrong' choice, and returns to the agreed moves afterwards. This is called 'tit for tat'. Please discuss pros and cons of this variant.

Outline answers to exercices for Imperfect competition (7)

3. Lerner's index of monopolistic power is defined as $(p^m – c'(q^m))/p^m$. It measures to what extent a monopolist can increase its price above the marginal cost (a price-taking firm cannot do it). The

price elasticity is defined as $\varepsilon=-dq/dp:(q/p)$. In other words, $dp/dq=-(1/\varepsilon):(q/p)$. The monopolistic price corresponds to the quantity q^m such that $MR(q^m)=MC(q^m)$. But $MR(q^m)=d(p(q)q)/dq|_{q=q_m}$. Hence $MR(q^m) = q^m dp(q)/dq|_{q=q_m}+p^m = -(1/\varepsilon)p^m+p^m$. In the monopolistic equilibrium we have: $-(1/\varepsilon)p^m+p^m = MC(x^m)$. Hence $1/\varepsilon = (p^m–c'(q^m))/p^m$.

4. T.8 reads: *In the oligopolistic Cournot model for $J\geq 1$ firms with a linear inverse demand function as in T.6 the Nash equilibrium is given by $q^*(J)=q_1^*=...=q_J^* = (a-c)/(b(J+1))$. The equilibrium price is then $p^*(J)=a-(a-c)J/(J+1)$, and the firms' profits are $\pi^*(J)=\pi_1^*=...=\pi_J^* = = (a-c)^2/(b(J+1)^2)$.*Thus the answer can be derived from T.8 almost directly. The only thing that requires computing is the demand function. In the exercise we have a demand function $q=80-10p$, while in theorem 6 it is $p(q)=a-bq$ (inverse demand function, i.e. a demand curve). The function we have can be rewritten as $p=8-q/10$, i.e. $a=8$, $b=1/10$. The additional parameters are: $c=2$, and $J=3$. The answer is $p^*=8-18/4=3.5$, $q_i^*=6/(4/10)=15$, and $\pi_i^*=6^2/(4^2/10)=22.5$.

5. In the absence of the cost of entry, SPNE must involve the lowest average cost firm. However, if the cost of entry is positive ($K>0$) then there might be an SPNE such that the only firm active in the second stage reveals a higher cost (if its monopolistic profit is higher than K).

6. As a result of collusion, the J oligopolists enjoy a profit of $\pi>0$ (if they charge the price $p>c$, or even the monopolistic profit π^m if they agreed to charge the monopolistic price p^m). That is each of them enjoys $\pi/J>0$. However each of them may contemplate cheating, i.e. charging a price lower than agreed upon. If an oligopolist cheats (by offering a slightly lower price than agreed) then its profit will change from $\pi/J+\pi/(J(1+\rho))+\pi/(J(1+\rho)^2)+\pi/(J(1+\rho)^3)+...$ to (almost) π (only now, and nothing in the future). The first number can be rewritten as $\pi/J(1+1/(1+\rho)+1/(1+\rho)^2+1/(1+\rho)^3+...)$. The geometric series in the parenthesis has the sum equal to $(1+\rho)/\rho$. Hence the first number is $\pi(1+\rho)/(J\rho)$. It is greater than π if and only if $\rho<1/(J-1)$. Therefore if this inequality holds cheating is not attractive. Otherwise it is attractive. The word "almost" can be avoided if one makes the argument fully precise by assuming that the price offered is $p-\varepsilon$ (where ε is sufficiently small to make the one-shot profit sufficiently close to π).

7. 'Tit for tat' was tested empirically in the 1970s. It proved to provide the players with higher payoffs than traditional Nash reversion strategies. Its disadvantage is that the rival who makes a 'wrong' move is not afraid that he or she will be punished forever. Thus incentives to play agreed strategies are weaker. On the other hand, the rival who punishes only once is not pushed into less profitable Nash strategies forever. Thus – in the long run – his or her payoffs can be higher.

8. Asymmetric information

1. In the signalling game an equilibrium in the labour market lets all workers whose productivity is higher than the opportunity cost of labour be employed because

[a] <u>thanks to the signalling, types of workers requiring certain wages can be identified</u>
[b] signalling – e.g. by demonstrating a certain education level – increases workers' productivity
[c] signalling – e.g. thanks to acquiring a certain education level – satisfies workers' aspirations despite a lower wage
[d] employers must compensate the signalling workers by offering a wage covering not only their (higher) productivity, but also the signalling cost
[e] none of the above

2. In the moral hazard model an optimum incentive compatible contract always

[a] should motivate the agent to make a maximum effort ($e^*=e_H$)

[b] should motivate the agent to make a minimum effort ($e^*=e_L$)

[c] gives the agent a wage at least as high as the cost of the maximum effort ($g(e_H)$)

[d] makes the principal pay the agent not more than the cost of the maximum effort ($g(e_H)$)

[e] <u>none of the above</u>

3. Let us assume that $\Theta=\{\theta_1,\theta_2\}$, with $\theta_1<\theta_2$, $r(\theta_1)=\theta_1$ and $r(\theta_2)<\theta_2$. What additional conditions imply that the market equilibrium with asymmetric information as in D.1 is Pareto optimal?

4. Let us assume that education – measured by the number of years spent in schools – does not change worker's productivity which depends on random factors and is not observable for employers. Schooling is not obligatory. There are two categories of workers with the productivity of 100, and 80, respectively. For the first category the cost of completing one year at school is 1, and for the second category – it is 3. Both the productivities and education costs are given in terms of present values so that they are fully comparable. Can the diploma of a primary school (8 years of education) signall that a worker belongs to the higher productivity category? Please explain.

5. A farmer can cultivate a rented piece of land making an effort $e_L=1$, or $e_H=2$. The value of crops $\pi\in[0,1]$, however, is a random variable whose distribution depends on the level of effort and its conditional distribution is $f(\pi|e)$, where $f(\pi|1)=0.5+\pi$, $f(\pi|2)=1.5-\pi$. The utility enjoyed by the farmer from the arrangement is $u(w,e)=w-c(e)$, where w is the wage, and $c(1)=1/12$ and $c(2)=1/3$. Please design a lease contract providing the landlord and the farmer a Pareto optimum. It is assumed that both are risk neutral, the landlord maximizes the expected net revenue (the value of crops net of the farmer's wage), and farmer maximizes the expected wage which must satisfy the aspiration level of $u^0=0{,}25$. What level of farmer's effort will such a contract motivate for?

6. How will the terms of the contract change if one assumes that the farmer is risk averse as reflected by the utility function $u(w,e)=\log_{10}(w+1)–c(e)$? All the other conditions as in the exercise 5.

7. The second-price auction (also known as Vickrey's auction) has better characteristics than the first-price auction. Both provide the principal with the same expected revenue, but they differ in terms of providing the agents with incentives to truthfully reveal their preferences. Why?

Outline answers to exercices for Asymmetric information (8)

3. Pareto optimality requires that those with productivity θ_2 are employed. Those with productivity θ_1 may work or stay unemployed. In order to check whether this can be achieved as a market equilibrium D.2, one needs to check five cases: (1) $w<\theta_1$, (2) $w=\theta_1$, (3) $\theta_1<w<\theta_2$, (4) $w=\theta_2$, (5) $w>\theta_2$. If (1) then nobody is employed, i.e. $\Theta=\emptyset$, and by definition $\{\emptyset,0\}$ is not a market equilibrium. In (2) and (3) all workers with productivity θ_1 are employed. However, it is unclear whether those with productivity θ_2 work as well. This depends on whether $r(\theta_2)<\theta_1$. If it is – they do. Otherwise they do not which violates Pareto optimality. In (4) and (5) everybody is employed but Θ is "too large". θ_1 workers receive wage which higher than their productivity. Hence there is no market equilibrium. Summing up these analyses, in order to determine whether

the market equilibrium is a Pareto optimum one needs to know, whether $r(\theta_2)<$ is smaller or not than θ_1. (we only know that $r(\theta_2)<\theta_2$, which is not sufficient).

4. According to the double inequality derived in T.6, the "separating" standard of education, e^* should be such that $(\theta_H-\theta_L)/c_L < e^* < (\theta_H-\theta_L)/c_H$. We need to check whether the numbers given in the exercise satisfy these inequalities. $\theta_H=100$, $\theta_L=80$; $c_H=1$, $c_L=3$; and $e^*=8$. L=20/3, and R=20. Both inequalities are satisfied.

5. The landlord's problem is to maximize the expected value of π-w, while the problem of the farmer is to maximize w-c(e) (provided that w-c(e)$\geq$1/4). π is random, while w is deterministic. A rational landlord should offer a wage that is just sufficient to meet the farmer's aspiration level, i.e. equal to c(e)+1/4, not more. In other words, the landlord maximizes the expected value π-c(e)-1/4. If $e=e_L=1$ then this value is ${}_0\int^1((1/2+\pi)\pi-1/3)d\pi=[\pi^2/4+\pi^3/3-\pi/3]_0{}^1=1/4+1/3-1/3=1/4$. If $e=e_H=2$ then this value is ${}_0\int^1((3/2-\pi)\pi-7/12)d\pi=[3\pi^2/4-\pi^3/3-7\pi/12]_0{}^1=3/4-1/3-7/12=-2/12=-1/6$ The conclusion is thus that an optimum contract should motivate the farmer to put low level of effort ($e= e_L=1$). A wage which does this can be anything higher than 1/12+1/4=1/3 but lower than 1/3+1/4=7/12. Given the fact that the landlord should have a positive outcome as well (an average crop under the low-effort scenario is 7/12), a feasible wage can be, say, w=1/2. Please note that this is a Pareto optimum; the revenues of the farmer can be improved only at the expense of the landlord and *vice versa*.

6. Farmer's risk aversion implies that the wage w=1/2 suggested in 5 has the utility of $log_{10}(1.5)\approx0.1761<1/12+1/4$, i.e. too little to accept the contract. A maximum wage that the landlord could pay is lower than 7/12. But even w=7/12 will not motivate the farmer to sign a contract ($log_{10}(7/12+1)\approx0.1996<1/12+1/4$). Hence, if the risk aversion is reflected by the formula $log_{10}(w+1)-c(e)$ then it is impossible to design an incentive compatible contract at all.

7. Both types of auction provide incentives not to bid too low, since the lower bid loses. However, if the winner is to pay the price he or she quoted, then he or she has some incentive to underbid. This (slight) incentive disappears in the second-price auction, since the price to be paid is quoted by the loser.

9. General equilibrium

1. Pareto-optimality of equilibrium in a pure exchange economy consisting of two consumers results from the fact that

[a] the equilibrium lets determine the prices uniquely

[b] <u>for each consumer his or her equilibrium demand is materialized in a bundle which is preferred over any other bundle drawn from the relevant budgetary set</u>

[c] in equilibrium the consumers enjoy the maximum sum od their utilities

[d] neither of the consumers would accept prices which do not allow him or her to attain the highest posssible indifference curve

[e] none of the above

2. Wealth transfers which allow to achieve a Pareto optimum (according to the second welfare economics theorem)

[a] allow a redistribution of initial endowments such that shares in firm profits are not changed

[b] allow a redistribution of shares in firm profits such that initial endowments are not changed
[c] correct wealth distribution resulting from ownership titles held
[d] let consumers meet their needs at least to the same extent as in the no-transfer case
[e] <u>none of the above</u>

3. The economy consists of two goods and two consumers. Their indifference curves are given by functions: $u_1(x_{11},x_{12})=x_{11}^a x_{12}^{1-a}$ and $u_2(x_{21},x_{22})=\min\{x_{21},x_{22}\}$. Initial endowments are $\omega_{11}=0$, $\omega_{12}=1$ and $\omega_{21}=1$, $\omega_{22}=0$. Assuming that there is no production, please calculate equilibrium prices and allocation.

4. Solve the problem # 3 setting a = 0.4. Is there a unique answer?

5. What needs to be checked in order to claim that an equilibrium computed in 3 is a Pareto optimum?

6. Let an economy satisfy D.5. Find equilibrium prices, profit and consumption assuming that production function is given by $f(z)=z^{1/2}$, utility function $u(x_1,x_2)=\mathit{ln}x_1+\mathit{ln}x_2$, and labour (leisure) endowment is $L^0=1$.

7. In a pure exchange economy model a following tax is imposed. Everybody whose income is higher than the average pays a half of this difference to a fund used to subsidize those whose income is lower than the average; the subsidies are proportional to these differences. Please calculate net incomes (i.e. incomes after the tax/subsidy) as functions of prices in an economy with two consumers, assuming that endowments were $\omega^1=(1,2)$, $\omega^2=(2,1)$. Will excess demand functions staisfy conditions (i)–(v) in T.19 if preferences are continuous, strictly convex and strictly monotonic?

8. Can a pure exchange economy model for two consumers be reconciled with the price-taking assumption?

Outline answers to exercices for General Equilibrium (9)

3. The first consumer has Cobb-Douglas preferences, while the second consumer has Leontiev preferences. The price ratio can thus be determined by looking at the MRS of the first consumer. In the initial allocation both consumers enjoy zero utilities. In the optimal allocation the second consumer should have the consumption of both goods identical ($x_{21}=x_{22}$). Consequently $x_{11}=x_{12}$ as well. These observations make the computational task easier. Let $x=x_{11}=x_{12}$ and hence $x_{21}=x_{22}=1-x$. Prices should make the revenues and expenditures, respectively, identical. What the first consumers spends on the first good should be equal to what he/she gets from selling the second good ($p_1x=p_2(1-x)$) and likewise what the second consumer spends on the second good should be equal to what he/she gets from selling the first good ($p_2(1-x)=p_1x$). Of course both equations are the same; they imply the same price ratio: $p_1/p_2=(1-x)/x$. Let us call this price ratio p. Hence $p=(1-x)/x$, or $x=1/(p+1)$. What needs to be calculated is thus either p, or x. Both can be determined by the MRS condition:
$p=-MRS_{12}$. Assuming that the utility from consuming the bundle (x_{11},x_{12}) is u, one can write $u=x_{11}^a x_{12}^{1-a}$ and consequently $x_{12}^{1-a}=u/x_{11}^a=ux_{11}^{-a}$; $x_{12}=u^{1/(1-a)}x_{11}^{a/(1-a)}$. The derivative of this function is $a/(1-a)u^{1/(1-a)}x_{11}^{a/(1-a)-1}$. But $x_{11}=x$, and $u=x_{11}^a x_{12}^{1-a}=x^a x^{1-a}=x$. Therefore

$-MRS=a/(1-a)x^{1/(1-a)}x^{a/(1-a)-1}=a/(1-a)x^{1/(1-a)}x^{a/(1-a)-1}=a/(1-a)$. Combining this with the earlier observation, one gets the equation (1-x)/x=a/(1-a), which can be solved for x: x=1-a. Finally p=a/(1-a).

The same result can be obtained by recalling that for the Cobb-Douglas preferences in an optimum bundle the expenditures for either good are proportional to exponents (i.e. a and 1-a, respectively). In other words, $p_1x:p_2x=a/(1-a)$, that is p=a/(1-a). The expenditure=revenue condition this implies that x=1/(p+1)=1-a.

4. If one sets a=0.4 the solution is x=0.6, and p=0.4/0.6=2/3. In other words, the first consumer gets the allocation (0.6, 0.6), the second one – (0.4, 0.4), and the price of the first good is 2/3 of the price of the second good. As far as allocations go, the solution is unique. However prices are not unique. They can be, for instance, $p_1=2/3$ and $p_2=1$, but also $p_1=2$ and $p_2=3$, $p_1=10$ and $p_2=15$, and so on.

5. According to T.2 – nothing. The theorem states that a market equilibrium in an Egeworth box is a Pareto optimum. What needs to be checked is thus only that the exercise satisfies the definition of a pure exchange in an Edgeworth box (D.1), but it (almost) does. We have two consumers, and two goods the supply of which comes from what the consumers brought. What remains to be checked is whether the consumers are price-takers (i.e. they cannot manipulate prices). Even though it looks as if we had a bilateral monopoly, they can be price-takers. It just requires to establish the position of an impartial Walrasian auctioneer who quotes price ratios the consumers are supposed to respond to (without a possibility of sending any other signals). If they maximize their utilities (i.e. they behave rationally), their offers should converge to a market equilibrium.

6. This is a Robinson Crusoe economy. The "firm" Robinson Crusoe Inc. maximizes its profit, $Max_{z\geq 0}\{pf(z)–wz\}=Max_{z\geq 0}\{pz^{1/2}–wz\}$. As a result of the maximization problem, a (secondary) demand for labour is determined z(p,w), the supply of the consumption good q(p,w) and profit $\pi(p,w)$. As a "consumer", Robinson Crusoe maximizes the utility subject to a budget constraint given by selling labour and enjoying the profit of the firm he is the sole owner of: $Max_{x1,x2\geq 0}\{u(x_1,x_2): px_2\leq w(L^0–x_1)+\pi(p,w)\}=Max_{x1,x2\geq 0}\{x_1x_2: px_2\leq w(1-x_1)+\pi(p,w)\}$ (please note that the maximization of $ln x_1+ln x_2$, is equivalent to the maximization of x_1x_2).

Function $pz^{1/2}–wz$ is concave (with respect to z), so it is sufficient to look for where its derivative vanishes. It does when $(1/2)pz^{-1/2}=w$, i.e. when $z=p^2/(4w^2)$. Consequently q=p/(2w), and $\pi=p^2/(2w)-p^2/(4w)=p^2/(4w)$. Let $x=x_1$, i.e. Robinson's leisure. Given the Cobb-Douglas preferences, $px_2:(wx)=1$, i.e. $x_2=wx/p$. According to D.5, the Walrasian (competitive) equilibrium is given by the system of two equations: $x_2(p^*,w^*)=q(p^*,w^*)$ and $z(p^*,w^*)=L^0–x_1(p^*,w^*)$, i.e. wx/p=p/(2w) and $p^2/(4w^2)=1-x$. If one defines p/w=b, then these two equations read: x/b=b/2 and $b^2/4=1-x$. Its solutions are $b=2/3^{1/2}$ and x=2/3. Robinson should devote 2/3 of his time to leisure and 1/3 to work.

7. Given the initial endowments, $w_1=p_1+2p_2$, and $w_2=2p_1+p_2$. The average income is $(3/2)p_1+(3/2)p_2$. The difference between actual and average income (tax base) $TB_1=(1/2)p_1-(1/2)p_2=(1/2)(p_1-p_2)$, and $TB_2=-(1/2)p_1+(1/2)p_2=(1/2)(p_2-p_1)$. If the price ratio is 1 (both prices are the same) then nobody pays taxes and nobody gets a subsidy. However, if the price ratio is lower than one, then the second consumer pays a tax $T_2=-(1/4)(p_1-p_2)$, and the first receives a subsidy $T_1=(1/4)(p_1-p_2)$. If the price ratio is higher than one, then it is the other way around. Incomes after transfers are thus $w'_1=(5/4)p_1+(7/4)p_2$, and $w'_2=(7/4)p_1+(5/4)p_2$.

In a pure exchange economy analysed in the class, excessive demand was defined as $\mathbf{z}^i(\mathbf{p})=x^i(\mathbf{p},\mathbf{p}^T\cdot\boldsymbol{\omega}^i)-\boldsymbol{\omega}^i$. If transfers are added, the excess demand needs to be redefined as $\mathbf{z}^i(\mathbf{p})=x^i(\mathbf{p},\mathbf{p}^T\cdot\boldsymbol{\omega}^i-T_i)-\boldsymbol{\omega}^i$. These new functions satisfy items (i)-(v) of T.19.

8. Yes, but this requires the assistance of the so-called 'impartial Walrasian auctioneer'. If there are only two agents (D.1-3) then they are likely to behave like a bilateral monopoly; they do not reveal their true preferences, but may wish to bargain, bluff, threaten, and otherwise manipulate prices. The procedure which guarantees price-taking is based on intermediation of an auctioneer who drives a *tâtonnement* process. The consumers are to be separated and they cannot communicate directly. The auctioneer announces a price ratio, and asks the consumers separately about their trading offers (neither knows the rival's offer). If the auctioneer finds these offers consistent (i.e. the amount offered for sale by the first is equal to the amount offered to buy by the second and *vice versa*), he or she allows the agents to make the exchange, and an equilibrium is reached. If the offers are not consistent, the auctioneer announces a new price ratio, and repeats the procedure. If he or she knows how to interpret the information obtained (for instance, when the amount supplied is higher than the amount demanded, the new price should be lowered) the procedure is likely to converge in a number of iterations. Please note that consumers are price-takers, since they have to accept prices as given (they cannot manipulate them).

10. Public choice theory

1. The May theorem states that in a dichotomous choice model a majority voting function

[a] <u>does not depend on an ordering of consumers</u>
[b] cannot be dictatorial
[c] cannot be defined for more than two consumers
[d] is not Paretian
[e] none of the above

2. The Arrow theorem states that – if the set of alternatives consists of at least 3 elements – the only social welfare functional which is Paretian and independent of irrelevant alternatives

[a] is a majority voting function
[b] determines the preferences of a median agent
[c] cannot be dictatorial
[d] is not defined for preferences when consumers are indifferent between some non identical alternatives
[e] <u>none of the above</u>

3. Analyze the so-called Condorcet paradox. There are three alternatives: C,K,S. The consumers' preferences are split into three categories. For one third of the consumers we have $C \geq K \geq S$, for another one third $K \geq S \geq C$, and $S \geq C \geq K$ for the rest. Please prove that a majority voting will not result in a rational aggregate preference relation.

4. Explain why in # 3 above T.5 does not apply.

5. Please prove that every Pareto optimum is a weak Pareto optimum, but not conversely.

6. Let us assume that the total cost of a project is $c>0$ and it has to be financed from taxes imposed on 3 consumers. Hence the set of alternatives is $X=\{(t_1,t_2,t_3)\geq 0: t_1+t_2+t_3=c\}$. A specific distribution of taxes is to be determined by a majority vote. Please prove that no distribution such that $t_1,t_2,t_3>0$ is a Condorcet winner.

Outline answers to exercices for Public choice theory (10)

3. Let the ordering C versus K be determined first. Since one third of the population prefers C>K>S, and one third prefers S>C>K, then two thirds prefer C over K. Let the ordering of K versus S be determined now. Since one third of the population prefers C>K>S, and one third prefers K>S>C, then two thirds prefer K over S. These two establish the sequence C>K>S. However, if one tried to check the ordering S versus C, then – again by majority voting – S over C would have been determined. Hence the sequence would read C>K>S>C which is nonsensical if transitivity is assumed. By choosing another pair (other than C and K) to begin with, different – equally nonsensical – sequences can be determined. In other words, majority voting cannot resolve the problem.

4. T.5 provides a method to aggregate individual preferences that are single-peaked with respect to a linear order. Irrespective of a linear order adopted, these three items (K, C, S) cannot lead to single-peaked preferences. To see this, let K, C, S be linearly ordered using e.g. the alphabet: C-K-S. One third of the population places K on the top, C in the middle, and S in the bottom. The preferences are thus rising from C to K, and declining from K to S. They are single-peaked with K being the only peak. Another third of the population places C on the top, S in the middle, and K in the bottom. The preferences are thus declining from C to K, and rising from K to S. They are not single-peaked, since C and S are two different peaks (with C being higher than S, which is not crucial in T.5). The last third of the population places S on the top, K in the middle, and C in the bottom. The preferences are thus rising from C to K, and from K to S. They are single-peaked with S being the only peak. No matter what ordering is adopted, there will always be at least one part of the population ('voter') with preferences that are not single-peaked. This demonstrates that T.5 does not apply in this case.

5. According to D.19, a weak Pareto optimum $(x_1,x_2,\ldots,x_I)$ requires that it is not possible to find an alternative $(y_1,y_2,\ldots,y_I)$, such that $y_1>x_1$, $y_2>x_2$, …, and $y_I>x_I$. In contrast, a Pareto optimum $(x_1,x_2,\ldots,x_I)$ requires that it is not possible to find an alternative $(y_1,y_2,\ldots,y_I)$, such that $y_1\geq x_1$, $y_2\geq x_2$, …, and $y_I\geq x_I$, and at least for one $i=1,\ldots,I$ the inequality is strict. Hence if it is not possible to find an alternative satisfying the second condition, it is impossible to find an alternative satisfying the first one. This demonstrates that a Pareto optimum is a weak Pareto optimum. Let us define $(y_1,y_2,\ldots,y_I)$ in the following way: $y_1=x_1$, $y_2=x_2$, …, $y_{I-1}=x_{I-1}$, and $y_I>x_I$. It contradicts the definition of Pareto optimality of $(x_1,x_2,\ldots,x_I)$, but not necessarily the definition of weak Pareto optimality.

6. A Condorcet winner (D.17) is an alternative which is preferred over any other alternative in a majority vote. We assume that everybody prefers to pay less taxes rather than more. Of course the majority of any two – e.g. 1 and 2 – can outvote the third one – e.g. 3 – by agreeing to zero taxes for themselves and placing the entire burden on the third ($t_3=c$; the Condorcet winning allocation is $(0,0,c)$). However, let $t_1,t_2,t_3>0$ be such that $t_1+t_2+t_3=c$. Any two – e.g. 1 and 2 – can agree to lower their taxes to zero, and to place the entire burden on the third. They will constitute a majority when the new allocation is to be voted. Hence $t_1,t_2,t_3>0$ cannot be a Condorcet winner.