Microeconomics for PhD candidates 2022/2023

- Economics is about how people make choices when they cannot have whatever they would like to have.
- Microeconomics explains how economic agents (consumers, firms, and governments) take decisions.
- Microeconomics can analyse the behaviour of large groups of economic agents, including entire economies.

1. Consumer theory

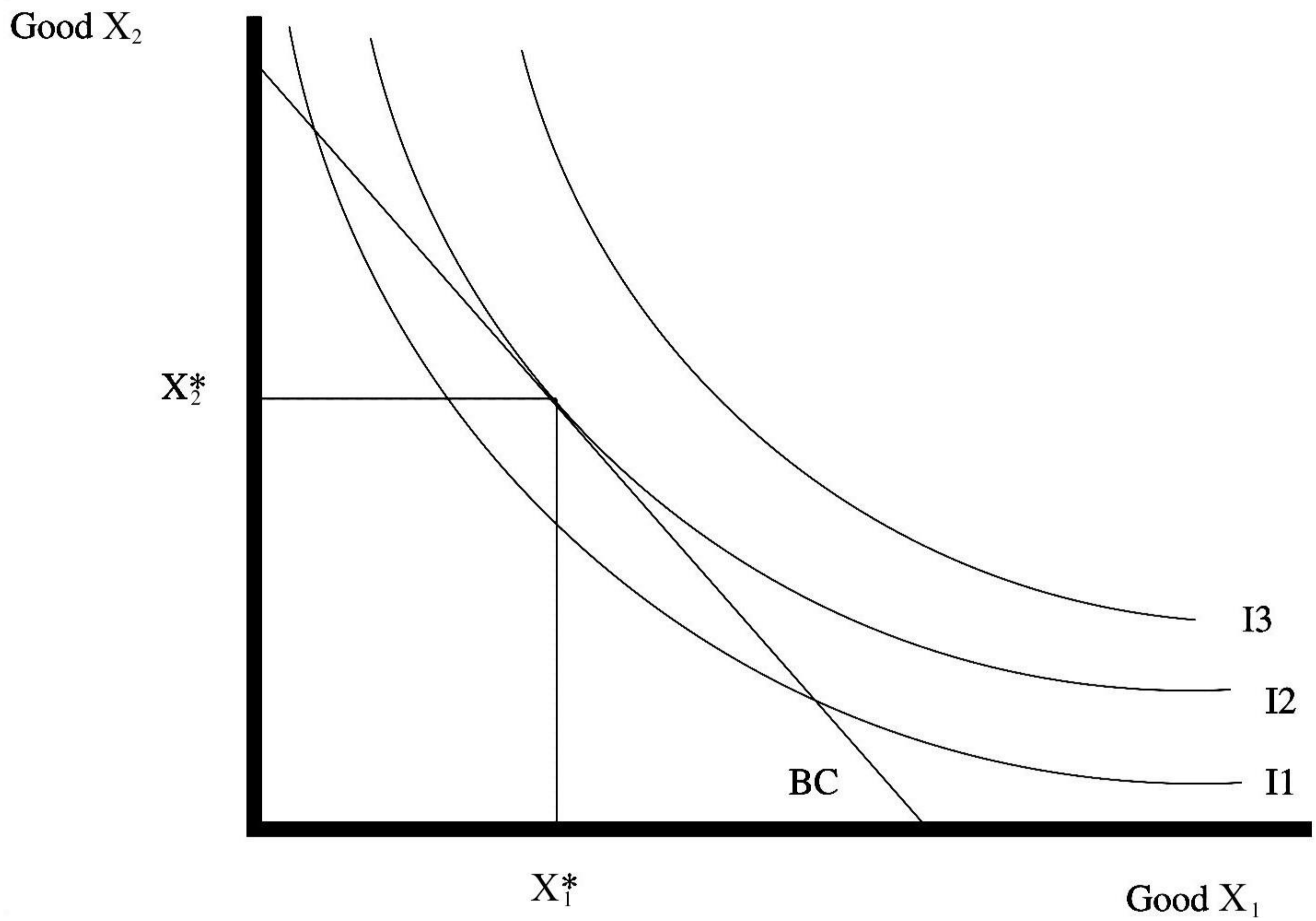
(D.1) Preference relation, $\geq$, defined on the set X of consumption alternatives (choice options)

(D.2) Proposition: Relation $\geq$ is rational, if

1. $\forall\ x,y \in X\ [x \geq y \vee y \geq x]$ (completeness); and
2. $\forall\ x,y,z \in X\ [(x \geq y \wedge y \geq z) \Rightarrow x \geq z]$ (transitivity).

(D.3) Strict preference relation, $>$:

$x > y \Leftrightarrow (x \geq y \wedge \neg\, y \geq x)$



Good X_2
X_2^*
I3
I2
I1
BC
X_1^*
Good X_1

Note: Preference (and strict preference) relations use the same symbols as arithmetic relations (greater than, and greater than or equal to). This, however, shall not lead to any misunderstandings, since it will always be clear from the context whether the formula "$x \geq y$" means "x is preferred over y" or "x is greater than or equal to y". If x and y are consumption alternatives (bundles), i.e. $x,y \in X$ then the formula reads "x is preferred over y", but if x and y are numbers then the formula reads "x is greater than or equal to y".

<u>(D.4) Indifference relation</u>, ~:

$x \sim y \Leftrightarrow (x \geq y \wedge y \geq x)$

<u>(T.1) Properties of preferences</u>: If $\geq$ is rational, then:

1. $\forall\ x \in X\ [x \geq x]$ (reflexivity of $\geq$)
2. $\forall\ x \in X\ [\neg\ x > x]$ (antireflexivity of $>$)
3. $\forall\ x,y,z \in X\ [(x > y \wedge y > z) \Rightarrow x > z]$ (transitivity of $>$)
4. $\forall\ x \in X\ [x \sim x]$ (reflexivity of $\sim$)
5. $\forall\ x,y,z \in X\ [(x \sim y \wedge y \sim z) \Rightarrow x \sim z]$ (transitivity of $\sim$)
6. $\forall\ x,y \in X\ [x \sim y \Rightarrow y \sim x]$ (symmetry of $\sim$)

   (4–6 imply that $\sim$ is an equivalence in X)
7. $\forall\ x,y,z \in X\ [(x > y \wedge y \geq z) \Rightarrow x > z]$

(D.5) Utility function, $u: X \rightarrow \Re$, represents the relation $\succeq$:
$\forall\ x,y \in X\ [x \succeq y \Leftrightarrow u(x) \geq u(y)]$

(T.2) If there is a utility function representing $\succeq$, then $\succeq$ is rational

(T.3) If u is a utility function representing $\succeq$, and f is strictly increasing, then f(u) is also a utility function representing the same relation $\succeq$.

(D.6) Walrasian (competitive) budget set, $B_{p,w}$:

$B_{p,w} = \{\mathbf{x}\in\Re_+^L: \mathbf{p}^T\cdot\mathbf{x} \leq w\}$ ($\{x\in\Re_+^L: \mathbf{p}^T\cdot\mathbf{x} = w\}$ is a budgetary hyperplane)

(T.4) $\forall\ \mathbf{p}\in\Re^L\ \forall\ w>0\ \forall\ \alpha>0\ [B_{p,w} = B_{\alpha p,\alpha w}]$

(D.7) Walrasian demand, $x(\mathbf{p},w)$:

Any $x(\mathbf{p},w)\subset B_{p,w}$, where $\mathbf{p}\in\Re_+^L$, $w\in\Re_+$

(D.8) Weak axiom of revealed preferences:

$\forall\ \mathbf{p},\mathbf{p}'\in\Re_+^L\ \forall\ w,w'>0\ [(\mathbf{p}^T\cdot x(\mathbf{p}',w')\leq w \wedge$

$x(\mathbf{p}',w')\neq x(\mathbf{p},w)) \Rightarrow p'^T\cdot x(\mathbf{p},w)>w']$

(D.9) Walrasian demand is homogeneous of degree 0, if:

$\forall\ \mathbf{p}\in\Re^L\ \forall\ w>0\ \forall\ \alpha>0\ [x(\alpha\mathbf{p},\alpha w) = x(\mathbf{p},w)]$

(D.10) Walrasian demand satisfies the Walras Law, if:

$\forall\ \mathbf{p}\in\Re_+{}^L\ \forall\ w>0\ [\mathbf{x}\in x(\mathbf{p},w) \Rightarrow \mathbf{p}^T\cdot\mathbf{x}=w]$

(D.11) Walrasian demand function, $x(\mathbf{p},w)\in B_{p,w}$

(D.12) Income effect

$(\partial x_1(\mathbf{p},w)/\partial w, ..., \partial x_L(\mathbf{p},w)/\partial w)^T = D_w x(\mathbf{p},w) \in \Re^L$

(D.13) Price effect

$$\begin{bmatrix} \partial x_1(\mathbf{p},w)/\partial p_1 & \ldots & \partial x_1(\mathbf{p},w)/\partial p_L \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \partial x_L(\mathbf{p},w)/\partial p_1 & \ldots & \partial x_L(\mathbf{p},w)/\partial p_L \end{bmatrix} = D_p x(\mathbf{p},w) \in \Re^{L \cdot L}$$

(T.5) If a Walrasian demand function is homogeneous of degree 0, then

$$\forall\ \mathbf{p} \in \Re_+^L\ \forall\ w>0\ [D_p x(\mathbf{p},w)\cdot\mathbf{p} + D_w x(\mathbf{p},w)w = \mathbf{0}]$$

<u>(D.14) Price and income elasticities</u>, $\varepsilon_{\ell k}(\mathbf{p},w)$, $\varepsilon_{\ell w}(\mathbf{p},w)$:

$$\varepsilon_{\ell k}(\mathbf{p},w) = \partial x_\ell(\mathbf{p},w)/\partial p_k : x_\ell(\mathbf{p},w)/p_k$$

$$\varepsilon_{\ell w}(\mathbf{p},w) = \partial x_\ell(\mathbf{p},w)/\partial w : x_\ell(\mathbf{p},w)/w$$

<u>(T.6)</u> If a Walrasian demand function is homogeneous of degree 0, then

$$\forall\ \ell=1,\ldots,L\ [\varepsilon_{\ell 1}(\mathbf{p},w)+\ldots+\varepsilon_{\ell L}(\mathbf{p},w)+\varepsilon_{\ell w}(\mathbf{p},w) = 0]$$

<u>(T.7)</u> If a Walrasian demand function satisfies the Walras Law, then

1. $\forall\ \mathbf{p}\in\Re_+^L\ \forall\ w>0\ [\mathbf{p}^T\cdot D_p x(\mathbf{p},w)+x(\mathbf{p},w)^T = \mathbf{0}^T]$
2. $\forall\ \mathbf{p}\in\Re_+^L\ \forall\ w>0\ [\mathbf{p}^T\cdot D_w x(\mathbf{p},w) = 1]$

(D.15) A price change from $\mathbf{p}$ to $\mathbf{p}'$ is compensated, if it is accompanied by an income change from w to w' such that $w'=\mathbf{p}'^T\cdot x(\mathbf{p},w)$

(T.8) If a Walrasian demand function is homogeneous of degree 0, the Walras Law is satisfied and the weak axiom of revealed preferences is satisfied too, then

$$\forall\ \mathbf{p},\mathbf{p}'\in\Re_+^L\ \forall\ w,w'>0\ [w'=\mathbf{p}'^T\cdot x(\mathbf{p},w) \Rightarrow (\mathbf{p}'-\mathbf{p})^T\cdot(x(\mathbf{p}',w')-x(\mathbf{p},w))\leq 0];$$

the first inequality is strict if $x(\mathbf{p},w)\neq x(\mathbf{p}',w')$

Proof

If $x(\mathbf{p}',w')=x(\mathbf{p},w)$, then the implication is obvious, because $(\mathbf{p}'-\mathbf{p})^T\cdot\mathbf{0}=0$. Thus let us assume that $x(\mathbf{p}',w')\neq x(\mathbf{p},w)$. Then the left hand side of the inequality reads

$$L = \mathbf{p}'^T(x(\mathbf{p}',w')-x(\mathbf{p},w)) - \mathbf{p}^T(x(\mathbf{p}',w')-x(\mathbf{p},w)).$$

The compensation condition implies that $w'=\mathbf{p}'^T\cdot x(\mathbf{p},w)$ (as well as – from the Walras Law (D.10) – that $w'=\mathbf{p}'^T\cdot x(\mathbf{p}',w')$). Hence the first element of L, $L_1=\mathbf{p}'^T(x(\mathbf{p}',w')-x(\mathbf{p},w)) = \mathbf{p}'^T\cdot x(\mathbf{p}',w') - \mathbf{p}'^T\cdot x(\mathbf{p},w)) =$ $= w'-w' = 0$. It is the second element of L, i.e. $L_2=\mathbf{p}^T(x(\mathbf{p}',w')-x(\mathbf{p},w))$ which needs to be calculated.

Proof (cont.)

If $\mathbf{p}'^T \cdot x(\mathbf{p},w) = w'$, it means that $x(\mathbf{p},w)$ can be purchased by prices $\mathbf{p}'$ and income w'. From the weak axiom of revealed preferences (D.8) we infer that $\mathbf{p}^T \cdot x(\mathbf{p}',w') > w$, i.e. $x(\mathbf{p}',w)$ cannot be purchased by prices $\mathbf{p}$ and income w. Since – again by the Walras Law – $\mathbf{p}^T \cdot x(\mathbf{p},w) = w$, then L_2 is the difference between the numbers $\mathbf{p}^T \cdot x(\mathbf{p}',w') > w$ and $\mathbf{p}^T \cdot x(\mathbf{p},w) = w$. Thus it is a positive number. Hence, if $L_1 = 0$ and $L_2 > 0$, then $L < 0$, which was to be proved.

(T.9) By the assumptions of T.8, the coordinates of a demand function are non increasing with respect to corresponding prices, as long as their changes are compensated ("The Law of Demand")

(T.10) By the assumptions of T.8 (substituting $w' - w = (\mathbf{p}' - \mathbf{p})^T \cdot x(\mathbf{p}, w)$ with $dw = x(\mathbf{p}, w)^T \cdot d\mathbf{p}$), if the demand function is differentiable, then $d\mathbf{p}^T \cdot (D_p x(\mathbf{p}, w) + D_w x(\mathbf{p}, w) \cdot x(\mathbf{p}, w)^T) \cdot d\mathbf{p} \leq 0$ (negative semi-definiteness)

<u>(D.16) Slutsky Matrix (substitution matrix)</u>, $S(\mathbf{p},w)$:

$S(\mathbf{p},w) = D_p x(\mathbf{p},w) + D_w x(\mathbf{p},w) \cdot x(\mathbf{p},w)^T$;

$S_{\ell k}(\mathbf{p},w) = \partial x_\ell(\mathbf{p},w)/\partial p_k + (\partial x_\ell(\mathbf{p},w)/\partial w) \cdot x_k(\mathbf{p},w)$

<u>(T.11)</u> $\forall\ \mathbf{p} \in \Re_+^L\ \forall\ w>0\ \forall\ \ell=1,\ldots,L\ [S_{\ell\ell}(\mathbf{p},w) \leq 0]$ (if the assumptions of T.8 are satisfied)

<u>(T.12)</u> By the assumptions of T.8:

$\forall\ \mathbf{p} \in \Re_+^L\ \forall\ w>0\ \forall\ \ell=1,\ldots,L\ [\partial x_\ell(\mathbf{p},w)/\partial p_\ell > 0 \Rightarrow \partial x_\ell(\mathbf{p},w)/\partial w < 0]$

(Giffen good must be *inferior*)

(D.17) Monotonicity of preferences

$\forall\ x,y\in\Re_+^L\ [x \gg y \Rightarrow x > y]$ ($\gg$ means that all coordinates satisfy >)

(D.18) Strong monotonicity of preferences

$\forall\ x,y\in\Re_+^L\ [(x\geq y \wedge x\neq y) \Rightarrow x > y]$ (> denotes strict preference relation)

(D.19) Local non-satiation of preferences

$\forall\ x\in\Re_+^L\ \forall\ \varepsilon>0\ \exists\ y\in\Re_+^L\ [\|y-x\|<\varepsilon \wedge y>x]$ (> denotes strict preference relation)

(D.20) Convexity of preferences

$\forall$ x,y,z$\in\Re_+^L$ $\forall$ $\alpha\in[0,1]$ [(y$\geq$x $\wedge$ z$\geq$x) $\Rightarrow$
αy+(1–α)z$\geq$x] ($\geq$ denotes preference relation)

({y$\in\Re_+^L$: y$\geq$x} are convex sets)

(D.21) Strictly convex preferences

$\forall$ x,y,z$\in\Re_+^L$ $\forall$ $\alpha\in(0,1)$ [(y$\geq$x $\wedge$ z$\geq$x $\wedge$ y$\neq$z) $\Rightarrow$
αy+(1–α)z>x] (> and $\geq$ denote strict preference and preference relations, respectively)

(D.22) Homothetic preferences

$\forall$ x,y$\in\Re_+^L$ $\forall$ $\alpha\geq0$ [x~y $\Rightarrow$ αx~αy]

(D.23) Quasi-linear preferences

1. $\forall$ $x,y \in \Re \times \Re_+^{L-1}$ $\forall$ $\alpha > 0$ $[x \sim y \Rightarrow$
$x+(\alpha,0,...0) \sim y+(\alpha,0,...0)]$
2. $\forall$ $x \in \Re \times \Re_+^{L-1}$ $\forall$ $\alpha > 0$ $[x+(\alpha,0,...0) > x]$

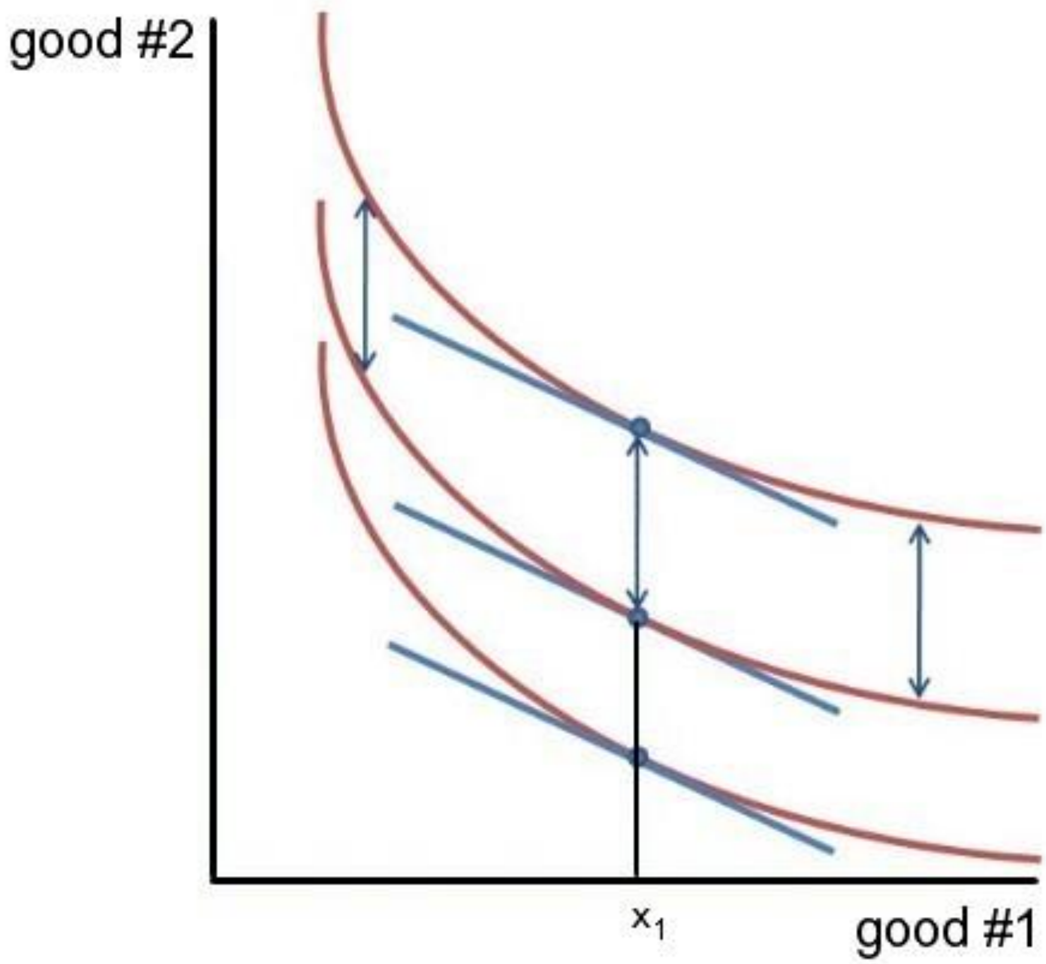
(D.24) Lexicographic preferences (for L=2)

$\forall$ $x,y \in \Re_+^2$ $[x \geq y \Leftrightarrow (x_1 > y_1 \vee (x_1 = y_1 \wedge x_2 \geq y_2))]$

(D.25) Continuous preferences

$\forall$ $n=1,2,...$ $\forall$ $x_n,y_n \in \Re_+^L$ $[(x_n \geq y_n \wedge x = \lim_{n\to\infty} x_n \wedge$
$y = \lim_{n\to\infty} y_n) \Rightarrow x \geq y]$

Quasi-linear preferences



(T.13) Lexicographic preferences are not continuous

Proof

Let us take the following example $x_n = (2/n,0)$ and $y_n = (1/n,1)$ – a sequence of pairs whose elements are lexicographically ordered. For each n=1,2,... $x_n \geq y_n$ holds. Of course $x_n \rightarrow (0,0)$ and $y_n \rightarrow (0,1)$. And yet $(0,1) > (0,0)$, i.e. the limit values satisfy a reverse relationship.

(T.14) Lexicographic preferences cannot be represented by a utility function

(T.15) If the relation ≥ is rational and continuous, then there exists a utility function which represents it

(D.26) Utility Maximization Problem (UMP)

$\text{Max}_x\{u(x)\text{: } x \in \Re_+^L \wedge \mathbf{p}^T \cdot \mathbf{x} \leq w\}$

(T.16) If $\mathbf{p} \gg \mathbf{0}$, and u is continuous, then UMP has a solution

(T.17) If u is a continuous function representing rational preferences $\geq$, which are locally non-satiated in $X=\Re_+^L$, then the Walrasian demand $x(\mathbf{p},w)$ has the following properties:

1. $\forall\ \mathbf{p}\in\Re_+^L\ \forall\ w>0\ \forall\ \alpha>0\ [x(\alpha\mathbf{p},\alpha w) = x(\mathbf{p},w)]$ (homogeneity of degree 0)

2. $\forall\ \mathbf{p},\mathbf{x}\in\Re_+^L\ \forall\ w>0\ [\mathbf{x}\in x(\mathbf{p},w) \Rightarrow \mathbf{p}^T\cdot\mathbf{x}=w]$ (The Walras Law)

3. If $\geq$ is convex (hence u is quasi-concave) then $x(\mathbf{p},w)$ is a convex set

4. If $\geq$ is strictly convex (hence u is strictly quasi-concave) then $x(\mathbf{p},w)$ may have just one element

<u>(T.18) Kuhn-Tucker conditions</u>

If u is differentiable and its derivative is a continuous function then each solution $\mathbf{x}^* \in x(\mathbf{p},w)$ of the UMP satisfies the following condition:

$$\exists\ \lambda \geq 0\ \forall\ \ell=1,\ldots,L\ [\partial u(\mathbf{x}^*)/\partial x_\ell \leq \lambda p_\ell \wedge (x^*_\ell > 0 \Rightarrow \partial u(\mathbf{x}^*)/\partial x_\ell = \lambda p_\ell)]$$

$$(D_x u(x^*) \leq \lambda \mathbf{p} \wedge (\mathbf{x}^* \gg \mathbf{0} \Rightarrow D_x u(x^*) = \lambda \mathbf{p}))$$

<u>(D.27) Internal solution</u>: $\mathbf{0} \ll \mathbf{x}^* \in x(\mathbf{p},w)$

(T.19) By the assumptions of T.18 an internal solution must satisfy the following conditions:

$\forall\ \ell,k=1,\dots,L\ [\partial u(\mathbf{x}^*)/\partial x_\ell : \partial u(\mathbf{x}^*)/\partial x_k = p_\ell : p_k]$

(D.28) Marginal Rate of Substitution, $MRS_{\ell k}$

$MRS_{\ell k} = \partial u(\mathbf{x}^*)/\partial x_\ell : \partial u(\mathbf{x}^*)/\partial x_k$

(D.29) Elements y of the set X can be interpreted as lotteries $L=(p_1,\dots,p_N)$, where $p_1+\dots+p_N=1$.
$\mathcal{L}$ is the set of such lotteries; their outcomes – numbered 1,...,N – are determined.

(T.20) A convex combination of lotteries is also a lottery (with probabilities calculated as convex combinations of probabilities from original lotteries)

(T.21) Preferences $\geq$ are continuous on $\mathcal{L}$ if for every $L, L', L'' \in \mathcal{L}$ the following sets are closed: $\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \geq L''\}$ and $\{\alpha \in [0,1]: L'' \geq \alpha L + (1-\alpha)L'\}$

(D.30) Preferences $\geq$ satisfy the independence axiom in $\mathcal{L}$, if for every $L, L', L'' \in \mathcal{L}$ and every number $\alpha \in (0,1)$

$$L \geq L' \text{ is satisfied if and only if}$$

$$\alpha L + (1-\alpha)L'' \geq \alpha L' + (1-\alpha)L''$$

(D.31) The von Neumann-Morgenstern (vNM) expected utility function, U:

U: $\mathcal{L} \to \Re$ has the "expected utility form", i.e.

$\exists\ u_1,\dots,u_N \in \Re\ \forall\ L=(p_1,\dots,p_N) \in \mathcal{L}\ [U(L) = u_1p_1+\dots+u_Np_N]$

(D.32) Bernoulli utilities

Numbers u_n from D.33 can be interpreted as utilities of "degenerated lotteries"

$$L^1=(1,0,\dots,0),\dots,L^N=(0,\dots,0,1)$$

<u>(T.22)</u> A utility function $U: \mathcal{L} \to \Re$ has the "expected utility form" if and only if

$\forall$ K=1,2,... $\forall L_1,L_2,...,L_K \in \mathcal{L}$ $\forall$ $\alpha_1,...,\alpha_K>0$

$[\alpha_1+...+\alpha_K=1 \Rightarrow$

$$U(\alpha_1 L_1+...+\alpha_K L_K) = \alpha_1 U(L_1)+...+\alpha_K U(L_K)]$$

<u>Proof</u> $\Leftarrow$

Let $L=(p_1,...,p_N)$. We define degenerated lotteries $L^1,...,L^N$ such that $L^i=(0,...,0,1,0,...,0)$; the *i*th probability is equal to 1. Then $L=p_1L^1+...+p_NL^N$ and $U(L) = U(p_1L^1+...+p_NL^N) = p_1U(L^1)+...+p_NU(L^N) = p_1u_1+...+p_Nu_N$, where the second equality holds by the assumption.

Proof $\Rightarrow$

Let us consider a combination of lotteries $(L_1,\ldots,L_k;\alpha_1,\ldots,\alpha_k)$, where $L_k=(p_1^k,\ldots,p_N^k)$. Let $L'=\alpha_1 L_1+\ldots+\alpha_k L_k$. Hence it can be calculated: $U(L') =$

$= U(\alpha_1 L_1+\ldots+\alpha_k L_k) =$

$= u_1(\alpha_1 p_1^1+\ldots+\alpha_k p_1^k)+\ldots+u_N(\alpha_1 p_N^1+\ldots+\alpha_k p_N^k) =$

$= \alpha_1(u_1 p_1^1+\ldots+u_N p_N^1)+\ldots+\alpha_k(u_1 p_1^k+\ldots+u_N p_N^k) =$

$= \alpha_1 U(L_1)+\ldots+\alpha_k U(L_k)$, where the second equality follows from the assumption.

(T.23) The expected utility theorem

If a preference relation $\geq$ in $\mathcal{L}$ is rational, and satisfies independence and continuity axioms then it can be represented by a vNM function, i.e. numbers u_n can be attributed to each outcome n=1,...N such that:

$\forall L=(p_1,...,p_N),L'=(p'_1,...,p'_N)\in\mathcal{L}\ [L\geq L' \Leftrightarrow u_1p_1+...+u_Np_N\geq u_1p'_1+...+u_Np'_N]$

(D.33) Lottery with monetary payoffs

The lottery has monetary payments $x_1,...,x_N$, and Bernoulli utilities are a function u: $\Re \rightarrow \Re$ of these payments: $u(x_1),...,u(x_N)$. *Certainty equivalent* of the lottery $L=(p_1,...,p_N)$ is a number $c(L,u)$ such that $u(c(L,u))=u(x_1)p_1+...+u(x_N)p_N$

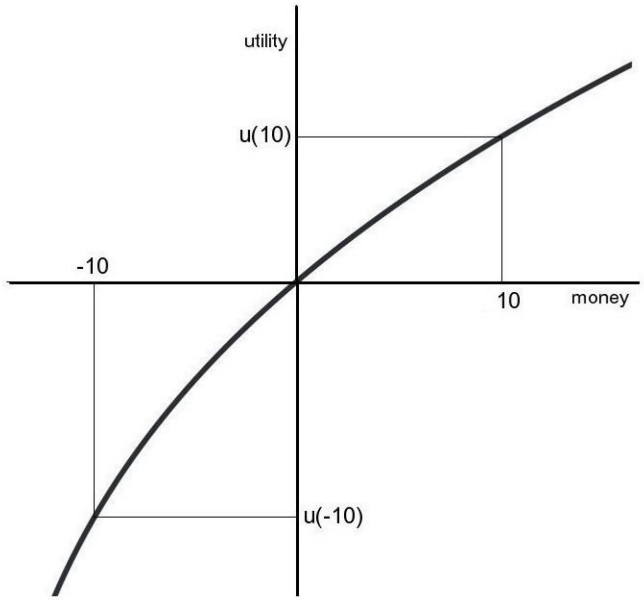
(D.34) Risk aversion and risk neutrality implied by Bernoulli utilities

$\forall L=(p_1,\ldots,p_N)\in\mathcal{L}\ [u(x_1)p_1+\ldots+u(x_N)p_N \leq u(x_1p_1+\ldots+x_Np_N)]$ (aversion)

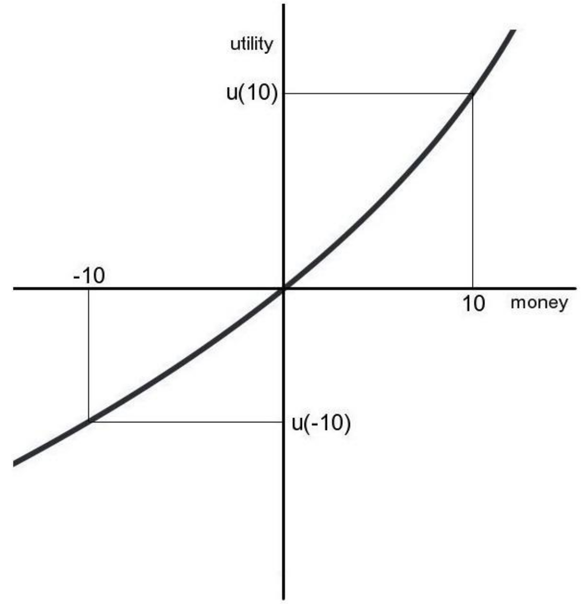
$\forall L=(p_1,\ldots,p_N)\in\mathcal{L}\ [u(x_1)p_1+\ldots+u(x_N)p_N = u(x_1p_1+\ldots+x_Np_N)]$ (neutrality)

(T.24) The following conditions are equivalent:

1. A consumer is risk averse
2. Function u is concave
3. $\forall L=(p_1,\ldots,p_N)\in\mathcal{L}\ [c(L,u) \leq x_1p_1+\ldots+x_Np_N]$



utility
u(10)
-10
10
money
u(-10)



utility
u(10)
-10
10
money
u(-10)

Note (***the Allais paradox***)

There are three outcomes of lotteries:

- x_1=0,
- x_2=1 million USD, and
- x_3=5 million USD

Experiment 1 (most people prefer L_1):
Choose between two lotteries:
L_1=(0,1,0) and L_2=(0.01,0.89,0.1)

Experiment 2 (most people prefer L_4):
Choose between two lotteries:
L_3=(0.89,0.11,0) and L_4=(0.9,0,0.1)

Theorem

People who choose L_1 in the first experiment and L_4 in the second one do not comply with the vNM theory.

Proof:

If the vNM theory was followed, then Bernoulli's utilities would have been applied:

- $u(x_1)=u_1$,
- $u(x_2)=u_2$, and
- $u(x_3)=u_3$.

Proof (cont.)

The outcome of the first experiment implies that:

$u_2 > 0.01u_1 + 0.89u_2 + 0.1u_3$.

The outcome of the second experiment implies that:

$0.9u_1 + 0.1u_3 > 0.89u_1 + 0.11u_2$.

These two inequalities contradict each other, since the second one can rewritten as:

$0.01u_1 + 0.1u_3 > 0.11u_2$, and consequently

$0.01u_1 + 0.1u_3 > u_2 - 0.89u_2$, or

$0.01u_1 + 0.89u_2 + 0.1u_3 > u_2$

which contradicts the first one.

Note (***the Machina paradox***)

- $x_1=0$,
- $x_2=10$ USD, and
- $x_3=10{,}000$ USD

- $u(x_1)<u(x_2)<u(x_3)$;
- hence $L_1 \leq L_2 \leq L_3$,
- where $L_1=(1,0,0)$, $L_2=(0,1,0)$, and $L_3=(0,0,1)$.
- By the independence axiom (D.32):
- $0.01L_2+0.99L_3 \geq 0.01L_1+0.99L_3$.

Note (cont.)

Most people choose otherwise, i.e.

- $0.01L_2+0.99L_3 \leq 0.01L_1+0.99L_3$.

Non-technical summary

A. Preference relations do not have to be rational in real life situations; empirical studies do not confirm transitivity in some circumstances

B. If weak axiom of revealed preferences holds then the Walrasian demand has to satisfy certain algebraic conditions – for instance the Slutsky matrix (substitution matrix) is negatively semi-definite

C. Rational and continuous preferences can be represented by utility functions; consumer choice problem boils down to mathematical programming (UMP); if the function is differentiable then the Kuhn-Tucker conditions determine MRS

D. In a probabilistic setting preferences can be represented by vNM expected utility functions; however, empirical research does not necessarily confirm that consumers act as predicted by the vNM theory

2. Individual and aggregate demand

(D.1) Expenditure Minimisation Problem (EMP)

$\text{Min}_x\{\mathbf{p}^T\cdot\mathbf{x}: \mathbf{x}\geq\mathbf{0} \wedge u(\mathbf{x})\geq u\}$ (it is assumed that $\mathbf{p}\gg\mathbf{0}$ and $u>u(\mathbf{0})$)

<u>(T.1)</u> Let us assume that u is a continuous utility function representing a locally non-satiated preference relation $\geq$ in $X=\Re_+^L$ and $\mathbf{p}\gg\mathbf{0}$. Then:

1. If $\mathbf{x}^*$ is a solution of an UMP for some $w>0$, then $\mathbf{x}^*$ is also a solution of the EMP for $u=u(\mathbf{x}^*)$. Moreover, the minimum expenditure is equal to w.
2. If $\mathbf{x}^*$ is a solution of an EMP for some $u>u(\mathbf{0})$, then $\mathbf{x}^*$ is also a solution of the UMP for $w=\mathbf{p}^T\cdot\mathbf{x}^*$. Moreover, the maximum utility is equal to u.

Proof:

1. Let us assume that $\mathbf{x}^*$ is not a solution to EMP (with $u \geq u(\mathbf{x}^*)$). It means that there exists $\mathbf{x'}$ such that $u(\mathbf{x'}) \geq u(\mathbf{x}^*)$ and $\mathbf{p}^T\mathbf{x'} < \mathbf{p}^T\mathbf{x}^* \leq w$. From the local non-satiation axiom we know that there exists $\mathbf{x''}$ close to $\mathbf{x'}$ such that $u(\mathbf{x''}) > u(\mathbf{x'})$ and still $p^T\mathbf{x''} < w$. But this implies that $\mathbf{x''} \in B_{\mathbf{p},w}$ and $u(\mathbf{x''}) > u(\mathbf{x}^*)$. This contradicts the assumption that $\mathbf{x}^*$ solved the UMP. Hence $\mathbf{x}^*$ must solve the EMP. The equality $\mathbf{p}^T\mathbf{x}^* = w$ results from T.17.2 (Walras Law proved in Lecture 1).

Proof (cont.)

2. If $u > u(\mathbf{0})$ then $\mathbf{x}^* \neq \mathbf{0}$. Consequently $\mathbf{p}^T\mathbf{x}^* > 0$. Let us assume that $\mathbf{x}^*$ is not an optimum for the UMP with $w = \mathbf{p}^T\mathbf{x}^*$. That is there exists $\mathbf{x'}$ such that $u(\mathbf{x'}) > u(\mathbf{x}^*)$ and $\mathbf{p}^T\mathbf{x'} \leq \mathbf{p}^T$. Let us take $\mathbf{x''} = \alpha\mathbf{x'}$, $\alpha \in (0,1)$. From the continuity of u, if α is sufficiently close to 1, then $u(x'') > u(x^*)$ and $p^T x'' < p^T x^*$. This, however, contradicts the optimality of x^* for EMP. Thus x^* must be an optimum for UMP with $w = p^T x^*$, so that $u(x^*)$ is the maximum utility. The fact that this is equal to u will result from T.3.2.

<u>(D.2) Expenditure function</u>, $e(\mathbf{p},u)=\mathbf{p}^T\cdot\mathbf{x}^*$, where $\mathbf{x}^*$ is a solution of EMP for some $\mathbf{p}\gg\mathbf{0}$ and $u>u(\mathbf{0})$

<u>(T.2)</u> By the assumptions of T.1 a function $e(\mathbf{p},u)$ is:

1. homogenous of degree 1 with respect to $\mathbf{p}$
2. strictly increasing with respect to u and non-decreasing with respect to p_ℓ for all $\ell=1,...,L$
3. concave with respect to $\mathbf{p}$
4. continuous with respect to $\mathbf{p}$ and u

<u>(D.3) Hicks (compensated) demand function</u>, h(**p**,u):

Set of solutions of EMP, h(**p**,u)$\subset\Re_+^L$, is called Hicksian (compensated) demand; if it consists of a single point, it defines a Hicksian demand function

(T.3) By the assumptions of T.1 Hicksian demand has the following properties:

1. homogeneity of degree 0 with respect to **p**, $(\forall \mathbf{p}\in\Re^L\ \forall\ u\in\Re\ \forall\ \alpha>0\ [h(\alpha\mathbf{p},u)=h(\mathbf{p},u)])$
2. no excess utility $(\forall x\in h(\mathbf{p},u)\ [u(\mathbf{x})=u])$
3. if $\geq$ is convex, then $h(\mathbf{p},u)$ is convex
4. if $\geq$ is strictly convex (hence u(.) is strictly quasi-concave), then $h(\mathbf{p},u)$ consists of a single point

(T.4) The Kuhn-Tucker conditions for EMP

$$\exists\ \lambda\geq 0\ [\mathbf{p}\geq\lambda D_x u(\mathbf{x}^*) \wedge \mathbf{x}^{*T}\cdot(\mathbf{p}-\lambda D_x u(\mathbf{x}^*))=0]$$

(D.4) Hicks-compensated income

$h(\mathbf{p},u) = x(\mathbf{p},e(\mathbf{p},u))$; when the prices $\mathbf{p}$ change for $\mathbf{p}'$: then $w_{Hicks}=e(\mathbf{p}',u)-w$

(T.5) The law of demand

By the assumptions of T.1, let $h(\mathbf{p},u)$ be a Hicksian demand function. Then

$\forall\ \mathbf{p}',\mathbf{p}''\in\Re^L\ [(\mathbf{p}''-\mathbf{p}')^T\cdot(h(\mathbf{p}'',u)-h(\mathbf{p}',u))\leq 0]$

(T.6) $(p''_\ell-p'_\ell)(h_\ell(\mathbf{p}'',u)-h_\ell(\mathbf{p}',u))\leq 0$ (note: the relationship may not hold for a Walrasian demand function)

<u>(T.7)</u> By the assumptions of T.1, let $\geq$ defined over $X=\Re_+^L$ be strictly convex. Then

$\forall\ \mathbf{p}\in\Re^L\ \forall\ u\in\Re\ [h(\mathbf{p},u) = D_p e(\mathbf{p},u)]$

$(\forall\ \mathbf{p}\in\Re^L\ \forall\ u\in\Re\ \forall\ \ell=1,\ldots,L\ [h_\ell(\mathbf{p},u)=\partial e(\mathbf{p},u)/\partial p_\ell])$

<u>(T.8)</u> In T.7 let h be differentiable in $(\mathbf{p},u)$, and its derivatives be continuous functions. Then

1. $D_p h(\mathbf{p},u) = D_p^2 e(\mathbf{p},u)$
2. $D_p h(\mathbf{p},u)$ is negatively semi-definite
3. $D_p h(\mathbf{p},u)$ is symmetric
4. $D_p h(\mathbf{p},u)\cdot\mathbf{p} = \mathbf{0}$

(T.9) Slutsky equation

In T.7 let $u=u(\mathbf{x}^*)$, where $\mathbf{x}^*$ is a solution to an UMP. Then

$$\forall\, \mathbf{p}\in\Re^L\ \forall\, w>0\ [D_p h(\mathbf{p},u) =$$
$$= D_p x(\mathbf{p},w)+D_w x(\mathbf{p},w)\cdot x(\mathbf{p},w)^T]$$

$$(\forall\, \mathbf{p}\in\Re^L\ \forall\, w>0\ \forall\, \ell,k=1,\ldots,L\ [\partial h_\ell(\mathbf{p},u)/\partial p_k =$$
$$= \partial x_\ell(\mathbf{p},w)/\partial p_k+x_k(\mathbf{p},w)\cdot\partial x_\ell(\mathbf{p},w)/\partial w])$$

(D.5) Strong axiom of revealed preferences (for a demand function)

$\forall$ N=1,2,... [$\forall$ $(\mathbf{p}^1,w^1),\ldots,(\mathbf{p}^N,w^N)\in\Re^{L+1}$ $\forall$ $n\le N-1$

$[(x(\mathbf{p}^{n+1},w^{n+1})\neq x(\mathbf{p}^n,w^n) \wedge \mathbf{p}^{nT}\cdot x(\mathbf{p}^{n+1},w^{n+1})\le w^n)] \Rightarrow$

$\mathbf{p}^{NT}\cdot x(\mathbf{p}^1,w^1)>w^N]$

(T.10) If a Walrasian demand function $x(\mathbf{p},w)$ satisfies strong axiom of revealed preferences, then there exists a rational preference relation $\geq$ which is consistent with $x(\mathbf{p},w)$, i.e.:

$\forall$ $\mathbf{p}\in\Re^L$ $\forall$ $w>0$ $\forall$ $\mathbf{y}\in B_{p,w}$ $[\mathbf{y}\neq x(\mathbf{p},w) \Rightarrow x(\mathbf{p},w)>\mathbf{y}]$

(D.6) Aggregate demand for I consumers each of which has a rational preference relation $\succeq_i$ and derived from it a Walrasian demand function $x_i(\mathbf{p},w_i)$ reads:
$x(\mathbf{p},w_1,\ldots,w_I) = x_1(\mathbf{p},w_1)+\ldots+x_I(\mathbf{p},w_I)$

(T.11) If functions in D.6 are differentiable then $x(\mathbf{p},w_1,\ldots,w_I) = x(\mathbf{p},w)$, where $w=w_1+\ldots+w_I$ if and only if for any good $\ell=1,\ldots,L$, any consumer $i,j=1,\ldots,I$ and any income distribution $(w_1,\ldots,w_I)$ the following equality holds:

$$\partial x_{\ell i}(\mathbf{p},w_i)/\partial w_i = \partial x_{\ell j}(\mathbf{p},w_j)/\partial w_j$$

(T.12) If the income distribution $(w_1,...,w_I)$ depends only on $w=w_1+...+w_I$ and $\mathbf{p}$, then the aggregate demand does not depend on this distribution, but only on w and $\mathbf{p}$:

$$x(\mathbf{p},w_1,...,w_I) = x(\mathbf{p},w)$$

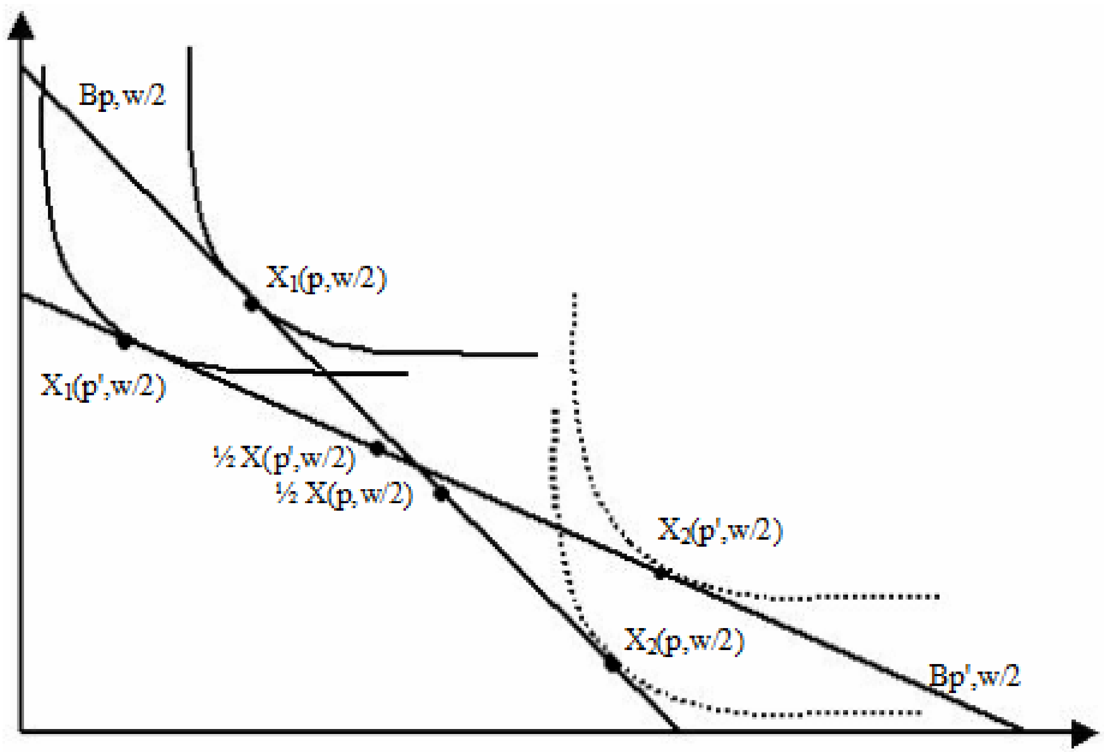
(T.13) Conditions of T.12 are satified when $w_i(\mathbf{p},w)=\alpha_i w$ ($\alpha_1+...+\alpha_I=1$, $\alpha_1,...,\alpha_I$ – parameters)

<u>(T.14)</u> The following characteristics of individual demands $x_i(\mathbf{p},w_i)$ carry on to aggregate demand $x(\mathbf{p},w_1,\ldots,w_I)$:

1. continuity,
2. homogeneity of degree 0,
3. Walras Law

(T.15) Weak axiom of revealed preferences satisfied by all individual demand functions $x_1(\mathbf{p},w_1),\ldots,x_I(\mathbf{p},w_I)$ – when income distribution is as in T.13 – does not imply this axiom for the aggregated demand function $x(\mathbf{p},w)$

Proof



Proof (cont.)

Let the market consist of two goods only. Two consumers (1 and 2) revealed preferences as a result of choosing bundles when prices were $\mathbf{p}$ and $\mathbf{p'}$. In both cases each of them had a half of the total income *w* both of them enjoy jointly. Let us denote their (identical) budget sets constructed with respective prices by $B_{p,w/2}$ and $B_{p',w/2}$. As the picture shows, the optimum bundles for the consumer 1 – i.e. individual demands for given prices and income – are $x_1(\mathbf{p},w/2)$ and $x_1(\mathbf{p'},w/2)$, respectively. For the consumer 2 optimum choices are $x_2(\mathbf{p},w/2)$ and $x_2(\mathbf{p'},w/2)$.

<u>Proof (cont.)</u>

Thus aggregate demand for prices $\mathbf{p}$ will read $x(\mathbf{p},w)= x_1(\mathbf{p},w/2)+x_2(\mathbf{p},w/2)$, and for prices $\mathbf{p'}$: $x(\mathbf{p'},w)= x_1(\mathbf{p'},w/2)+x_2(\mathbf{p'},w/2)$. The picture above shows halves of this aggregate demand, i.e. average bundles. As the pictures shows, these average bundles cannot satisfy the weak axiom of revealed preferences. It is evident that $\mathbf{p}^T x(\mathbf{p'},w)/2<w/2$, i.e. $\mathbf{p}^T x(\mathbf{p'},w)<w$ and $\mathbf{p'}^T x(\mathbf{p},w)/2<w/2$, i.e. $\mathbf{p'}^T x(\mathbf{p},w)<w$. In other words, the bundle $x(\mathbf{p'},w)$ can be afforded by prices $\mathbf{p}$, but $x(\mathbf{p},w)$ is chosen. Likewise $x(\mathbf{p},w)$ can be afforded by $\mathbf{p'}$, but $x(\mathbf{p'},w)$ is chosen instead. The first fact reveals the preference $x(\mathbf{p},w)>x(\mathbf{p'},w)$ while the second one: $x(\mathbf{p'},w)>x(\mathbf{p},w)$.

(D.7) Demand function (whether individual or aggregate) $x(\mathbf{p},w)$ satisfies the Uncompensated Law of Demand (ULD) when:

$\forall\ \mathbf{p},\mathbf{p}'\in\Re^L\ \forall\ w>0$

$$[(\mathbf{p}'-\mathbf{p})^T\cdot(x(\mathbf{p}',w)-x(\mathbf{p},w))\leq 0 \wedge (x(\mathbf{p}',w)\neq x(\mathbf{p},w) \Rightarrow (\mathbf{p}'-\mathbf{p})^T\cdot(x(\mathbf{p}',w)-x(\mathbf{p},w))<0)]$$

(T.16) If individual demand functions $x_1(\mathbf{p},w_1),\ldots,x_I(\mathbf{p},w_I)$ satisfy D.7, and income distribution is determined as in T.13, aggregate demand $x(\mathbf{p},w)$ satisfies D.7 too.

<u>(T.17)</u> By T.16 assumptions aggregate demand function satisfies the weak axiom of revealed preferences, i.e. $\forall\ \mathbf{p},\mathbf{p}'\in\Re_+^L\ \forall\ w,w'>0$ $[(\mathbf{p}^T\cdot x(\mathbf{p}',w')\leq w \wedge x(\mathbf{p}',w')\neq x(\mathbf{p},w)) \Rightarrow \mathbf{p}'^T\cdot x(\mathbf{p},w)>w']$

<u>(D.8) Indirect utility function</u>, $v(\mathbf{p},w)=u(\mathbf{x}^*)$, where $\mathbf{x}^*$ solves UMP (with $\mathbf{p}\gg\mathbf{0}$ and $w>0$)

<u>(T.18)</u> If u is a continuous utility function representing preference relation $\geq$, which is locally non-satiated in $X=\Re_+^L$, then indirect utility function v has the following properties:

1. v is homogenous of degree 0
2. v is strictly increasing with respect to w and non-increasing with respect to p_ℓ (for any ℓ=1,...,L)
3. v is quasi-convex (i.e. sets $\{(\mathbf{p},w)\in\Re_+^{L+1}: v(\mathbf{p},w)\leq v_0\}$ are convex for any v_0)
4. v is continuous with respect to **p** i w

<u>(D.9)</u> For an aggregate demand function $x(\mathbf{p},w)$ there exists a representative consumer in a <u>positive sense</u>, if there is a rational preference relation $\succeq$ in $\Re_+^L$ that yields this demand function, i.e. if:

$$\forall\ \mathbf{x},\mathbf{p}\in\Re_+^L\ \forall\ w\in\Re\ [(\mathbf{p}^T\cdot\mathbf{x}\leq w \wedge \mathbf{x}\neq x(\mathbf{p},w)) \Rightarrow x(\mathbf{p},w)\succ\mathbf{x}]$$

<u>(D.10) Bergson-Samuelson social welfare function</u>, $W: \Re^I\rightarrow\Re$, which aggregates individual utilities

(D.11) Bergson-Samuelson problem, (BSP):
Max $\{W(v_1(\mathbf{p},w_1),...,v_I(\mathbf{p},w_I)): w_1+...+w_I \leq w\}$, where $\mathbf{p}$,w are arbitrary, and W – a social welfare function

(T.19) Let us assume that for any $\mathbf{p}$ and w, $w_1(\mathbf{p},w),...,w_I(\mathbf{p},w)$ are solutions to BSP (from D.11). Then the value of W of this solution, $v(\mathbf{p},w)$, is an indirect utility function of the consumer who is representative (in a positive sense) for the aggregate demand function $x(\mathbf{p},w)=x_1(\mathbf{p},w_1)+...+x_I(\mathbf{p},w_I)$

<u>(D.12)</u> Let us assume that for an aggregate function $x(\mathbf{p},w)=x_1(\mathbf{p},w_1(\mathbf{p},w))+...+x_I(\mathbf{p},w_I(\mathbf{p},w))$ there exists a representative consumer in a positive sense and the corresponding preference relation is $\geq$. If for arbitrary $\mathbf{p}$ and w income distribution $w_1(\mathbf{p},w),...,w_I(\mathbf{p},w)$ solves BSP – and thus the values of these solutions are an indirect utility function for $\geq$ – then the consumer is called representative in a <u>normative sense</u> for the function W.

<u>(T.20)</u> If the income distribution $w_1(\mathbf{p},w),\ldots,w_I(\mathbf{p},w)$ which solves a BSP depends on w, and not on **p**, then every consumer representative in positive sense is also representative in normative case for W

<u>(T.21)</u> If consumers have homothetic preferences representing the same utility function which is homogeneous of degree 1, and $\alpha_1+\ldots+\alpha_I=1$, $\alpha_1,\ldots,\alpha_I>0$, then for the Bergson-Samuelson function given by the formula $W(u_1,\ldots,u_I)=\alpha_1 \mathit{ln} u_1+\ldots+\alpha_I \mathit{ln} u_I$ each consumer who is representative in a positive sense is also representative in a normative sense

Non-technical summary

A. Consumer's optimum choice can be seen from two points of view: utility maximisation subject to a budget constraint (UMP) or expenditure minimisation subject to a minimum utility level (EMP). It turns out that if the utility function is continuous and represents locally non-satiated preferences then both approaches imply the same choices

B. UMP implies a Walrasian (uncompensated) demand function x(**p**,w), and EMP a compensated (Hicksian) demand function, h(**p**,u). The former is defined by observable variables while the latter contains a non-observable one (u). Nevertheless the Slutsky equation lets characterize the latter by the former, and First Order Conditions for internal solutions are identical in both cases.

C. Hicksian demand function satisfies the "Law of Demand" which is satisfied by Walrasian demand functions only when (by additional assumptions) income is compensated

D. Strong axiom of revealed preferences implies that the Walrasian demand reflects some rational preferences; weak axiom of revealed preferences is not sufficient for such an argument.

E. Not all the properties of individual demand functions are inherited by the aggregate demand function. For instance, continuity, homogeneity of degree 0, Walras Law, and the Law of Uncompensated Demand are inherited, but weak axiom of revealed preferences is not.

F. A representative consumer in a positive sense is sought for an aggregate demand function in order to understand this demand as a result of this consumer's preferences. In addition, if a Bergson-Samuelson social welfare function is defined, one can analyze if this consumer's choices maximise the function. If yes, the consumer is called representative in a normative sense as well.

Exercise

Weak axiom of revealed preferences satisfied by a Walrasian demand function does not imply strong axiom of revealed preferences

Counterexample. Let us consider three price systems: $\mathbf{p}^1$, $\mathbf{p}^2$, $\mathbf{p}^3$; and three corresponding bundles: $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$ selected by a consumer according to a Walrasian demand function x (if x is a demand function it means that for given prices there exists only one bundle that the consumer prefers over other ones in the budget set):

$\mathbf{p}^1=(1,2,2)\mathbf{p}^2=(2,2,1)\mathbf{p}^3=(2,1,2)$

$\mathbf{x}^1=(2,2,1)\mathbf{x}^2=(2,1,2)\mathbf{x}^3=(1,2,2)$

We assume that in each case the consumer's income is the same: $w^1=w^2=w^3=w=8$. It is easy to see that in each of the three cases budget constraint is satisfied and entire income is spent, i.e. $(\mathbf{p}^{iT}\cdot\mathbf{x}^i)=8$.

3. Production Theory

<u>(D.1) Production set</u>, $Y \subset \Re^L$: set of all possible production combinations (if $\mathbf{y} \in Y$ and $y_\ell < 0$, then the good ℓ is a net input)

<u>(D.2) (Production) transformation function</u>, $F: \Re^L \to \Re$ such that $Y = \{\mathbf{y} \in \Re^L: F(\mathbf{y}) \leq 0\}$. The set

$$\{\mathbf{y} \in \Re^L: F(\mathbf{y}) = 0\}$$

is called a production possibility frontier

<u>(D.3)</u> If F is differentiable and $F(\mathbf{y}^0)=0$, then

$$MRT_{\ell k}(\mathbf{y}^0) = \partial F(\mathbf{y}^0)/\partial y_\ell : \partial F(\mathbf{y}^0)/\partial y_k$$

is called the Marginal Rate of Transformation of good ℓ in good k

<u>(T.1)</u> If L=2, then $MRT_{\ell k}(\mathbf{y}^0)$ corresponds to the direction of the frontier in $\mathbf{y}^0$ (in a y_ℓ–y_k plane)

Proof (using the Leibniz-Newton notation)

Economists (and engineers) apply the symbol "dy" in order to denote so-called differential of y. This notation was widely used by 17th and 18th century mathematicians who laid foundations for differential calculus. Later on it was used only in symbols such as "dy/dx", and examples were provided to prove that de-coupling "dy" from "dx" can lead to a nonsense. Indeed it can. Nevertheless, when applied with caution, it can simplify formulae. In what follows we will use this notation (dy) instead of simple

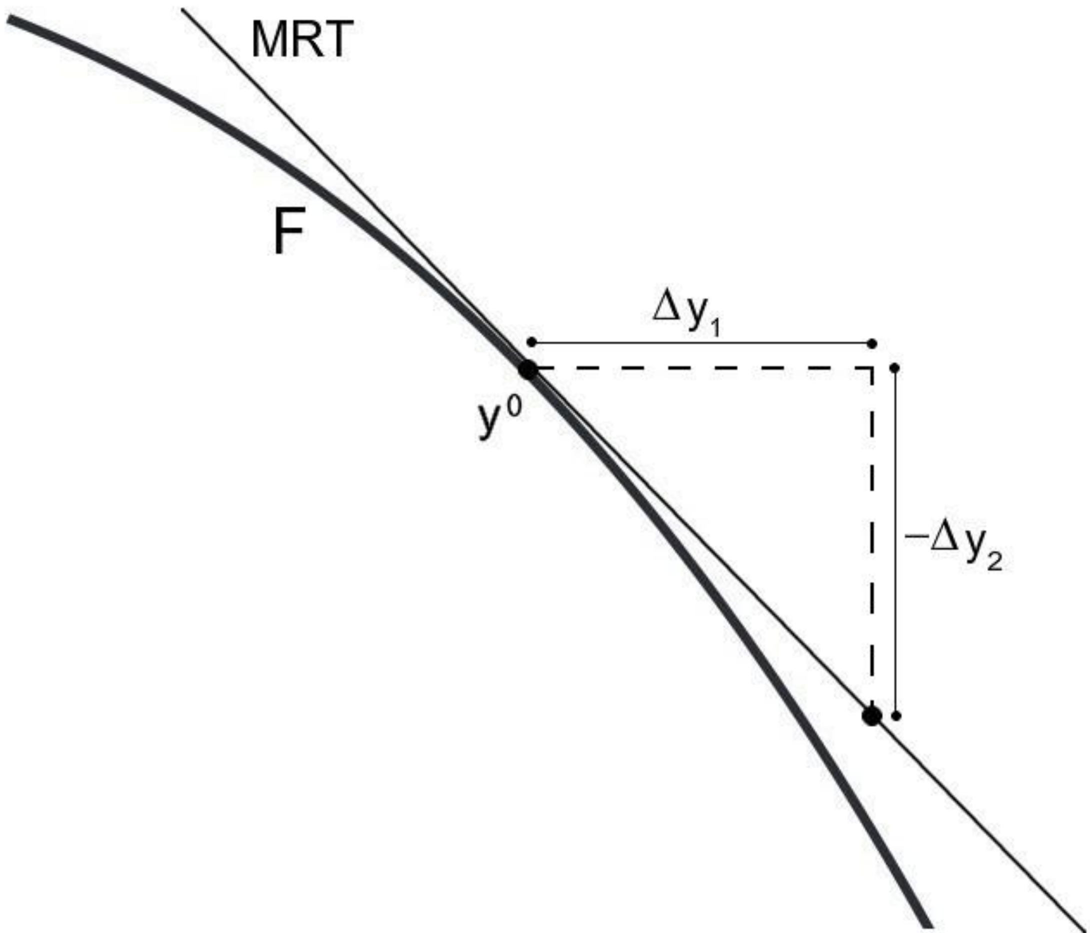
Proof (cont.)

derivatives. For practical purposes, in order to calculate df(x), one needs to calculate df(x)/dx=g(x) and "multiply" both sides of the equation by dx.

As we move along an isoquant, dF=0. Therefore (by the so-called complete differentiation formula): $(\partial F(\mathbf{y}^0)/\partial y_\ell)dy_\ell + (\partial F(\mathbf{y}^0)/\partial y_k)dy_k = 0$. Doing the Leibniz-Newton trick, we get:

$$(\partial F(\mathbf{y}^0)/\partial y_\ell) : (\partial F(\mathbf{y}^0)/\partial y_k) = -dy_k/dy_\ell.$$

i.e. what was to be proved (note that $\partial F(\mathbf{y}^0)/\partial y_\ell$ means the value of a derivative of F calculated for $\mathbf{y}=\mathbf{y}^0$).



MRT
F
Δy1
y0
−Δy2

(D.4) Notational convention. If there is only one good (coordinate) considered an output, y_L, and all the remaining ones are considered inputs $y_1,\ldots,y_{L-1}$, then we apply the following notation: $\mathbf{y}=(-z_1,\ldots,-z_{L-1},q)$, $F(y_1,\ldots,y_L)=q-f(z_1,\ldots,z_{L-1})$, and $f\colon \Re^{L-1}\to\Re$. Then $Y=\{(-z_1,\ldots,-z_{L-1},q)\colon q-f(z_1,\ldots,z_{L-1})\le 0$ and $z_1,\ldots,z_{L-1}\ge 0\}$.

Marginal Rate of Technical Substitution,

$$\mathrm{MRTS}_{\ell k}(\mathbf{z}^0) = \partial f(\mathbf{z}^0)/\partial z_\ell : \partial f(\mathbf{z}^0)/\partial z_k$$

(D.5) Production set axioms

1. Non-emptiness, $Y \neq \emptyset$
2. Closedness,
$$\forall \mathbf{y} \in \Re^L \ \forall \mathbf{y}^1, \mathbf{y}^2, \ldots \in Y \ [\mathbf{y} = \lim_{n\to\infty} \mathbf{y}^n \Rightarrow \mathbf{y} \in Y]$$
3. "No free lunch", $\forall \mathbf{y} \in Y \ [\mathbf{y} \geq \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}]$
4. "No sunk costs", $\mathbf{0} \in Y$
5. "Free disposal", $\forall \mathbf{y}, \mathbf{y}' \in \Re^L \ [(\mathbf{y} \in Y \wedge \mathbf{y}' \leq \mathbf{y}) \Rightarrow \mathbf{y}' \in Y]$
6. Irreversibility, $\forall \mathbf{y} \in Y \ [\mathbf{y} \neq \mathbf{0} \Rightarrow -\mathbf{y} \notin Y]$
7. Non-increasing scale effects,
$$\forall \mathbf{y} \in Y \ \forall \alpha \in [0,1] \ [\alpha \mathbf{y} \in Y]$$

8. Non-decreasing scale effects,

$$\forall \mathbf{y} \in Y \ \forall \alpha \geq 1 \ [\alpha \mathbf{y} \in Y]$$

9. Constant scale effects,

$$\forall \mathbf{y} \in Y \ \forall \alpha \geq 0 \ [\alpha \mathbf{y} \in Y] \text{ (i.e. Y is a cone)}$$

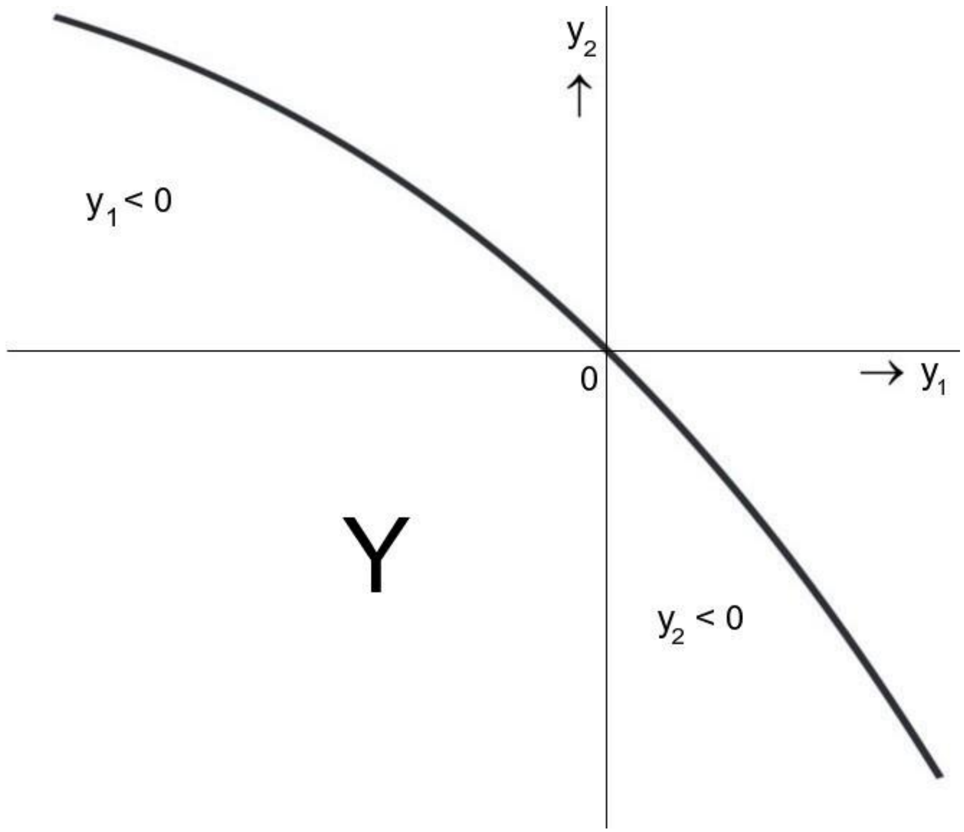
10. Additivity ("free entry"), $\forall \mathbf{y},\mathbf{y}' \in Y \ [\mathbf{y}+\mathbf{y}' \in Y]$
11. Convexity,

$$\forall \mathbf{y},\mathbf{y}' \in Y \ \forall \alpha \in [0,1] \ [\alpha \mathbf{y}+(1-\alpha)\mathbf{y}' \in Y]$$

12. Y is a convex cone,

$$\forall \mathbf{y},\mathbf{y}' \in Y \ \forall \alpha,\beta \geq 0 \ [\alpha \mathbf{y}+\beta \mathbf{y}' \in Y]$$

Production set Y



(T.2) Relations among D.5 axioms

1. (7) $\Rightarrow$ (4)
2. (11) $\wedge$ (4) $\Rightarrow$ (7)
3. (4) $\Rightarrow$ ((11) $\Leftrightarrow$ f is a concave function)
4. ((9) $\wedge$ (11)) $\Rightarrow$ (12)
5. ((10) $\wedge$ (7)) $\Leftrightarrow$ (12)

(D.6) Profit Maximisation Problem (PMP):

$\mathrm{Max}_{\mathbf{y}}\{\mathbf{p}^T\cdot\mathbf{y}: \mathbf{y}\in Y\} = \mathrm{Max}_{\mathbf{y}}\{\mathbf{p}^T\cdot\mathbf{y}: F(\mathbf{y})\leq 0\}$.

Profit function, $\pi(\mathbf{p}) = \mathrm{Max}_{\mathbf{y}}\{\mathbf{p}^T\cdot\mathbf{y}: \mathbf{y}\in Y\}$.

Firm's (net) supply, $y(\mathbf{p}) = \{\mathbf{y}\in Y: \mathbf{p}^T\cdot\mathbf{y}=\pi(\mathbf{p})\}$

(T.3) Axiom (8) from D.5 implies:

$$\forall \mathbf{p}\ [\pi(\mathbf{p})\leq 0 \vee \pi(\mathbf{p})=+\infty]$$

Proof:

$\pi(\mathbf{p})=\mathrm{Max}_{\mathbf{y}}\{\mathbf{p}^T\cdot\mathbf{y}\colon \mathbf{y}\in Y\}$. Thus $\forall \mathbf{y}^0\in Y\ [\mathbf{p}^T\cdot\mathbf{y}^0\leq 0] \Rightarrow \pi(\mathbf{p})\leq 0$. If however there is $\mathbf{y}^0\in Y$ such that $\mathbf{p}^T\cdot\mathbf{y}^0>0$ then we obviously have $\mathbf{p}^T\cdot\mathbf{y}^0<\mathbf{p}^T\cdot(\alpha\mathbf{y}^0)$, and – by D.5.8 – $(\alpha\mathbf{y}^0)\in Y$ for $\alpha\geq 1$. This means that by moving from $\mathbf{y}^0$ to $\alpha\mathbf{y}^0$ profit will increase proportionally (and it cannot be constrained by any finite number).

(T.4) Kuhn-Tucker conditions for PMP:

$$\mathbf{y}^* \in y(\mathbf{p}) \Rightarrow \exists\lambda\geq 0\ [p_\ell = \lambda\cdot\partial F(\mathbf{y}^*)/\partial y_\ell;\ \ell=1,\ldots,L]$$

$$(\mathbf{y}^* \in y(\mathbf{p}) \Rightarrow \exists\lambda\geq 0\ [\lambda\cdot D_y F(\mathbf{y}^*)=\mathbf{p}])$$

(T.5) $MRT_{\ell k}(\mathbf{y}^*) = p_\ell/p_k$

(T.6) Following the notational convention from D.4:

PMP can be written as: $\mathrm{Max}_{z\geq 0}\{pf(\mathbf{z})-\mathbf{w}^T\cdot\mathbf{z}\}$, where

$$\mathbf{p} = [w_1,\ldots,w_{L-1},p]^T,$$

and the Kuhn-Tucker conditions read:

$$p\cdot\partial f(\mathbf{z}^*)/\partial z_\ell \leq w_\ell \text{ and}$$

$$(p\cdot\partial f(\mathbf{z}^*)/\partial z_1 - w_1)z^*_1 + \ldots + (p\cdot\partial f(\mathbf{z}^*)/\partial z_{L-1} - w_{L-1})z^*_{L-1} = 0$$

$$(\mathbf{p}^T\cdot D_z f(\mathbf{z}) \leq \mathbf{w} \wedge (\mathbf{p}^T\cdot D_z f(\mathbf{z}) - \mathbf{w})\cdot\mathbf{z}^* = 0)$$

$z^*_\ell, z^*_k > 0 \Rightarrow \mathrm{MRTS}_{\ell k} = w_\ell/w_k$

(T.7) If Y is a convex set, then Kuhn-Tucker conditions (T.4) are sufficient for finding optimum, $\mathbf{y}^*$

(T.8) Let us assume that Y is a closed set, and disposal is free (D.5.2 and D.5.5) Then:

1. π is homogeneous of degree 1
2. π is convex
3. Y is convex $\Rightarrow$ Y = $\{\mathbf{y}\in\nabla^L\colon \forall\mathbf{p}\gg\mathbf{0}\ [\mathbf{p}^T\cdot\mathbf{y}\leq\pi(\mathbf{p})]\}$
4. y is homogeneous of degree 0
5. Y is convex $\Rightarrow$ $\forall\mathbf{p}$ [y($\mathbf{p}$) is a convex set]
6. Y is strictly convex $\Rightarrow$ $\forall\mathbf{p}$ [y($\mathbf{p}$)$\neq\emptyset$ $\Rightarrow$ y($\mathbf{p}$) consists of a single point]

7. Hotelling lemma:
   $y(\mathbf{p}^0)$ consists of a single point $\Rightarrow \pi$ is differentiable and $D_p\pi(\mathbf{p}^0)=y(\mathbf{p}^0)$
8. $\forall \mathbf{p},\mathbf{p}' \ \forall \mathbf{y}\in y(\mathbf{p}),\mathbf{y}'\in y(\mathbf{p}') \ [(\mathbf{p}-\mathbf{p}')^T\cdot(\mathbf{y}-\mathbf{y}')\geq 0]$
   (the "Law of Supply")
9. $\forall \mathbf{p},\mathbf{p}' \ \forall \mathbf{y}\in y(\mathbf{p}),\mathbf{y}'\in y(\mathbf{p}') \ [(\forall k\neq\ell \ \ p_k=p'_k)$
   $\Rightarrow (p_\ell-p'_\ell)(y_\ell-y'_\ell)\geq 0]$
   (the supply moves where the prices move)
10. If y is differentiable in $\mathbf{p}^0$, then the substitution matrix, $D_p y(\mathbf{p}^0) = D^2{}_p\pi(\mathbf{p}^0)$, is symmetric and positively semi-definite with $D_p y(\mathbf{p}^0)\cdot\mathbf{p}^0 = \mathbf{0}$

<u>Proof</u> (of the item 2 only):

Let $\mathbf{y} \in y(\alpha\mathbf{p}+(1-\alpha)\mathbf{p'})$. Then $\pi(\alpha\mathbf{p}+(1-\alpha)\mathbf{p'}) =$
$= (\alpha\mathbf{p}+(1-\alpha)\mathbf{p'})^T\mathbf{y} = \alpha\mathbf{p}^T\mathbf{y} + (1-\alpha)\mathbf{p'}^T\mathbf{y} \leq$
$\leq \alpha\, \pi(\mathbf{p})+(1-\alpha)\, \pi(\mathbf{p'})$.

<u>(T.9)</u> By the assumption T.8(10):

1. $\forall \ell=1,\ldots,L\ [\partial y_\ell(\mathbf{p})/\partial p_\ell \geq 0]$ (non-negativity of own substitution effects)
2. $\forall \ell,k=1,\ldots,L\ [\partial y_\ell(\mathbf{p})/\partial p_k = \partial y_k(\mathbf{p})/\partial p_\ell]$ (symmetry of substitution effects)

<u>(D.8) Cost Minimization Problem (CMP)</u> (only for a single output and $\mathbf{z}\geq\mathbf{0}$, $\mathbf{w}\gg\mathbf{0}$):

$\mathrm{Min}_z\{\mathbf{w}^T\cdot\mathbf{z}: f(\mathbf{z})\geq q\}$

If $\mathbf{z}^*$ solves CMP, then $c(\mathbf{w},q) = \mathbf{w}^T\cdot\mathbf{z}^*$ is called a <u>cost function</u>, and $z(\mathbf{w},q) = \mathbf{z}^*$ is called a <u>conditional (secondary) demand</u> for production factors.

(T.10) Kuhn-Tucker conditions for CMP:

If $\mathbf{z}^*$ solves CMP, and f is a differentiable function, then:

$\exists\lambda\geq 0\ \forall\ell=1,\ldots,L-1\ [w_\ell \geq \lambda\cdot\partial f(\mathbf{z}^*)/\partial z_\ell \wedge (z^*_\ell > 0 \Rightarrow w_\ell = \lambda\cdot\partial f(\mathbf{z}^*)/\partial z_\ell)]$

If Y is convex (f is concave), then the condition above is sufficient for $\mathbf{z}^*$ to solve CMP

(T.11) Analogues to T.2, T.3, T.7 and T.8 from the lecture 2

Keeping the notation of D.8 and assumptions to T.8:

1. c is homogeneous of degree 1 with respect to **w** and non-decreasing with respect to q
2. c is concave with respect to **w**
3. If sets $\{\mathbf{z} \geq \mathbf{0}: f(\mathbf{z}) \geq q\}$ are convex for every q, then $Y = \{(-\mathbf{z}, q): \mathbf{w}^T \cdot \mathbf{z} \geq c(\mathbf{w}, q), \mathbf{w} \gg \mathbf{0}\}$
4. z is homogeneous of degree 0 with respect to **w**
5. If the set $\{\mathbf{z} \geq \mathbf{0}: f(\mathbf{z}) \geq 0\}$ is convex, then $z(\mathbf{w}, q)$ is also a convex set. Moreover if the set $\{\mathbf{z} \geq \mathbf{0}: f(\mathbf{z}) \geq 0\}$ is strictly convex, then $z(\mathbf{w}, q)$ consist of a single point

6. Shepard's Lemma:
   if $z(\mathbf{w}^0,q)$ consists of a single point, then c is differentiable with respect to $\mathbf{w}$ in $\mathbf{w}^0$, and $D_w c(\mathbf{w}^0,q)=z(\mathbf{w},q)$
7. If z is differentiable in $\mathbf{w}^0$, then $D_w z(\mathbf{w}^0,q)=D^2{}_w c(\mathbf{w}^0,q)$ is a symmetric, negatively semi-definite matrix, and $D_w z(\mathbf{w}^0,q)\cdot\mathbf{w}^0 = \mathbf{0}$
8. If f is homogeneous of degree 1 (i.e. its returns of scale are constant), then c and z are homogeneous of degree 1 with respect to q
9. If f is concave, then function c convex with respect to q (i.e. the marginal cost is a non-decreasing function of q)

(T.12) PMP can be also phrased as the following problem:

$$\mathrm{Max}_{q\geq 0}\{pq - c(\mathbf{w},q)\},$$

the first order condition of which reads:

$p - \partial c(\mathbf{w},q^*)/\partial q \leq 0$ and if $q^* > 0$, then $p - \partial c(\mathbf{w},q^*)/\partial q = 0$

(D.9) Aggregate supply $y(\mathbf{p})$ from J firms characterized by production possibility sets $Y_1,...,Y_J$ and satisfying axioms D.5 (1), (2) i (5) with respective maximum profits $\pi_j(\mathbf{p})$ and supplies $y_j(\mathbf{p})$ (for $j=1,...,J$):

$$y(\mathbf{p}) = y_1(\mathbf{p})+...+y_J(\mathbf{p}) = \{\mathbf{y}\in\Re^L\colon \mathbf{y}=\mathbf{y}_1+...+\mathbf{y}_J$$

for some $\mathbf{y}_j\in y_j(\mathbf{p})$, $j=1,...,J\}$

Aggregate production set

$$Y = Y_1+...+Y_J = \{\mathbf{y}\in\Re^L\colon \mathbf{y}=\mathbf{y}_1+...+\mathbf{y}_J$$

for some $\mathbf{y}_j\in Y_j$, $j=1,...,J\}$

$$\pi^*(\mathbf{p}),\ y^*(\mathbf{p})$$

– optimum profit and supply for the production set Y

<u>(T.13)</u> For any $\mathbf{p} \gg \mathbf{0}$

1. $\pi^*(\mathbf{p}) = \pi_1(\mathbf{p})+...+\pi_J(\mathbf{p})$
2. $y^*(\mathbf{p}) = y_1(\mathbf{p})+...+y_J(\mathbf{p})$

<u>(D.10)</u> Production (supply) $\mathbf{y} \in Y$ is called <u>efficient</u>, if there is no $\mathbf{y}' \in Y$ such that $\mathbf{y}' \geq \mathbf{y}$ an $\mathbf{y}' \neq \mathbf{y}$

<u>(T.14)</u> If $\mathbf{y} \in Y$ solves a PMP for some $\mathbf{p} \gg \mathbf{0}$, then $\mathbf{y}$ is efficient

<u>(T.15)</u> If Y is convex and $\mathbf{y} \in Y$ is efficient, then $\mathbf{y}$ solves a PMP for some $\mathbf{p} \geq \mathbf{0}$

Non-technical summary

A. Production set axioms reflect alternative assumptions about production conditions and they are subject to empirical verification.

B. There is an analogy between optimisation problems of consumer theory (UMP and EMP) and optimisation problems of production theory (PMP and CMP). The fundamental formal distinction between UMP and PMP is that a firm does not have a budget constraint (hence there are no 'income effects' in PMP). Consequently there may be no finite solution **y*** by non-decreasing returns to scale.

C. A more comprehensive analogy is between EMP and CMP. There is a correspondence between the utility function u and the production function f, expenditure function e and cost function c, as well as between Hicksian demand h, and conditional demand for inputs z. As a result, properties of e and c, and h and z are similar. For instance, T.7 from the lecture 2 and Shepard's Lemma (T.11.6) from lecture 3.

D. In consumption theory there is no strict equivalent of T.13 on aggregate supply, as there is no obvious utility aggregation procedure (such as the obvious procedure for profit aggregation).

E. T.14 and T.15 can be interpreted as theorems on the optimality of market allocation for the system consisting of J price-taking firms.

4. Game theory

(D.1) Extended form game,

$$\Gamma_E = [\mathcal{X}, \mathcal{A}, I, p, \alpha, \mathcal{H}, H, \iota, \rho, u], \text{ where:}$$

1. A finite set of nodes $\mathcal{X}$, a finite set of possible actions $\mathcal{A}$, and a finite set of players $\{1,...,I\}$

2. A function $p: \mathcal{X} \to \mathcal{X} \cup \{\varnothing\}$ specifying a single immediate <u>predecessor</u> of each node x; for every $x \in \mathcal{X}$, p(x) is non-empty except for one, designated as the <u>initial node</u>, x_0. The immediate successor nodes of x are thus $s(x)=p^{-1}(x)$. All predecessors and all successors of x can be found by iterating operations p and s. It is assumed that for any x and for any k=1,2,... $p(x) \cap s^k(x)=\varnothing$ (if $s^k(x)$ is defined). The set of <u>terminal nodes</u> $T=\{x \in \mathcal{X}: s(x)=\varnothing\}$. All other nodes ($\mathcal{X} \setminus T$) are called <u>decision nodes</u>.

3. A function α: $\mathcal{X}\backslash\{x_0\} \rightarrow \mathcal{A}$ giving the action that leads to any non-initial node x from its immediate predecessor p(x) and satisfying the condition that:

   if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$.

   The set of <u>choices</u> available at decision node c is

   $c(x) = \{a \in \mathcal{A}: a = \alpha(x') \text{ for some } x' \in s(x)\}$

4. A <u>family of information sets</u> $\mathcal{H}$ and a function H: $\mathcal{X} \to \mathcal{H}$ assigning each decision node to an <u>information set</u> H(x)$\in\mathcal{H}$. Thus, the information sets in $\mathcal{H}$ form a partition of $\mathcal{X}$. It is required that all decision nodes assigned to the same information set have the same choices available, i.e.:

$$H(x)=H(x') \Rightarrow c(x)=c(x').$$

Hence choices available at information set H can be defined as

$$C(H) = \{a\in\mathcal{A}: a\in c(x) \text{ for } x\in H\}$$

5. A function $\iota:\mathcal{H} \to \{0,1,...,I\}$ assigning each information set in $\mathcal{H}$ to a player (or to nature formally identified as the player 0), who moves at the decision nodes in that set. We can define the <u>family of player's *i* information sets</u> as

$$\mathcal{H}_i=\{H\in \mathcal{H}: i=\iota(H)\}$$

6. A function $\rho:\mathcal{H}_0\times\mathcal{A} \to [0,1]$ assigning probabilities to actions at information sets where <u>nature moves</u> and satisfying $\rho(H,a)=0$ if $a\notin C(H)$ and $\sum_{a\in C(H)}\rho(H,a)=1$ for all $H\in\mathcal{H}_0$

7. A family of <u>payoff functions</u> $u=(u_1,\ldots,u_I)$ assigning utilities to the players for each terminal node that can be reached, $u_i:T \rightarrow \Re$. In order to be consistent with the theory of expected utility, values of the functions u_i should be interpreted as Bernoulli utilities.

<u>(D.2)</u> Let $\mathcal{H}_i$ denote the family of information sets of the payer *i*, $\mathcal{A}$ – set of possible actions, and $C(H)\subset\mathcal{A}$ – the set of actions available at information set H. A strategy for player *i* is a function $s_i:\mathcal{H}_i \rightarrow \mathcal{A}$ such that

$$s_i(H)\in C(H) \text{ for all } H\in\mathcal{H}_i$$

Example: Tic Tac Toe (classic)

X		

X	O	

X	O	
X		

X	O	
X		
O		

X	O	X
X		
O		

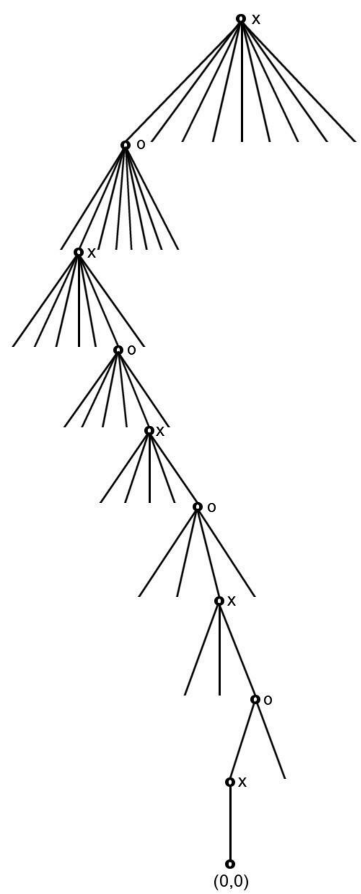
X	O	X
X		
O		O

X	O	X
X		
O	X	O

X	O	X
X	O	
O	X	O

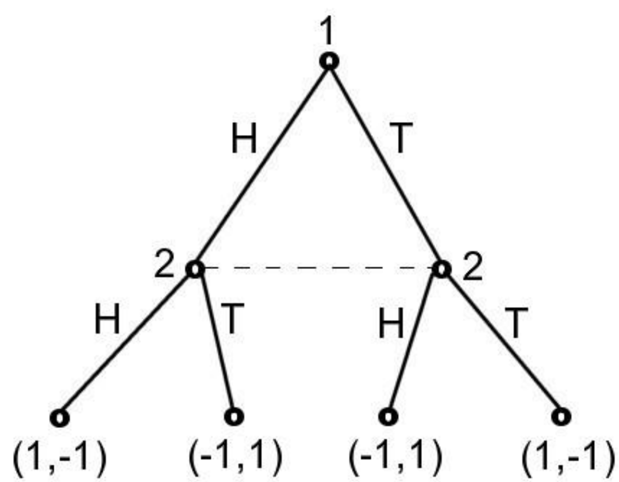
X	O	X
X	O	X
O	X	O

Example: Tic Tac Toe (mathematical)



<u>Example</u> (matching pennies)

One player chooses Heads or Tails. The other one independently (perhaps simultaneously) chooses Heads or Tails as well. They disclose their choices.



<u>Example</u> (cont.)

In case there is HH or TT, the second pays 1 to the first; otherwise the first pays 1 to the second. Independence (perhaps simultaneity) of both choices is reflected by the fact that the nodes where the second player makes the choice belong to the same information set. In the diagram ('game tree') this is reflected by the dotted line.

<u>(D.3) Game in a normal (condensed, strategic) form</u>,

$$\Gamma_N = [I, S_1 \times \ldots \times S_I, (u_1,\ldots,u_I)], \text{ where}$$

1. S_i – set of strategies of player *i* ($s_i \in S_i$)
2. $u_i(s_1,\ldots,s_I)$ – a payoff function whose values can be interpreted as expected utilities (in the von Neumann-Morgenstern sense) of outcomes (perhaps probabilistic ones)

<u>(T.1)</u> For every extended form game there is a unique normal form game, but not vice versa (i.e. the same normal form game may correspond to several extended form games)

Example

The 'Matching pennies' game depicted in the tree above corresponds to the following payoff matrix:

		Second	
		H	T
First	H	(1,-1)	(-1,1)
	T	(-1,1)	(1,-1)

<u>(D.4)</u> Strategies $s_i(H)$ from D.2 are called <u>pure strategies</u>. Any function $\sigma_i : S_i \rightarrow [0,1]$ such that $\sigma_i \geq 0$ and $\sum_{si \in Si} \sigma_i(s_i) = 1$ is called a <u>mixed strategy</u>. Numbers $\sigma_i(s_i)$ are interpreted as probabilities of choosing a (pure) strategy s_i. Game with mixed strategies is written as

$$\Gamma_N = [I, \Delta(S_1)\times\ldots\times \Delta(S_I), (u_1,\ldots,u_I)]$$

<u>(D.5)</u> Notational convention: $s_{-i} = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_I)$, $\sigma_{-i} = (\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_I)$, $S = S_1\times\ldots\times S_I$, $S_{-i} = S_1\times\ldots\times S_{i-1}\times S_{i+1}\times\ldots\times S_I$

<u>(D.6) Strictly dominant strategy</u>, $s_i \in S_i$:

$$\forall s'_i \neq s_i \; \forall s_{-i} \in S_{-i} \; [u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})]$$

<u>(D.7) Strictly dominated strategy</u>, $s_i \in S_i$:

$$\exists s'_i \neq s_i \; \forall s_{-i} \in S_{-i} \; [u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})].$$

Strategy s'_i strictly dominates over strategy s_i

<u>(D.8) Nash equilibrium</u>, strategy profile $s=(s_1,...,s_I)$ such that:

$$\forall i=1,...I \ \forall s'_i \in S_i \ [u_i(s_i,s_{-i}) \geq u_i(s'_i,s_{-i})].$$

For mixed strategies, strategy profile $\sigma=(\sigma_1,...,\sigma_I)$ such that:

$$\forall i=1,...I \ \forall \sigma'_i \in \Delta(S_i) \ [u_i(\sigma_i,\sigma_{-i}) \geq u_i(\sigma'_i,\sigma_{-i})]$$

<u>(T.2)</u> Every game $\Gamma_N = [I, \Delta(S_1) \times ... \times \Delta(S_I), (u_1,...,u_I)]$, where sets $S_1,...,S_I$ have finite number of elements has a Nash equilibrium in mixed strategies

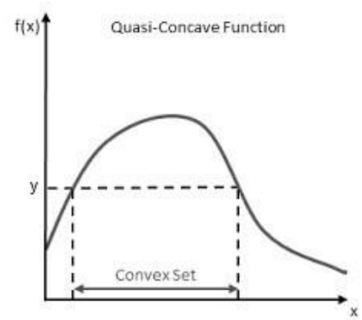
<u>Example</u> (Prisoner's Dilemma)

		Second	
		C	D
First	C	(-12,-12)	(0,-18)
	D	(-18,0)	(-1,-1)

Nash equilibrium is for (C,C) which is the very worst outcome for two players; i.e. Nash equilibrium does not necessarily 'optimize' the global outcome (which – in this case – would be (D,D))

(T.3) Game $\Gamma_N = [I, \Delta(S_1)\times\ldots\times \Delta(S_I), (u_1,\ldots,u_I)]$ has a Nash equilibriumin pure strategies if for every $i=1,\ldots,I$:

1. S_i is a non-empty, convex compact subset of $\Re^M$ (for some M); and
2. $u_i(s_1,\ldots,s_I)$ is continuous with respect to $(s_1,\ldots,s_I)$ and quasi-concave with respect to s_i



(D.9) Sequential rationality principle: each player's strategy should contain actions that are optimal for every node (taking into account other players' strategies)

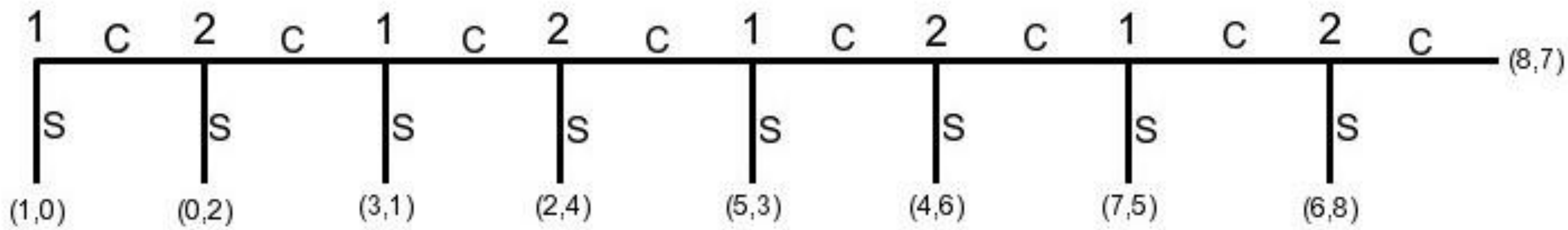
(D.10) Finite game with perfect information: every information set contains only one node and the number of nodes is finite

(T.4) Backward induction. Sequential rationality principle is satisfied if an optimum action for p(x) is determined once an optimum action for x is determined (i.e. anticipating an optimum solution at x)

(T.5) For finite games with perfect information backward induction boils down to determining outcomes for all terminal nodes x, determining optimum actions for preceding nodes p(x), assigning them payoffs that result from these optimum actions, and eliminating remaining strategies. This procedure is then iterated for earlier nodes p(p(x)) and so on, until all the nodes are exhausted.

<u>(T.6) Zermelo Theorem</u>. Every finite game with perfect information Γ_E has a Nash equilibrium in pure strategies which can be found using backward induction. Moreover if none of the players has identical payoffs in two different terminal nodes, this is the only Nash equilibrium that can be found in this way.

Example (Centipede game)



This 2-person game lasts 2k periods (where k=1,2,3,...), it has 2k decision nodes, and 2k+1 terminal nodes. The players move in turns, starting with the player number 1. In every node either of the two decisions can be taken: Stop (S) or Continue (C). If the game is stopped at node i=1,2,3,...,2k then the payoffs are: $P(i)=i-(1+(-1)^i)$, $D(1)=0$, and for $i>1$ $D(i)=i-(1-(-1)^i)$. If the game is not stopped by any player at any node, it terminates with the payoffs $P(2k)=2k$, and $D(2k)=2k-1$.

(D.11) Subgame of an extended form game Γ_E – a subset satisfying the following two conditions:

1. It starts with an information set consisting of a single decision node x, contains all subsequent nodes, s(x), s(s(x)) and so on, and does not contain other nodes;
2. If the node x is in the subgame then every $x' \in H(x)$ is in it as well (i.e. a subgame does not contain 'incomplete' information sets).

(T.7) The entire game Γ_E is also a subgame

(T.8) Every decision node in a finite game with perfect information can 'start' a subgame

(D.12) A strategy profile $\sigma=(\sigma_1,...,\sigma_I)$ in an I-person game Γ_E induces a Nash equilibrium in a subgame of this game, if actions defined by σ for information sets of this subgame (understood as a separate game) make a Nash equilibrium in it

(D.13) A strategy profile $\sigma=(\sigma_1,...,\sigma_I)$ in an I-person game Γ_E is called Subgame Perfect Nash Equilibrium, SPNE, if it induces a Nash equilibrium in its every subgame

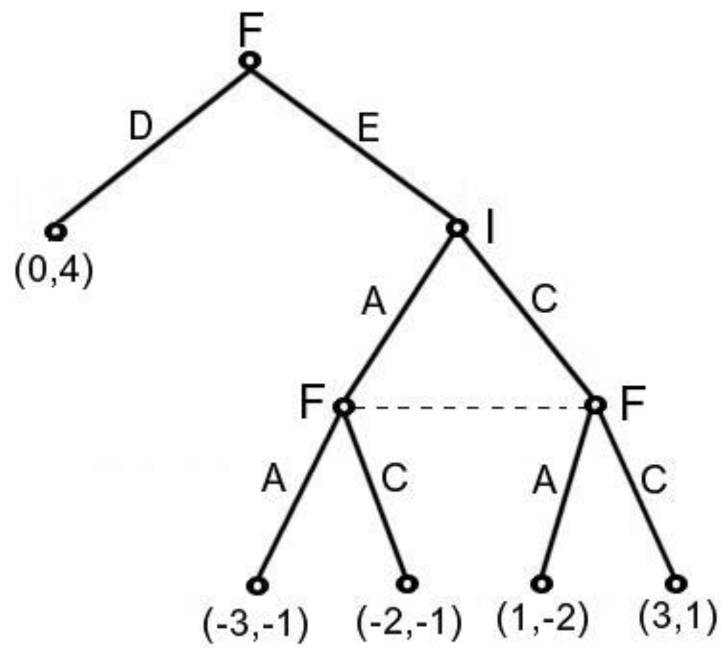
<u>(T.9)</u> Every finite game with full information Γ_E has a SPNE in pure strategies. Moreover, if none of the players has identical payoffs in two different terminal nodes, this is the only SPNE.

Example

A market is served by one firm called Incumbent (I). Another firm (F) contemplates entering the market. It has two strategies: Enter (E) and Do Not Enter (D). If it chooses D, the payoffs are [0,4] (its payoff is 0, and the payoff of the Incumbent is – as before – 4). If it chooses E, the two firms will play a duopolistic game which boils down either to attacking (A) e.g. by a price war or to cooperating (C), e.g. à la Cournot or Stackelberg (or by sharing the market). If both attack then the payoffs are [-3,-1], if they cooperate then they are [3,1].

Example (cont.)

If I attacks and F cooperates it is [-2,-1], and if I cooperates and F attacks it is [1,-2]. The story can be reflected by a game Γ_E defined by the tree:



Example (cont.)

If F chooses E, then I has two strategies: A and C. Choosing its own strategy, F does not know what strategy has been chosen by I. This is reflected by the two nodes where F moves belonging to the same information set (the two nodes are linked by a dotted line). The game Γ_E has two subgames: the entire game and, say, Γ_O the oligopolistic rivalry part following the decision E taken by F.

Example (cont.)

It can be demonstrated that the number of subgames is the difference between the number of non-terminal nodes (here 4) and the number of nodes 'hidden' in non-single node information sets (here 2). Hence the number of subgames is 2. Since by T.7 Γ_E is also its subgame, then Γ_O is the only other subgame.

<u>Example</u> (cont.)

The game Γ_E has the following strategic (normal) form Γ_N:

		Incumbent (I)	
		C if F enters	A if F enters
Entering Firm (F)	D&C	0,4	**0,4**
	D&A	0,4	**0,4**
	E&C	**3,1**	-2,-1
	E&A	1,-2	-3,-1

<u>Example</u> (cont.)

This strategic-form game has three Nash equilibria (D&C,A), (D&A,A), and (E&C,C) with payoffs (0,4), (0,4), and (3,1), respectively. The first two do not satisfy the sequential rationality principle (I's threat that it will attack if F enters is not credible).

Example (cont.)

The subgame Γ_O has the following strategic form (payoff matrix):

		Incumbent (I)	
		C	A
Entering Firm (F)	C	**3,1**	-2,-1
	A	1,-2	-3,-1

The only Nash equilibrium in this subgame is (C,C).

Non-technical summary

A. Game theory is an analytical method of predicting behaviour linked to conflict or cooperation. When taking decisions, players do not necessarily know their partners' decisions, but they know their space of choice and understand consequences of alternative decisions.

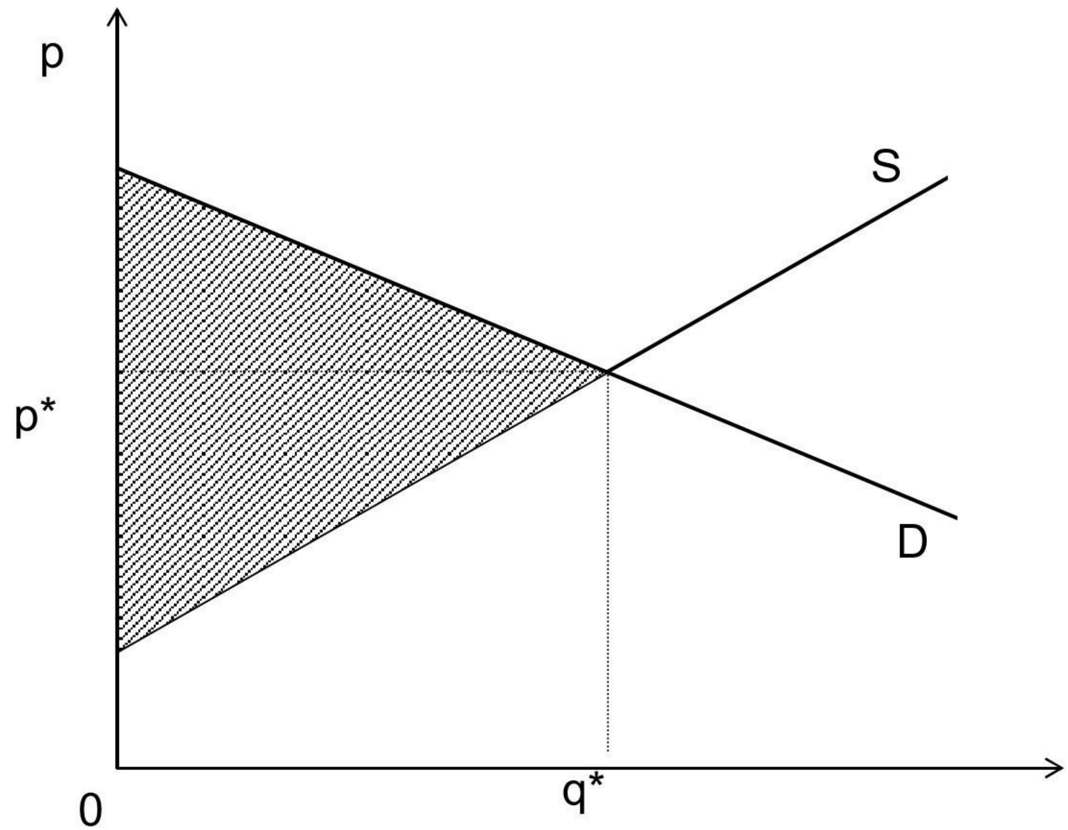
B. The central concept of the game theory is Nash equilibrium. Nash equilibrium denotes a situation (strategy profile) such that no player has a motivation to unilaterally change his or her strategy.

C. Nash equilibrium can be realized in a situation which is not at all optimal for the players.

D. Games with a finite number of nodes and when players do not take decisions simultaneously (i.e. they *a priori* know decisions of their partners) are called finite games with perfect information. In such cases actions can be predicted simply by the backward induction method. By applying this method the only Nash equilibria that can be found are those which satisfy the sequential rationality principle.

E. The concept of Subgame Perfect Nash Equilibrium (SPNE) reflects the sequential rationality principle.

5. Competitive equilibrium



(D.1) Feasible allocation, any combination $(\mathbf{x}_1,\ldots,\mathbf{x}_I,\mathbf{y}_1,\ldots,\mathbf{y}_J) \in X_1\times\ldots\times X_I\times Y_1\times\ldots\times Y_J$ satisfying $x_{\ell 1}+\ldots+x_{\ell I} \leq \omega^{\ell} + y_{\ell 1}+\ldots+y_{\ell J}$ for $\ell=1,\ldots,L$; notation as in lectures 1 and 3, and symbol $\omega^{\ell} \geq 0$ stands for an initial allocation (*endowment*) of the good ℓ in the economy

<u>(D.2) Pareto optimality</u>, any combination satisfying D.1, if there is no other feasible allocation

$$(\mathbf{x}'_1,\ldots,\mathbf{x}'_I,\mathbf{y}'_1,\ldots,\mathbf{y}'_J) \text{ such that:}$$

$\forall i=1,\ldots,I\ [u_i(\mathbf{x}'_i)\geq u_i(\mathbf{x}_i)] \wedge \exists i=1,\ldots,I\ [u_i(\mathbf{x}'_i)>u_i(\mathbf{x}_i)]$

<u>(D.3) Ownership in economy</u>. It is assumed that the initial endowment of every good ℓ is entirely owned by consumers: $\omega^{\ell} = \omega_{\ell 1}+\ldots+\omega_{\ell I}$, (vector of initial endowments owned by a consumer *i* is denoted as $\boldsymbol{\omega}_i = (\omega_{1i},\ldots,\omega_{Li})^T$). It is also assumed that every consumer *i* has share of θ_{ij} in firm *j*, with $\theta_{1j}+\ldots+\theta_{Ij} = 1$ for every $j=1,\ldots,J$

(T.1) By the notation of D.3 every consumer *i* has a *de facto* monetary income

$$w_i=\mathbf{p}^T\cdot\boldsymbol{\omega}_i + \theta_{i1}\mathbf{p}^T\cdot\mathbf{y}_1+\ldots+\theta_{iJ}\mathbf{p}^T\cdot\mathbf{y}_J$$

implied by a (possible) sale of the initial endowments and shares in firms

(D.4) Competitive (Walrasian) equilibrium, any allocation $(\mathbf{x}^*_1,...,\mathbf{x}^*_I,\mathbf{y}^*_1,...,\mathbf{y}^*_J)$ satisfying D.1 coupled with a price vector $\mathbf{p}^*\in\Re^L$, which satisfy the following conditions:

1. Profit maximization of firms: for any j=1,...,J $\mathbf{y}^*_j$ solves a PMP

   $\mathrm{Max}_{yj}\{\mathbf{p}^{*T}\cdot\mathbf{y}_j: \mathbf{y}_j\in Y_j\}$

2. Utility maximization of consumers: for every i=1,...,I $\mathbf{x}^*_i$ solves a UMP

   $\mathrm{Max}_{xi}\{u_i(\mathbf{x}_i): \mathbf{p}^{*T}\cdot\mathbf{x}_i \le \mathbf{p}^{*T}\cdot\boldsymbol{\omega}_i + \theta_{i1}\mathbf{p}^{*T}\cdot\mathbf{y}^*_1+...+\theta_{iJ}\mathbf{p}^{*T}\cdot\mathbf{y}^*_J \wedge \mathbf{x}_i\in X_i\}$

3. Market clearing: for every ℓ=1,...,L

   $x^*_{\ell 1}+...+x^*_{\ell I} \le \omega^\ell + y^*_{\ell 1}+...+y^*_{\ell J}$

(T.2) If u_i are continuous utility functions representing relations $\geq_i$, each of which is locally non-satiated, then condition 3 in D.4 is satisfied as equality

(T.3) If allocation $(\mathbf{x}^*_1,\ldots,\mathbf{x}^*_I,\mathbf{y}^*_1,\ldots,\mathbf{y}^*_J)$ coupled with a price vector $\mathbf{p}^* \gg \mathbf{0}$ is a competitive equilibrium, then for every $\alpha>0$ the same allocation coupled with the price vector $\alpha\mathbf{p}^*$ is also a competitive equilibrium. Thus, without a loss of generality it can be assumed that one of the prices is equal to 1.

(T.4) By the assumptions of T.2, if allocation $(\mathbf{x}_1,\dots,\mathbf{x}_I,\mathbf{y}_1,\dots,\mathbf{y}_J)$ coupled with a price vector $\mathbf{p} \gg \mathbf{0}$ satisfies the condition 3 in D.4 in the form of equality for all goods $\ell \neq k$ and if every consumer is on his/her budget line, then the condition 3 in D.4 is satisfied in the form of equality for the good k too.

(D.5) Marshallian analysis of partial equilibrium.

1. A market is analyzed for a single good ℓ, which makes a small part of the entire economy. Consequently (a) income effects of price changes of this good, and (b) impact of this market on prices of other goods can be neglected. Therefore all other goods can be treated as one compound good (*numeraire*). Its price is equal to 1, and p denotes the price of the good ℓ.

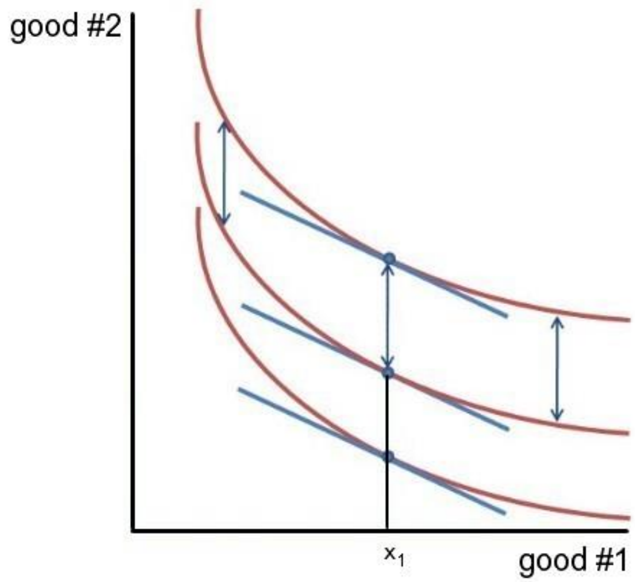
2. Every consumer *i* has a quasi-linear utility function $u_i(x_i,m_i) = \varphi_i(x_i)+m_i$, where x_i denotes the consumption of the good ℓ, and m_i is the consumption of the compound good. The set of consumption alternatives is $X_i=\Re_+\times\Re$ (negative consumption of the compound good is accepted in order to rule out corner solutions). It is assumed that the function φ_i is constrained from above and twice differentiable with $\varphi_i'(x_i)>0$ and $\varphi_i''(x_i)<0$ for all $x_i\geq0$. Allowing for a parallel translation (see T.3 from lecture 1), it can be assumed that $\varphi_i(0)=0$.

3. The quantity of the compound good used by the firm *j* in order to produce $q_j \geq 0$ units of the good ℓ is $c^j(q_j)$ (where c_j is a cost function as defined in D.8 in lecture 3). Denoting by z_j the use of of the compound good in the firm *j*, its production set is defined as

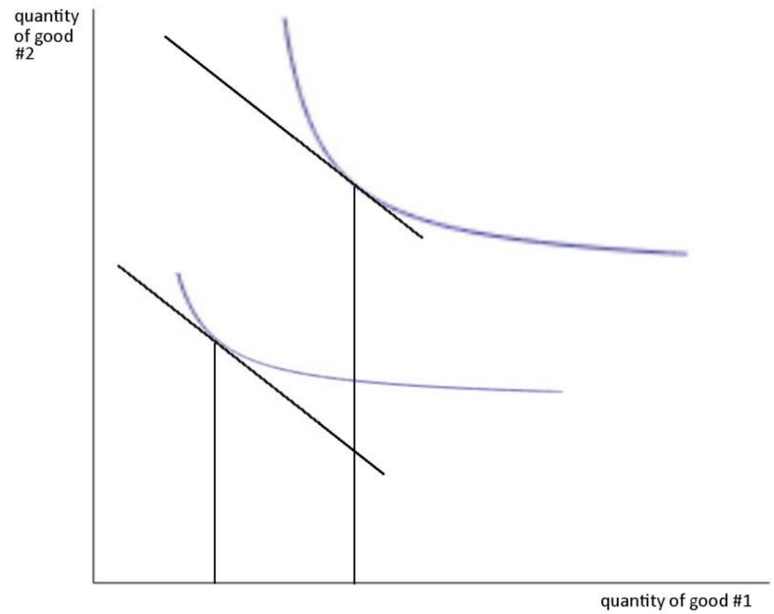
$$Y_j = \{(-z_j, q_j) : q_j \geq 0 \wedge z_j \geq c_j(q_j)\}.$$

It is assumed that the function c_j is twice differentiable with $c_j'(q_j) > 0$ and $c_j''(q_j) \geq 0$ for $q_j \geq 0$.

4. There is no initial endowment of the good ℓ (its supply comes from the current production only). Initial endowments of the compound good are $\omega_{mi} > 0$, and – by definition – $\omega_{m1} + \ldots + \omega_{mI} = \omega_m$.



Quasi-linear preferences: fixed price (irrespective of the budget constraint) implies the same demand for good #1 (x_1)



Not quasi-linear preferences: fixed price may imply different demand for good #1 (depending on the budgetary constraint)

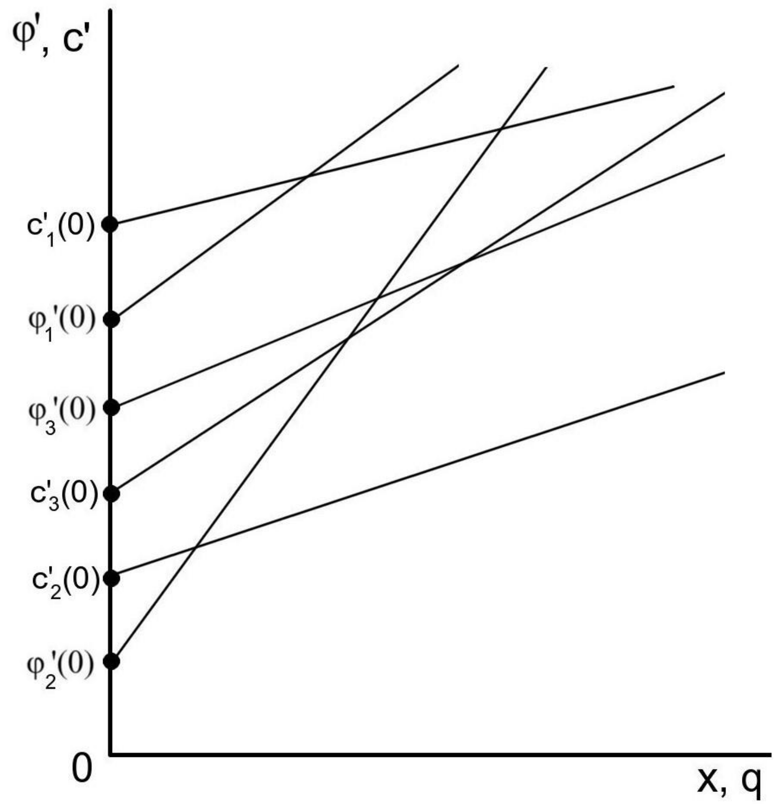
(T.5) In the Marshallian analysis (D.5) the necessary and sufficient conditions are:

1. For profit maximization (D.4.1) $\text{Max}\{p^* q_j - c_j(q_j): q_j \geq 0\}$:
   $p^* \leq c_j'(q^*_j)$, with equalities for $q^*_j > 0$
2. For utility maximization (D.4.2)
   $\text{Max}\{m_i + \varphi_i(x_i): p^* x_i \leq \omega_{mi} + \theta_{i1}(p^* q^*_1 - c_1(q^*_1)) + \ldots + \theta_{iJ}(p^* q^*_J - c_J(q^*_J))\}$:
   $\varphi_i'(x^*_i) \leq p^*$, with equalities for $x^*_i > 0$
3. Market clearing (D.4.3):
   $x^*_1 + \ldots + x^*_I = q^*_1 + \ldots + q^*_J$

Note: conditions 1–3 are independent of initial allocation and shares in firms

<u>(T.6)</u>

If $\text{Max}_i \varphi_i'(0) > \text{Min}_j c_j'(0)$, then $x^*_1 + \ldots + x^*_I,\ q^*_1 + \ldots + q^*_J > 0$



(T.7) Aggregate demand for the good ℓ,

$x(p) = x_1(p)+...+x_I(p)$, where $x_i(p)$, the Walrasian demand function, is defined as the inverse to φ_i'.

Note: because of quasi-linearity of functions u_i, functions x_i do not depend on w_i.

(T.8) The demand function in T.7 is continuous and non-increasing for $p>0$; at the same time it is strictly decreasing for $p<\mathrm{Max}_i\varphi_i'(0)$. For $p\geq\mathrm{Max}_i\varphi_i'(0)$ we have: $x(p)=0$.

<u>(T.9) Aggregate supply function of the good ℓ</u>, $q(p)=q_1(p)+...+q_J(p)$, where the number $q_j(p)$ satisfies the equality from the condition T.5.1 for $p \geq c_j'(0)$. If c_j is strictly convex (then c_j' is a strictly increasing function), then there exist only one such number; therefore q_j (and hence also q) is a function.

<u>(T.10)</u> The supply function from T.9 is continuous and non-decreasing for $p>0$; at the same time, it is strictly increasing for $p>\mathrm{Min}_j c_j'(0)$.

<u>(T.11)</u> If at least one c_j is a linear function (constant scale effects), $c_j(q_j)=c_jq_j$ (thus it is not strictly convex, but it satisfies the condition $c_j''\geq 0$ (D.5.3)), then for $p>c_j$ the value of $q_j(p)=+\infty$ is not well-defined.

<u>(T.12)</u> By the assumptions of T.7 and T.9, if

$$\mathrm{Max}_i\varphi_i'(0) > \mathrm{Min}_jc_j'(0),$$

then there exists a unique equilibrium price $p^*\in(\mathrm{Min}_jc_j'(0),\mathrm{Max}_i\varphi_i'(0))$. Individual consumption and production levels are then set by $x^*_i=x_i(p^*)$ and $y^*_j=y_j(p^*)$, for $i=1,...,I$, $j=1,...,J$.

(T.13) If for every $j=1,...,J$ there is $c>0$ such that $c_j(q_j)=cq_j$, the assumptions of T.7 are satisfied, and $\mathrm{Max}_i \varphi_i'(0) > c$, then the equilibrium price $p^*=c$

(D.6) Inverse supply function, q^{-1}, can be interpreted as the marginal cost function of industry, $C'(.)=q^{-1}(.)$

(D.7) Inverse demand function, $x^{-1}(.)$ is denoted as $P(.)$

(T.14) In the Marshallian analysis dynamics of the equilibrium price p^* resulting from imposing a tax t on every unit of the good ℓ is given by the following differential equation:

$$p^{*\prime}(t) = x'(p^*(t)+t)/(q'(p^*(t))-x'(p^*(t)+t))$$

(D.8) Marshallian Aggregate Surplus

$$MAS(x_1,\ldots,x_I,q_1,\ldots,q_J) = \varphi_1(x_1)+\ldots+\varphi_I(x_I) - (c_1(q_1)+\ldots+c_J(q_J))$$

(T.15) In the Marshallian analysis allocation $(x^*_1,\ldots,x^*_I,q^*_1,\ldots,q^*_J)$ is Pareto optimal if and only if it solves the problem

$$\text{Max } \{MAS(x_1,\ldots,x_I,q_1,\ldots,q_J):\ x_1+\ldots+x_I-(q_1+\ldots+q_J)=0,\quad x_1,\ldots,x_I,q_1,\ldots,q_J\geq 0\}$$

Proof:

$\Leftarrow$

It needs to be proved that maximizing MAS implies Pareto optimality. To this end let us assume that $(x_1^0,\ldots,x_I^0,q_1^0,\ldots,q_J^0)$ maximizes this surplus and let us consider the utilities achieved by consumers $u_i(x_i^0,m_i^0)=\varphi_i(x_i^0)+m_i^0$, where the quantities m_i^0 of the

Proof (cont.)

compound good left once the cost of producing the analyzed good satisfy the equation:

$$m_1^0+...+m_I^0 = \omega_m-(c_1(q_1^0)+...+c_J(q_J^0))$$

(notation as in D.5). Therefore the sum of these utilities

$u_1(x_1^0,m_1^0)+...+u_I(x_I^0,m_I^0) =$

$= \varphi_1(x_1^0)+m_1^0+...+\varphi_I(x_I^0)+m_I^0 =$

$= \varphi_1(x_1^0)+...+\varphi_I(x_I^0)+ \omega_m-(c_1(q_1^0)+...+c_J(q_J^0)) =$

$= \varphi_1(x_1^0)+...+\varphi_I(x_I^0) - (c_1(q_1^0)+...+c_J(q_J^0))+ \omega_m =$

$= MAS(x_1^0,...,x_I^0,q_1^0,...,q_J^0)+ \omega_m.$

By the assumption this surplus is the maximum of all possible surpluses, and ω_m is a constant. Consequently $(x_1^0,\dots,x_I^0,q_1^0,\dots,q_J^0)$ must be a Pareto optimum, since it is not possible to increase a consumer's utility by choosing an alternative allocation, unless the sum of utilities of other consumers is decreased.

$\Rightarrow$

Now we will demonstrate that if an allocation does not maximize the surplus, it cannot be Pareto optimal. Therefore let us assume that

Proof (cont.)

$$\varphi_1(x_1^0)+...+\varphi_I(x_I^0) - (c_1(q_1^0)+...+c_J(q_J^0)) < \varphi_1(x_1')+...+\varphi_I(x_I') - (c_1(q_1')+...+c_J(q_J')).$$

This will imply that $(x_1^0,...,x_I^0,q_1^0,...,q_J^0)$ cannot be Pareto optimal. By adding to both sides of this inequality the number ω_m, and by defining the total quantity of the compound good at the left hand side as $m_1^0+...+m_I^0 = \omega_m-(c_1(q_1^0)+...+c_J(q_J^0))$ and $m_1'+...+m_I' = \omega_m-(c_1(q_1')+...+c_J(q_J'))$ at the right hand side, we get the following sequence of inequalities:

$$\varphi_1(x_1^0)+...+\varphi_I(x_I^0)+ \omega_m - (c_1(q_1^0)+...+c_J(q_J^0)) < \varphi_1(x_1')+...+\varphi_I(x_I')+ \omega_m - (c_1(q_1')+...+c_J(q_J')),$$

Proof (cont.)

$$\varphi_1(x_1^0)+\ldots+\varphi_I(x_I^0)+m_1^0+\ldots+m_I^0 < \varphi_1(x_1')+\ldots+\varphi_I(x_I')+m_1'+\ldots+m_I',$$

$$\varphi_1(x_1^0)+m_1^0+\ldots+\varphi_I(x_I^0)+m_I^0 < \varphi_1(x_1')+m_1'+\ldots+\varphi_I(x_I')+m_I',$$

that is

$$u_1(m_1^0,x_1^0)+\ldots+u_I(m_I^0,x_I^0) < u_1(m_1',x_1')+\ldots+u_I(m_I',x_I'),$$

or in short:

$$u_1^0+\ldots+u_I^0 < u_1'+\ldots+u_I'.$$

Proof (cont.)

Since the inequality is a strict one, let $\alpha>0$ be a number such that

$$\alpha = u_1'+...+u_I'-(u_1^0+...+u_I^0).$$

To complete the proof let us define the following distribution of utilities $(u_1'',...,u_I'')$. Namely, let $u_i''=u_i^0$ for $i=2,3,...,I$ and $u_1''=u_1^0+\alpha$, which will make it obvious that $(u_1^0,...,u_I^0)$ was not a Pareto optimum. The distribution will be defined as follows. The starting point is $(u_1^0,...,u_I^0)$:

$$u_i^0 = u_i^0+u_i'-u_i' = u_i'+u_i^0-u_i' = u_i'+T_i = u_i'',$$

Proof (cont.)

where $T_i = u_i^0 – u_i'$ for $i=2,3,...,I$; and $u_1'' = u_1^0+\alpha = u_1^0+u_1'–u_1'+\alpha = u_1'+ u_1^0–u_1'+\alpha = u_1'+T_1$, where $T_1 = u_1^0–u_1'+\alpha$. The feasibility of defining such a distribution can be proved by checking that $T_1+T_2+...+T_I = 0$ (in other words, it can accompany the allocation $(x_1',...,x_I',q_1',...,q_J')$ by transferring the compound good appropriately). Indeed

$$T_1+T_2+...+T_I = u_1^0–u_1'+\alpha+u_2^0–u_2'+...+ u_I^0–u_I' = u_1^0+...+u_I^0–(u_1'+...+u_I')+\alpha = –\alpha+\alpha = 0.$$

(T.16) The first theorem of welfare economics

In the Marshallian analysis, if an allocation $(x^*_1,\ldots,x^*_I,q^*_1,\ldots,q^*_J)$ coupled with the price p^* make a competitive equilibrium then the allocation is Pareto optimal

Proof:

T.5 on competitive equilibrium in Marshallian analysis implies that for every $i=1,\ldots,I$ and $j=1,\ldots,J$:

1. $p^* \leq c_j'(q^*_j)$, and equality holds if $q^*_j > 0$,
2. $\varphi_i'(x^*_i) \leq p^*$, and equality holds if $x^*_i > 0$,
3. $x^*_1+\ldots+x^*_I = q^*_1+\ldots+q^*_J$.

Proof (cont.)

Let us substitute $\lambda = p^*$, then – by the Kuhn-Tucker theorem for the problem of maximizing the surplus – the conditions are both necessary and sufficient for solving this problem. T.15 implies then that $(x^*_1,\dots,x^*_I,q^*_1,\dots,q^*_J)$ is a Pareto optimum.

(T.17) The second theorem of welfare economics

In the Marshallian analysis, for any Pareto optimum $(u^0_1,...,u^0_I)$ transfers $(T_1,...,T_I)$ of the compound good which satisfy the condition $T_1+...+T_I=0$ can be made such that a competitive equilibrium triggered by the initial allocation $(\omega_{m1}+T_1,...,\omega_{mI}+T_I)$ and by a price p^* lets achieve this Pareto optimum.

Proof:

The theorem predicts achieving the Pareto optimum following some transfers, i.e. enjoying the original level of utility by all consumers; this does not imply, however, that in the equilibrium every consumer will consume the original (initial) amount of the analyzed good and will enjoy the original (initial) amount of the compound good. Nevertheless T.15 implies that every Pareto optimum lets enjoy the same – maximum – sum of utilities. Let $(u^0_1,\dots,u^0_I)$ be a Pareto optimum where $u^0_i = u_i(x_i^0,m_i^0) = \varphi_i(x_i^0)+m_i^0$ (quantities m_i^0 of the compound good left once the cost of production of the analyzed good are paid satisfy the equation $m_1^0+\dots+m_I^0 = \omega_m-(c_1(q_1^0)+\dots+c_J(q_J^0)))$. T.15 implies that $(x^0_1,\dots,x^0_I,q^0_1,\dots,q^0_J)$ maximizes the surplus.

Proof (cont.)

Now let $(x^*_1,\dots,x^*_I,q^*_1,\dots,q^*_J)$ be an arbitrary competitive equilibrium. Its existence results from the assumptions of the Marshallian analysis. Namely, if $\mathrm{Max}_i\varphi_i'(0)>\mathrm{Min}_jc_j'(0)$, then the equilibrium exists by T.6. Otherwise let us put $p^*=(\mathrm{Min}_jc_j'(0)+\mathrm{Max}_i\varphi_i'(0))/2$. It is easy to check that by this price there is the following trivial equilibrium $x^*_1=\dots=x^*_I=q^*_1=\dots=q^*_J=0$. In both cases the existence of equilibrium is guaranteed, and T.5 implies that its conditions (including equilibrium prices) do not depend on the quantity of the compound good ω_m or its distribution $(\omega_{m1},\dots,\omega_{mI})$. Hence it is possible to change the quantity of the compound good or its distribution without affecting the price p^*. For that reason it can be assumed that in the equilibrium the sum

Proof (cont.)

of utilities (which depends on the quantity of the compound good, among other things) is the same as in the original Pareto optimum:

$$u^0_1+...+u^0_I = u^*_1+...+u^*_I, \text{ and } u^*_i=\varphi_i(x_i^*)+m_i^* \text{ for } i=1,...,I.$$

Now we will define transfers $T_1,...,T_I$ by the formula $T_i = \varphi_i(x_i^0)+m_i^0–(\varphi_i(x_i^*)+m_i^*)$. Their sum

$$T_1+...+T_I = u^0_1+...+u^0_I – (u^*_1+...+u^*_I) = 0.$$

The utility achieved by the *i*-th consumer in the equilibrium following the transfer is $\varphi_i(x_i^*)+m_i^*+T_i = \varphi_i(x_i^*)+m_i^*+\varphi_i(x_i^0)+m_i^0–(\varphi_i(x_i^*)+m_i^*) = \varphi_i(x_i^0)+m_i^0$, i.e. exactly what it was in the original Pareto optimum.

(D.9) Marshallian welfare model

In Marshallian analysis we assume that for any consumption level x of good ℓ its allocation is optimal (i.e. if $x=x_1+...+x_I$, then $\varphi_i'(x_i)=P(x)$ for every *i*). Moreover we assume that for any production level q of the good ℓ its allocation is optimal (i.e. if $q=q_1+...+q_J$, then $c_j'(q_j)=C'(q)$ for every *j*).

<u>(T.18)</u> Conditions of D.9 are satisfied if all the consumers are pricetakers and they face the same price, and if all the producers are pricetakers and they face the same price. (Note: the price faced by the consumers does not have to be the same that is faced by the producers.)

<u>(T.19)</u> In the Marshallian welfare model

$$S(x) = S_0 + \int_0^x (P(s) - C'(s))ds, \text{ where:}$$

$$S(x) = MAS(x_1,...,x_I,q_1,...,q_J), \text{ and}$$

$$S_0 = MAS(0,...,0,0,...0)$$

(T.20) Surplus decomposition in the Marshallian welfare model with a tax (notation as in D.8, D.9 and T.14):

$$DWL = S^*(x^*(0)) – S^*(x^*(t)) =$$

$$– \int_{x^*(0)}^{x^*(t)} (P(s) – C'(s))ds =$$

$$CS(p^*(0)) – CS(p^*(t)+t) + \Pi(p^*(0)) – \Pi(p^*(t)) – tx^*(t) =$$

$$\int_{p^*(0)}^{p^*(t)+t} x^*(s)ds + \int_{p^*(t)}^{p^*(0)} q^*(s)ds – tx^*(t),$$

where:

DWL – Deadweight welfare loss resulting from the tax,

$x^*(t)$ – equilibrium demand after the tax t,

$q^*(t)$ – equilibrium supply after the tax t,

$p^*(t)$ – equilibrium price after the tax t,

CS(p) – aggregate consumer surplus by the price p,

$\Pi(p)$ – aggregate producer surplus by the price p

(D.10) Long term competitive equilibrium

Triple (p^*, q^*, J^*), where

J^* – number of firms active in the market (i.e. with positive production).

Assumptions:

1. All firm are pricetakers and identical
2. For every firm $c(0)=0$
3. Every firm is a profit maximizer:

   q^* solves the PMP, $\mathrm{Max}_{q\geq 0}\{p^*q - c(q)\}$
4. The demand meets the supply: $x(p^*) = J^*q^*$
5. There is free entry: $p^*q^* - c(q^*) = 0$

<u>(T.21)</u> In the model defined by D.10:

1. If the production has strictly decreasing scale effects (the function c is strictly increasing and convex) and $x(c'(0))>0$, then there is no equilibrium defined by D.10
2. If the production has constant scale effects ($c(q)=c_0q$, c_0=const>0), then $p^*=c_0$ and $J^*q^*=x(c_0)$
3. If there is $q_0>0$ such that $c(q_0)/q_0=\mathrm{Min}_{q\geq 0}\{c(q)/q\}$, and $\forall q\neq q_0$ $[c(q_0)/q_0<c(q)/q]$, $c_0=c(q_0)/q_0$ and $x(c_0)>0$, then $(c_0,q_0,x(c_0)/q_0)$ is the only triple which satisfies D.10

Non-technical summary

A. The competitive equilibrium model assumes full "commercialization" of the consumption: consumption expenditures are equal to revenues from selling endowments and revenues from profits made by (co-)owned firms.

B. The Marshallian analysis of partial equilibrium rests on the assumption that – thanks to the quasi-linearity of the utility function – it is possible to confine the analysis to changes in one market only, and looking at other prices and quantities as fixed.

C. Definition of Pareto optimum is independent of allocation and production mechanism (in particular it does not have to be a market mechanism).

D. Welfare economics theorems suggest that Pareto optima and competitive equilibria are equivalent.

E. In the Marshallian analysis of partial equilibrium changes in welfare – e.g. resulting from imposing a tax – can be assessed by analysing economic surplus.

F. Long-term equilibrium is determined by the zero-profit condition (*free entry*).

6. External effects and public goods

(D.1) Externality: activity of an economic agent affects a consumer's welfare or a firm's production possibility directly.

(D.2) Model of a bilateral externality

Let I=2. It is assumed that utilities depend not only on the bundle purchased, but also on the activity $h \geq 0$ undertaken by the consumer i=1: $u_i(x_{1i},\ldots,x_{Li},h)$ for i=1,2, and $\partial u_2/\partial h \neq 0$. In equilibrium the consumers enjoy utilities $v_i(\mathbf{p},w_i,h) = \mathrm{Max}_{x_i \geq 0}\{u_i(\mathbf{x}_i,h): \mathbf{p}^{*T}\cdot\mathbf{x}_i \leq w_i\}$. It is also assumed that utilities are quasi-linear functions with respect to the compound good, so $v_i(\mathbf{p},w_i,h) = \varphi_i(\mathbf{p},h)+w_i$. Functions $\varphi_i(\mathbf{p},.)$ are strictly concave with respect to h (i.e. $\partial^2\varphi_i/\partial h^2<0$).

(T.1) The first order condition for Pareto optimality of the externality is: $\partial\varphi_1(\mathbf{p},h^0)/\partial h \leq -\partial\varphi_2(\mathbf{p},h^0)/\partial h$, and if $h^0>0$, then the equality holds

(T.2) Existence of the bilateral externality (D.2) implies that the market equilibrium h^* can be non Pareto optimal. Let $h^*,h^0>0$ (internal solutions). If h^0 is a Pareto optimum then:

$$\partial\varphi_2/\partial h<0 \Rightarrow h^*>h^0 \text{ and } \partial\varphi_2/\partial h>0 \Rightarrow h^*<h^0$$

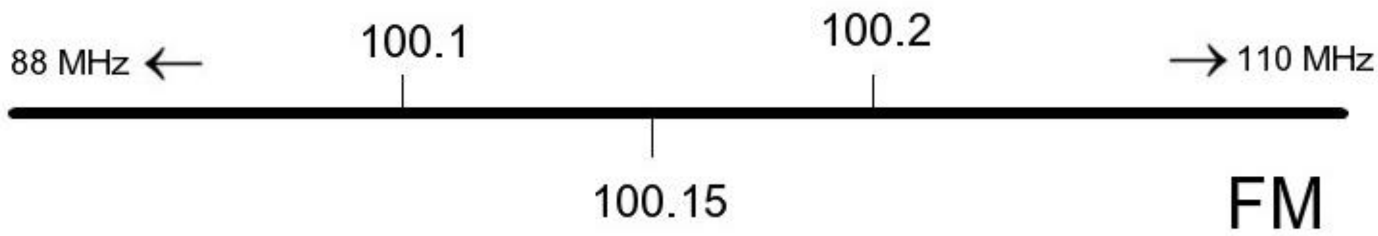
(D.3) Pigouvian tax

Tax rate: $t_h = -\partial\varphi_2(\mathbf{p},h^0)/\partial h$; tax due: $t_h(h-h_o)$, where h_o – tax threshold

(T.3) For any threshold, a Pigouvian tax for a bilateral externality (D.2) causes that market equilibrium is Pareto optimal

(T.4) Coase Theorem

A bilateral externality (D.2) holds, but consumers – by offering compensations – can agree on its level without paying transaction costs. If the consumer 2 has the right to prevent the consumer 1 from generating the externality, then the consumer 1 can motivate the consumer 2 to voluntarily waive this right in exchange for a money compensation. If the consumer 1 has the right to generate the externality, then the consumer 2 can motivate the consumer 1 to voluntarily waive this right in exchange for a money compensation. In both cases the same Pareto optimum level of the externality will be achieved.



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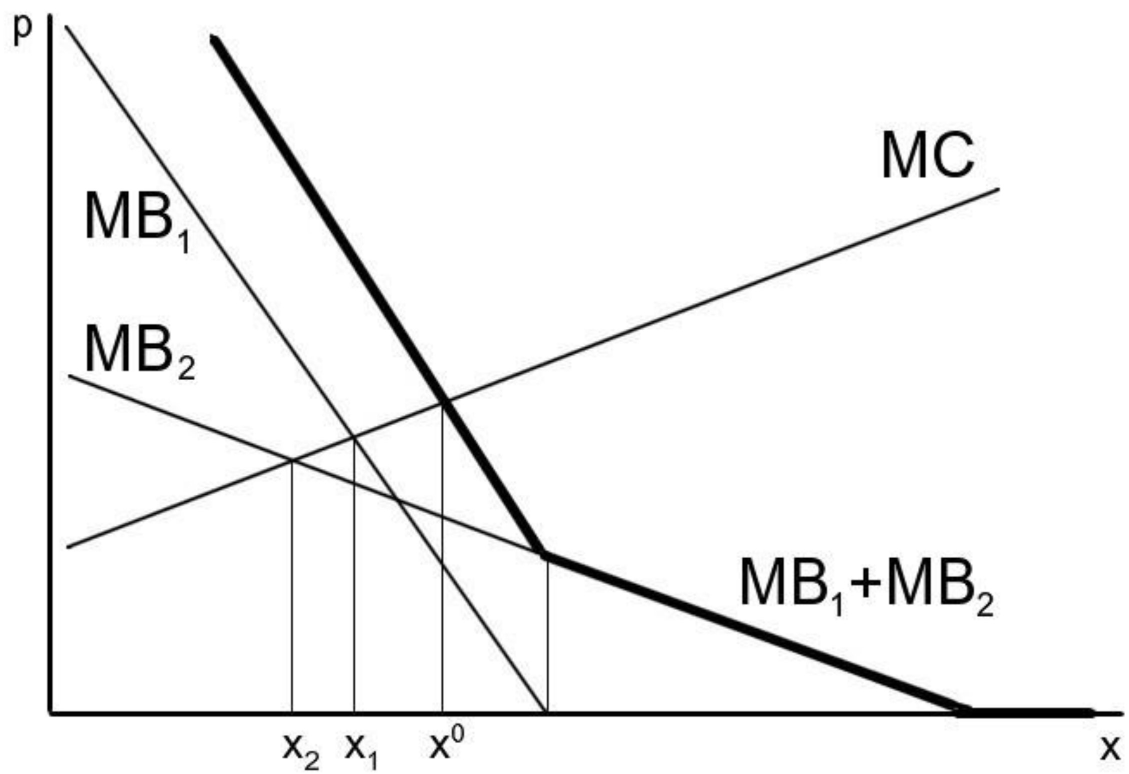
(T.5) If the bilateral externality (D.2) can be traded, then market equilibrium will be established in a Pareto optimum.

(D.4) Public good: a good that satisfies two conditions: (1) its consumption by one user does not preclude its consumption by another user (so-called non-rivalry principle), and (2) if a unit of the good was supplied, then nobody can be excluded from using it (so-called non-exclusion principle). Goods which violate both conditions are called private ones (such goods were analyzed in previous lectures)


p
MC
MB1
MB2
MB1+MB2
x2
x1
x0
x

(D.5) Public good model

Let I consumers use a public good x (in addition to private ones x_ℓ; ℓ=1,…,L). It is assumed that their utilities are quasi-linear with respect to the compound good representing all private goods (their prices do not depend on the public good consumption). In equilibrium the consumers achieve utilities $v_i(\mathbf{p},x,w_i,c_i) = \mathrm{Max}_{x_i \geq 0}\{u_i(\mathbf{x}_i,x,w_i): \mathbf{p}^T \cdot \mathbf{x}_i \leq w_i - c_i(x)\}$, where $c_i(x)$ is a part of the public good provision financed by the consumer *i* and $\mathbf{x}_i=[x_{i1},\ldots, x_{iL}]^T$ is vector of all private goods consumed by *i*. Please note that a specific optimum combination of private goods

Public good model (cont.)

may depend on prices **p** (that are common for all the consumers), and utility specification u_i (different for every consumer).Thus $v_i(x,w_i,c_i) = \varphi_i(\mathbf{p},x)+w_i-c_i(x)$. It is assumed that for every **p** functions $\varphi_i(\mathbf{p},.)$ are twice differentiable with $\partial^2\varphi_i/\partial x^2<0$ for $x\geq0$. Function c is twice differentiable with $c''>0$ for $x\geq0$. If $\partial\varphi_i/\partial x>0$ and $c'>0$, then the public good is a desired one and its production cost is positive. It is also conceivable that $\partial\varphi_i/\partial x<0$ and $c'<0$; the good is then an undesired one, and its reduction requires expenditures.

(T.6) The quantity x^0 of the public good satisfying D.5 is Pareto optimal, if $\partial\varphi_i/\partial x>0$ and $c'>0$, it solves the problem $\text{Max}_{x\geq 0}\{\varphi_1(\mathbf{p},x)+...+\varphi_I(\mathbf{p},x)-c(x)\}$. The following first order condition is both necessary and sufficient: $\partial\varphi_1(\mathbf{p},x^0)/\partial x+...+\partial\varphi_I(\mathbf{p},x^0)/\partial x \leq c'(x^0)$, and if $x^0>0$, then equality holds.

(T.7) Let prices **p** be fixed. Then utilities achieved by consuming the public good in D.5 (for $\partial\varphi_i/\partial x>0$ and $c'>0$) can be written briefly as $\varphi_i(x)$. Let us assume that units of the public good can be individually purchased at the price p, so that $x_1+...+x_I=x$ is the aggregate demand for this good purchased – unit by unit – by the consumers. In equilibrium the consumers solve the optimization problem $\mathrm{Max}_{x_i\geq 0}\{\varphi_i(x_i+\sum_{k\neq i}x_k^*)-p^*x_i\}$, under the assumption that the equilibrium price p^* is fixed and so are equilibrium quantities x_k^* purchased by other consumers (Nash equilibrium). The first order conditions (necessary and sufficient) are $\varphi_i'(x_i^*+\sum_{k\neq i}x_k^*) \leq p^*$, and if $x_i^*>0$, then equality holds.

(T.8) By the assumptions of T.7, if the supply of the public good is provided by a profit-maximizing producer – who solves the problem $\mathrm{Max}_{q\geq 0}\{p^*q - c(q)\}$ – then the first order condition (necessary and sufficient) is $p^* \leq c'(q^*)$, with the equality whenever $q^*>0$.

(T.9) By the assumptions of T.7 and T.8, if $I>1$ and $x^0>0$, then $\varphi_1'(q^*)+...+\varphi_I'(q^*) > c'(q^*)$, which precludes the Pareto optimality of q^*. Moreover: $q^*<x^0$.

(T.10) If in T.7 the equality $\varphi_i'(x^*) = p^*$, is satisfied for the *i*th consumer only, then $x_1^* = \ldots = x_{i-1}^* = x_{i+1}^* = \ldots = x_I^* = 0$. A free riding takes place.

(T.11) The non-optimality of the equilibrium from T.9 can be interpreted as a result of a positive externality (ignored by the market) which emerges when somebody purchases a unit of the public good individually.

(D.6) Lindahl equilibrium

The public good is sold individually (as if it were a private good), and it is 'dedicated' to individual consumers. Hence every consumer may pay a different price p_i^{**}. By the notation of T.7, in equilibrium consumers solve optimization problems $\mathrm{Max}_{x_i \geq 0}\{\varphi_i(x_i) - p_i^{**} x_i\}$ with the first order condition (necessary and sufficient) $\varphi_i'(x_i^{**}) \leq p_i^{**}$ with equality for $x_i^{**} > 0$.

Lindahl equilibrium (cont.)

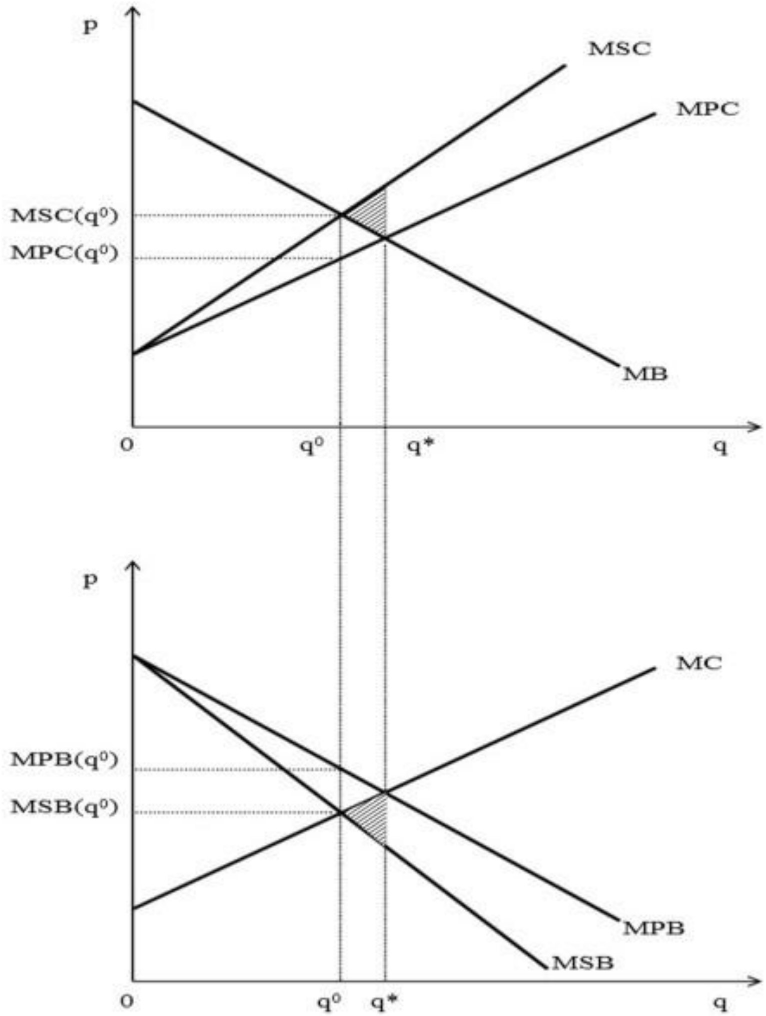
It is assumed further that the bundle of these 'dedicated' public goods is produced by a price-taking firm with a technology characterized by full complementarity: $x_1=...=x_I=q$. Thus in equilibrium, the firm solves the optimization problem $\text{Max}_{q\geq 0}\{(p_1^{**}+...+p_I^{**})q-c(q)\}$ with first order condition (necessary and sufficient) $p_1^{**}+...+p_I^{**}\leq c'(q^{**})$ with equality for $q^{**}>0$.

<u>(T.12)</u> In the Lindahl equilibrium (D.6 with market clearing condition $x_i^{**}=q^{**}$ for i=1,...,I) $q^{**}=x^0$ (i.e. the equilibrium is Pareto optimal).

<u>(D.7)</u> A multilateral externality is called <u>depletable</u> or private, if its impact on one agent diminishes its impact on other agents. If its impact does not depend on the number of affected agents, then it is called <u>non-depletable</u> or public.

(T.13) T.5 is valid for depletable multilateral externalities, but not necessarily for non-depletable ones.

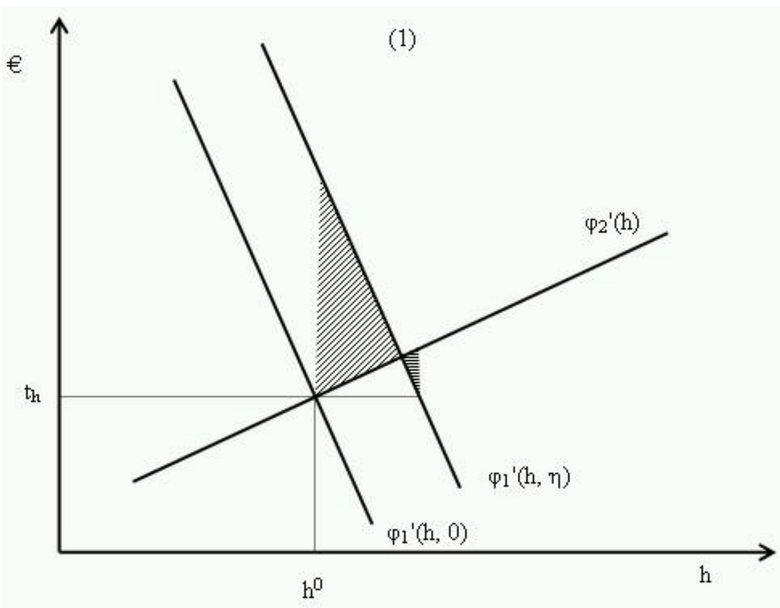
Market failure (negative externalities)

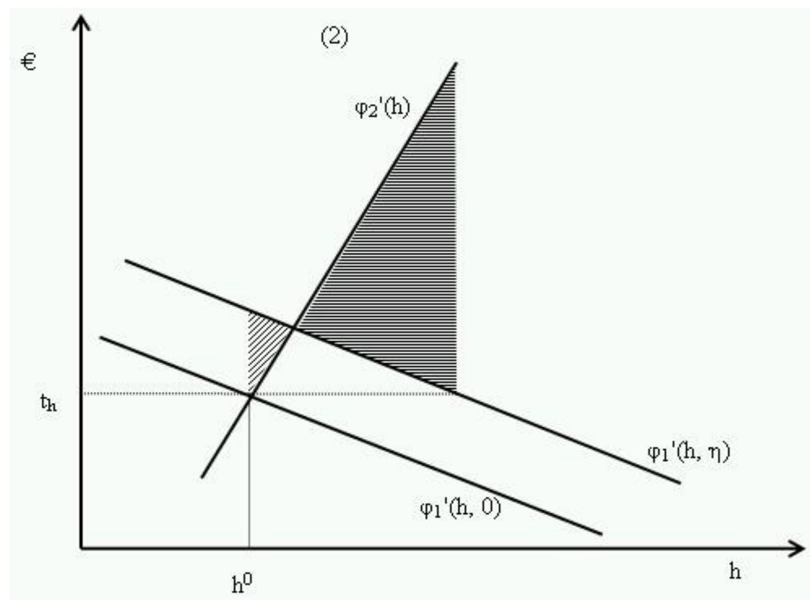


(T.14) Weitzman rule

Case I: $|\partial^2\varphi_1(\mathbf{p},h^0,\eta)/\partial h^2| > |-\partial^2\varphi_2(\mathbf{p},h^0)/\partial h^2|$

Case II: $|\partial^2\varphi_1(\mathbf{p},h^0,\eta)/\partial h^2| < |-\partial^2\varphi_2(\mathbf{p},h^0)/\partial h^2|$





(T.14) Weitzman rule (cont.)

In the bilateral externality model (D.2) it is assumed that φ_1 depends on some parameter $\eta \in \Re$ such that $\partial\varphi_1(\mathbf{p},h,\eta)/\partial h = \partial\varphi_1(\mathbf{p},h,0)/\partial h + \eta$, but φ_2 does not depend on it. The true value of $\eta \neq 0$ is known to the consumer 1, but the government thinks it is 0. Based on this (inaccurate) knowledge the government wishes to arrive at the optimum level of the externality h, i.e. to satisfy the equality $\partial\varphi_1(\mathbf{p},h^0,\eta)/\partial h = -\partial\varphi_2(\mathbf{p},h^0)/\partial h$ (assuming that $h^0 > 0$). If $|\partial^2\varphi_1(\mathbf{p},h^0,\eta)/\partial h^2| > |-\partial^2\varphi_2(\mathbf{p},h^0)/\partial h^2|$, then the loss of MAS caused by the (inaccurate) regulation caused by the Pigouvian tax t_h (D.3) is lower than the loss caused by requiring h^0. And *vice versa* if the inequality is reversed.

(T.15) The Groves-Clarke tax

In the bilateral externality model (D.2) let us assume that the externality can be either h=0 or h=1. Let $\varphi_1(\mathbf{p},0)=\varphi_2(\mathbf{p},1)=0$, $\varphi_1(\mathbf{p},1)=b$, $\varphi_2(\mathbf{p},0)=c$, and b is known to the consumer 1 only, and c – to the consumer 2 only. The consumers ought to reveal b and c by quoting numbers b' and c'. The government declares that it will mandate h=0, if $c'\geq b'$ or h=1, if $c'<b'$. Moreover in the latter case it will impose a tax c' on the consumer 1 and it will give a subsidy b' to the consumer 2. If such a mechanism is deployed both consumers are better of when they reveal b'=b and c'=c (i.e. truthfully reveal their preferences).

Non-technical summary

A. Externalities – both positive and negative – take place when someone's consumption activities have a direct impact on someone else's utility. Equilibrium does not have to be Pareto optimal.

B. All the corollaries on externalities involving consumers apply to producers as well. Utilities are to be substituted with profits.

C. In the presence of an externality a Pareto optimum can be attained as a result of government intervention either by a direct (quantity) regulation or by indirect regulation using a Pigouvian tax.

D. In special cases (a bilateral, i.e. a depletable externality) a Pareto optimum can be attained by trading externalities.

E. A public good and a non-depletable externality linked to it cannot be traded like a depletable one. It is difficult to operationalize a Lindahl equilibrium which theoretically exists under these circumstances.

F. Maximization of the economic surplus under externalities and imperfect information demonstrates the asymmetry of quantity and tax regulation (Weitzman rule). Information can be acquired by the government by applying certain incentives like the Groves-Clarke tax.

7. Imperfect competition

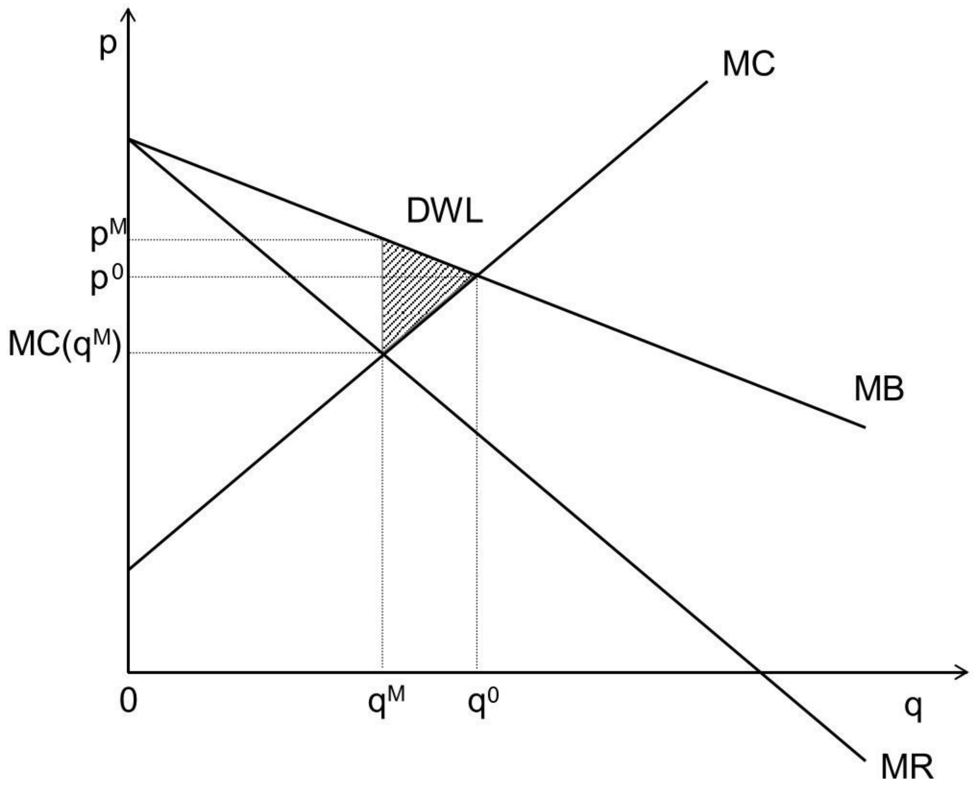
<u>(D.1) Monopoly optimization problem</u>

$\text{Max}_p\{px(p)–c(x(p))\}$ or $\text{Max}_{q\geq 0}\{p(q)q–c(q)\}$, where x(.) is continuous and strictly decreasing for p such that x(p)>0; p(.) is an inverse demand function (we define $p(0)=p_0$, where p_0 is the lowest price such that x(p)=0). It is also assumed that p(.) and c(.) are twice differentiable for $q\geq 0$, $p(0)>c'(0)$, and there exists a unique $q^0\in(0,\infty)$, such that $p(q^0)=c'(q^0)$ (this is the supply which maximizes economic surplus).

(T.1) The solution of the monopoly optimization problem (D.1), q^m, satisfies the following first order condition (necessary): $p'(q^m)q^m+p(q^m)=c'(q^m)$. This also implies that $p(q^m)>c'(q^m)$ and $q^m<q^0$. The deadweight welfare loss is then equal to $\int_{q^m}^{q^0}(p(s)-c'(s))ds$.

(T.2) If in the monopoly model D.1 $p(q)=a-bq$ and $c(q)=cq$, where $a>c\geq0$, $b>0$ then $q^0=(a-c)/b$, $p^0=p(q^0)=c$, $q^m=(a-c)/2b$ and $p^m=(a+c)/2$.

Monopoly – DWL



Duopoly

- y_i – supply from the *i*th rival
- y – aggregate supply
- p_i – price announced by the *i*th rival
- p – price determined by the market

Duopoly (cont.)

1. Bertrand: both rivals make price decisions simultaneously
2. Cournot: both rivals make quantity decisions simultaneously
3. Price leadership: one makes a price decision first
4. Quantity leadership (Stackelberg): one makes a quantity decision first

Duopoly (cont.)

- Determine a demand that the market will reveal: D(p)=y
- Let $p_i y_i$-TC(y_i) be the profit enjoyed by the *i*th firm (TC – total cost)
- Let p_i^* and y_i^* be profit maximizing decisions of the *i*th firm
- Firm *i* makes its decisions which maximize its profit, taking into account expected decisions of its rivals
- We look for an equilibrium in a sense that all firms make decisions exactly as they were expected by their rivals

(D.2) Bertrand model

A duopoly acts in the market characterized by a demand function x(.). The function is continuous and strictly decreasing for p such that $x(p)>0$, and there exists $p_0<\infty$ such that $x(p)=0$ for $p\geq p_0$. Both firms display constant scale effects with unit production cost $c>0$. It is assumed that $x(c)\in(0,\infty)$. The firms compete with each other by simultaneously announcing prices for their products, p_1 and p_2, respectively. The sales are then denoted as: $x_j(p_j,p_k)=x(p_j)$ if $p_j<p_k$, $x_j(p_j,p_k)=x(p_j)/2$ if $p_j=p_k$ and $x_j(p_j,p_k)=0$ if $p_j>p_k$. The profits of the firms are given by the formula $\pi_j(p_j,p_k)=(p_j-c)x_j(p_j,p_k)$.

(T.3) There is the unique Nash equilibrium in a Bertrand duopoly model: $(p_1^*, p_2^*) = (c, c)$.

(D.3) Cournot model

A duopoly acts in the market characterized by an inverse demand function p(.). The function is differentiable and $p'(q)<0$ for $q \geq 0$. Both firms display constant scale effects with unit production cost $c>0$. It is assumed that $p(0)>c$ and there exists the unique production level $q^0 \in (0,\infty)$ such that $p(q^0)=c$ (i.e. $q^0=x(c)$). The firms compete with each other by simultaneously taking supply decisions: q_1 and q_2, respectively. The price is then calculated as $p(q_1+q_2)$, and firms' profits are $\pi_j(q_j,q_k)=(p(q_1+q_2)-c)q_j$.

(T.4) In the Cournot model each Nash equilibrium (q_1^*, q_2^*) must satisfy the following conditions:

- $p'(q_1^*+q_2^*)q_1^*+p(q_1^*+q_2^*)\leq c$ and equality holds if $q_1^*>0$
- $p'(q_1^*+q_2^*)q_2^*+p(q_1^*+q_2^*)\leq c$ and equality holds if $q_2^*>0$, and
- $p'(q_1^*+q_2^*)(q_1^*+q_2^*)/2+p(q_1^*+q_2^*)=c$.

Proof:

Both inequalities result from the Kuhn-Tucker theorem for optimization problems of both duopolists (i.e. $\mathrm{Max}_{q_i}\{(p(q_1+q_2)-c)q_j.\}$). It will be demonstrated that both solutions have to be positive. Let us assume that – conversely – for instance $q_1^*=0$. Then the first inequality is reduced to $p(q_2^*)\leq c$. By analyzing the second inequality, it will be used to demonstrating that $q_2^*=0$ as well. If it is assumed that – on the contrary – $q_2^*>0$, then $p'<0$ (condition listed in D.3) implies $p'(q_2^*)q_2^*+p(q_2^*)<c$, and consequently $q_2^*=0$, and thus a contradiction. Hence $q_2^*=0$.

Proof (cont.)

But then both the first and the second inequality reduces to $p(0) \leq c$, which contradicts another condition of the Cournot model, i.e. $p(0) > c$. Therefore $q_1^* > 0$. Likewise it can be proved that $q_2^* > 0$. Consequently both inequalities from the theorem turn out to be equalities. By adding these equalities and dividing into 2, we get final equality from the theorem.

<u>(T.5)</u> In the Cournot model the Nash equilibrium price is higher than c (the competitive price), but lower than the monopoly price

<u>Proof</u>:

The inequality $p^*>c$ results from T.4. It needs to be proved that $q_1^*+q_2^*>q^m$ (which will imply that $p^*<p^m$). Let us – on the contrary – assume that $q_1^*+q_2^*<q^m$. Then the firm 1 could increase its supply to $q^m-q_2^*$ (higher than q_1^*). The sum of profits will incresase (by the definition of the monopolistic quantity). However, the price would decrease resulting in a lower profit of the firm 2.

Proof (cont.)

This implies that the profit of the firm 1 would increase. Hence (q_1^*, q_2^*) could not have been a Nash equilibrium. Consequently $q^m \leq q_1^* + q_2^*$. Yet the equality from T.4 excludes $q^m = q_1^* + q_2^*$ (the latter equality implies $p'(q^m)q^m/2 + p(q^m) = c$ which contradicts the condition from T.1)

(T.6) In the Cournot model with a linear inverse demand function $p(q) = a - bq$ (with $a > c \geq 0$ and $b > 0$) the Nash equilibrium involves $q_1^* = q_2^* = (a-c)/3b$ and the price $p(q_1^* + q_2^*) = (a+2c)/3 \in (c, p^m)$

(D.4) Oligopolistic Cournot model

The definition D.3 has an obvious generalization for J firms ($J \geq 1$) such that it corresponds to D.3 for $J=2$

(T.7) Let Q_J^* be an aggregate supply in an oligopolistic Cournot model. The necessary condition for the Nash equilibrium is then: $p'(Q_J^*)Q_J^*/J+p(Q_J^*)=c$.

<u>(T.8)</u> In the oligopolistic Cournot model for $J \geq 1$ firms with a linear inverse demand function as in T.6 the Nash equilibrium is given by

$$q^*(J)=q_1^*=\ldots=q_J^* = (a-c)/(b(J+1)).$$

The equilibrium price is then

$$p^*(J)=a-(a-c)J/(J+1),$$

and the firms' profits are

$$\pi^*(J)=\pi_1^*=\ldots=\pi_J^* = (a-c)^2/(b(J+1)^2).$$

<u>(T.9)</u> In the T.8 model p^* and π^* are decreasing functions of J, $\lim_{J\to\infty} p^*(J)=c$, $\lim_{J\to\infty}\pi^*(J)=0$ and $\lim_{J\to\infty} Jq^*(J)=(a-c)/b$

(D.5) Two-stage entry game

Stage I. All firms take simultaneous decisions whether to enter. Every entering firm pays an entry cost $K>0$. *Stage II*. The firms who entered the market play an oligopolistic game (e.g. as in D.4)

(T.10) In a two-stage entry game as in D.5 in every SPNE the number of firms J^* which enter the market satisfies: $\pi^*(J^*)\geq K$ and $\pi^*(J^*+1)<K$

<u>(T.11)</u> If in a two-stage entry game the second stage game is as in T.8, then $J^* = [(a-c)/(bK)^{1/2}-1]$

<u>Proof</u>:

By T.8 the following inequalities must be satisfied (otherwise J^* cannot be the number of firms in SPNE): $(a-c)^2/(b(J+1)^2) \geq K$ and $(a-c)^2/(b(J+2)^2) < K$. By taking square roots of both sides one gets $(a-c)/(b^{1/2}(J+1)) \geq K^{1/2}$ and $(a-c)/(b^{1/2}(J+2)) < K^{1/2}$. By rearranging these expressions one gets $J+1 \leq (a-c)/(bK)^{1/2}$ and $J+2 > (a-c)/(bK)^{1/2}$. In other words

$$(a-c)/(bK)^{1/2}-1 < J+1 \leq (a-c)/(bK)^{1/2}$$

i.e. $(a-c)/(bK)^{1/2}-2 < J \leq (a-c)/(bK)^{1/2}-1$. There is only one integer satisfying these conditions: $J=[(a-c)/(bK)^{1/2}-1]$ (square parenthesis stands for the 'entier' function).

(D.6) Oligopolistic Bertrand model

Definition D.2 can be generalized in an obvious way for J firms ($J \geq 1$) such that for $J=2$ it is satisfied as a special case.

(T.12) If in a two-stage entry game, the game played in the second stage is like in D.6, then – assuming that $\pi^m > K$ – we have: $J^* = 1$

<u>(T.13)</u> If in a two-stage entry game, the game played in the second stage is like in T.8, then J^0, the number of firms which maximizes MAS, i.e.

$$W(J) = \int_0^Q p(s)ds - J(cq^*(J)+K),$$

can be calculated from the condition $W'(J^r)=0$, that is $(J^r+1)^3=(a-c)^2/(bK)$. If J^r is integer then $J^0=J^r$. Otherwise $J^0=[J^r]$ or $J^0=[J^r]+1$.

<u>(D.7) Dynamic Bertrand model</u>

A duopoly satisfies the Bertrand conditions (D.2), but decisions are taken many times – over and over again. Having completed their sales as a result of playing a single stage game, the firms can (simultaneously) announce new prices. It is assumed that firms (j=1,2) maximize the sum of discounted future profits using the discount rate $\rho>0$:

$$\pi_{j1}+\pi_{j2}/(1+\rho)+\pi_{j3}/(1+\rho)^2+\pi_{j4}/(1+\rho)^3+...$$

The game is finite if the firms play it T times, or infinite if the firms play it infinitely many times.

Discounting

A discount rate (ρ) – a parameter used in economics in order to compare amounts (e.g. profits) occurring in different time moments. The amount x to occur a year from now corresponds to $x/(1+\rho)$ presently (so-called "present value").

(T.14) If the game in a dynamic Bertrand model (D.7) is finite then there exists the unique SPNE. It includes Nash strategies for a single stage, i.e. (c,c).

(D.8) Nash reversion strategy

In a dynamic Bertrand model we define

$$H_{t-1}=\{(p_{11},p_{21}),(p_{12},p_{22}),(p_{13},p_{23}),\ldots,(p_{1t-1},p_{2t-1})\}$$

(history of applied strategies). In stage t firms choose strategies $p_{jt}(H_{t-1})=p^m$ if $t=1$ or if H_{t-1} includes only the elements (p^m,p^m); otherwise $p_{jt}=c$.

<u>(T.15)</u> If the game in a dynamic Bertrand model is infinite then the Nash reversion strategy (D.8) is SPNE if and only if $\rho \leq 1$

<u>(T.16)</u> If the game in a dynamic Bertrand model is infinite and $\rho \leq 1$, then every consistent choice of any fixed price $p \in [c, p^m]$ enforced by the Nash reversion strategy is SPNE.

<u>(T.17)</u> If the game in a dynamic Bertrand model is infinite and $\rho > 1$, then the only SPNE includes the choice of the price $p = c$ in each stage by both partners.

Non-technical summary

A. If firms are not pricetakers then the market equilibrium depends on the way the firms compete with each other. For instance, in the short run price competition leads to an outcome similar to a competitive equilibrium, and quantity competition leads to an outcome further from a competitive equilibrium.

B. In the long run the rivalry can be interpreted as a two-stage game: in the first stage firms take decisions whether to enter the market, and in the second stage they decide how to compete once they meet each other in the market.

C. By applying the two-stage game model one can determine an equilibrium number of firms (in an SPNE) which can survive in the market. Linear models with identical firms determine this number uniquely, but they do not determine which specific firm will be in the market.

D. In the two-stage game model the equilibrium number of firms active in the market (in the SPNE sense) does not have to coincide with the socially optimum number of firms (in the sense of MAS maximization). For instance, if the second stage corresponds to the oligopolistic Cournot model, the equilibrium number of firms is higher than that maximizing the MAS.

E. In a finite dynamic game, any SPNE consists of Nash strategies for every stage. In an infinite dynamic game there may be SPNEs which do not confine to Nash strategies for single stages if any 'wrong' choice triggers the Nash reversion strategy.

8. Asymmetric information

Market failure in a competitive market

"Asymmetric information" = the buyer has less information about the commodity than the seller or *vice versa*; acquiring information is possible, but costly.

Consequences of asymmetric information:

- Equilibrium does not have to be a Pareto optimum;
- Equilibrium may not exist.

Examples of asymmetric information:

- Used cars,
- Insurance policies,
- Employment contracts.

<u>(D.1)</u> Competitive equilibrium in a labour market where the productivity θ of workers is known to themselves only, but employers know their statistical distribution is a pair:

$$\Theta^* = \{\theta: r(\theta) \leq w^*\}$$

$$w^* = E(\theta | \, \theta \in \Theta^*),$$

where r – the opportunity cost of labour (the value of leisure), w – wage, Θ – the set of all workers (characterized by a specific productivity); it is assumed that $\Theta^* \neq \varnothing$.

(T.1) If $r(\theta)=r=const$, then in Pareto optimum workers of the productivity $\theta \geq r$ work, and those of the productivity $\theta < r$ do not work

(T.2) If $r(\theta)=r=const$, $\theta \in [\theta_{min}, \theta_{max}]$, and $\theta_{min} < r < \theta_{max}$, then a competitive equilibrium (D.1) will not be a Pareto optimum

(D.2) Adverse selection: a behaviour of the agent who possesses information adversely affecting another agent who does not possess this information

(T.3) If $r(\theta)\leq\theta$ for $\theta\in[\theta_{min},\theta_{max}]$, then in a Pareto optimum all workers should be employed

(T.4) Let $r(\theta)=2\theta/3$ for $\theta\in[0,1]$, with θ being distributed uniformly on $[0,1]$. Then $w^*=0$ and $\Theta^*=\{0\}$, and since this is the only equilibrium, adverse selection will follow

Proof:

The pair $(0,\{0\})$ makes an obvious equilibrium. We will demonstrate that this is a unique equilibrium. An equilibrium wage cannot be negative. Thus let us assume that $w>0$. Then $\Theta = \{\theta: r(\theta)\leq w\} = \{\theta: 2\theta/3\leq w\} =$
$= \{\theta: \theta\leq 3w/2\} = [0,3w/2]$. At the same time $E(\theta|\ \theta\in\Theta) =$
$= E(\theta|\ \theta\leq 3w/2) = 1/(3w/2-0)\ {}_0\!\int^{3w/2}\theta d\theta =$
$= 2/(3w)\ [\theta^2/2]_0^{3w/2} = 3w/4 < w$. Therefore no $w>0$ can be an equilibrium wage.

<u>(D.3)</u> The distribution of reservation wages $r(\theta)$ is known, and so is known the distribution of probabilities $Pr\{[\theta_1,\theta_2]\}$. A competitive equilibrium with adverse selection in labour market served by 2 firms is the outcome of a two-stage game. In the stage I the firms offer wages w_1 and w_2, respectively. In stage II workers make decisions whether to work for the firm who offers a higher wage if $Max\{w_1,w_2\}\geq r(\theta)$. If $w_1=w_2$, the choice of the firm is random with equal probabilities (1/2,1/2).

<u>(T.5)</u> Let $r(\theta)\leq\theta$, r is strictly increasing on $[\theta_{min},\theta_{max}]$, and its density distribution function $f(\theta)$ satisfies $f>0$ on $[\theta_{min},\theta_{max}]$. Let W^* be the set of wages that satisfy the competitive equilibrium (D.1) and $w^*=\mathrm{Max}\{w: w\in W^*\}$. We assume that an equilibrium is attained as a result of a two-stage game as in D.3. Then:

1. If $w^*>r(\theta_{min})$ and $\exists\varepsilon>0\ \forall w'\in(w^*-\varepsilon,w^*)\ [E(\theta| r(\theta)\leq w')>w']$, then there exists a unique (in pure strategies) SPNE in this two-stage game. In this SPNE workers employed receive the wage w^*, and all the people from the set $\Theta(w^*)=\{\theta: r(\theta)\leq w^*\}$ are employed.

2. If $w^*=r(\theta_{min})$, then there exist many SPNEs (in pure strategies). In each such an equilibrium both firms enjoy zero profits, and every worker with the productivity θ gets the revenue $r(\theta)$ (either as a wage or the value of leisure).

Proof:

We adopt the convention that if $r(\theta)=w$, then the worker accepts the work (even though he/she is indifferent between work and leisure).

Proof (cont.)

1. Let $w^* > r(\theta_{min})$. First we observe that in SPNE both firms must have zero profits. This is because if it were otherwise in an SPNE then by offering a wage w'' and jointly employing M workers, they would enjoy a joint profit

$$\Pi = M(E(\theta|r(\theta)\leq w'')-w'') > 0.$$

The positivity of this number implies the positivity of M (somebody was employed), and consequently the inequality $w'' \geq r(\theta_{min})$. Let us assume further that one of the firms – say, the number 1 – enjoys the profit not larger than $\Pi/2$. However this firm could enjoy the profit at least $M(E(\theta|r(\theta)\leq w''+\alpha)-w''-\alpha)$ by

Proof (cont.)

offering the wage $w''+\alpha$ for some $\alpha>0$ (claiming all the workers of the second firm and thus having at least M employees). Yet $E(\theta|r(\theta)\leq w)$ – as an integral – is continuous with respect to w, and choosing a sufficiently small α one can make that the profit of this firm is close to Π, i.e. higher than $\Pi/2$. Therefore firm 1 has a motivation to leave the wage offer w'', thus contradicting our assertion that w'' was an element of SPNE. It follows from here that $\Pi\leq 0$. But no firm can have a negative profit in an SPNE (since by offering a zero wage it would enjoy the zero profit, i.e. better than a negative one).

Proof (cont.)

Therefore $\Pi=0$ and both firms must have zero profits in any SPNE. Knowing this, if w'' is the higher wage offered by one of the firms in an SPNE, it must be either $w''\in W^*$ (w'' must be an equilibrium wage satisfying D.1) or $w''<r(\theta_{min})$ (w'' is too low to be accepted by anybody). Let us suppose that $w''<w^*$. But then – by the assumption – each firm could yield a positive profit by changing its offer from w'' to $w'\in(w^*-\varepsilon,w^*)$, which contradicts that w'' was an element of SPNE. Hence the higher wage offered in an SPNE must be w^*.

Proof (cont.)

In order to complete this part of the proof, we will demonstrate that offering the wage w^* by both firms and accepting the employment by workers consistent with the rules listed make an SPNE. From the previous steps of the proof it follows that both firms make zero profits in an SPNE. Neither of the firms has an incentive to unilaterally lower the wage as this would result in losing all the employees and non-attaining a positive profit. Now let us check that offering a higher wage is not profitable either, as $E(\theta|r(\theta)\leq w)<w$ for $w>w^*$. By the assumption w^* is the highest competitive wage. Therefore there is no

Proof (cont.)

$w>w^*$, such that $E(\theta|r(\theta)\leq w)=w$. It follows from here – again by the continuity of $E(\theta|r(\theta)\leq w)$ with respect to w – the number $E(\theta|r(\theta)\leq w)-w$ must have the same sign (either positive or negative) for all $w>w^*$. It cannot be positive, since when $w\to\infty$ then $E(\theta|r(\theta)\leq w)\to E(\theta)$, and the latter is a finite number; that is, for sufficiently large w, the difference must be negative or – equivalently –$E(\theta|r(\theta)\leq w)<w$ which completes this part of the proof.

2. Let us assume that $w^*=r(\theta_{min})$, that is for any $w>w^*$ we have $E(\theta|r(\theta)\leq w)<w$ (offering a wage higher than w^* implies a loss). On the other hand, offering a

Proof (cont.)

price $w \leq w^*$ leads to zero profits for both firms. Thus the set of wage offers which can be included in an SPNE is $\{(w_1,w_2): w_1 \leq w^* \text{ and } w_2 \leq w^*)\}$. They can be multiple, but in every case both firms have zero profits, and every worker gets the same amount, namely $w^*=r(\theta_{min})$, that is $r(\theta)$ (i.e. $r(\theta_{min})$ if he/she belongs to the lowest productivity category and is employed; otherwise he/she receives $r(\theta)$ enjoying leisure).

(D.4) Signalling game

The productivity of workers (not observable for employers) takes two values: θ_L and θ_H, with $\theta_L<\theta_H$. It is assumed that r=0 for all workers. Each worker may acquire a formal education e>0 (observable for employers) whose cost is c_He for a high productivity worker and c_Le for a low productivity worker, with $c_H<c_L$. Acquiring the education e does not change the worker's productivity.

(T.6) If the level of education $e=e^*$ in D.4 is set such that

$$(\theta_H - \theta_L)/c_L < e^* < (\theta_H - \theta_L)/c_H,$$

then workers of the productivity $\theta=\theta_H$ – and only they – acquire this level of education (e^*) thus "signalling" to the employers their productivity and getting the wage $w_H=\theta_H$. Other workers are employed with the wage $w_L=\theta_L$. The market equilibrium achieved in this way is called a separating equilibrium.

Adding an adjective to a noun may change its meaning badly: "chair" $\rightarrow$ "electric chair".

"Equilibrium" $\rightarrow$ "separating equilibrium" ?

- Guarantees (in a second-hand car market) do not imply unnecessary cost.

- Signalling assumes that education is irrelevant for economic efficiency (a_1 and a_2 do not depend on whether somebody undertook the education or not).

(T.7) If the productivity θ of workers and the opportunity cost of employment r are as in D.4, then employers can distinguish which category they belong to by offering contracts such that wages depend on tasks that are easier to be achieved by high-productivity workers (screening game).

In a signalling game workers assign themselves to either category to acquiring a diploma. Here workers assign themselves to either category by choosing a more or less ambitious employment contract.

(D.5) The principal-agent problem

Stating the terms of an employment contract in order to overcome the asymmetric information caused by the fact that the "principal" cannot observe the amount of labour spent by the "agent".

(D.5) The principal-agent problem (elementary version)

- x – employee's effort
- $y=f(x)$ – product (we assume that its price is equal to 1)
- $s(y)$ or $s(x)$ – employee's salary
- $c(x)$ – cost born by the employee
- u^0 – employee's aspiration level: $s(f(x))-c(x) \geq u^0$ (*participation constraint*)

(D.5) The principal-agent problem (cont.)

The incentive compatibility constraint is:

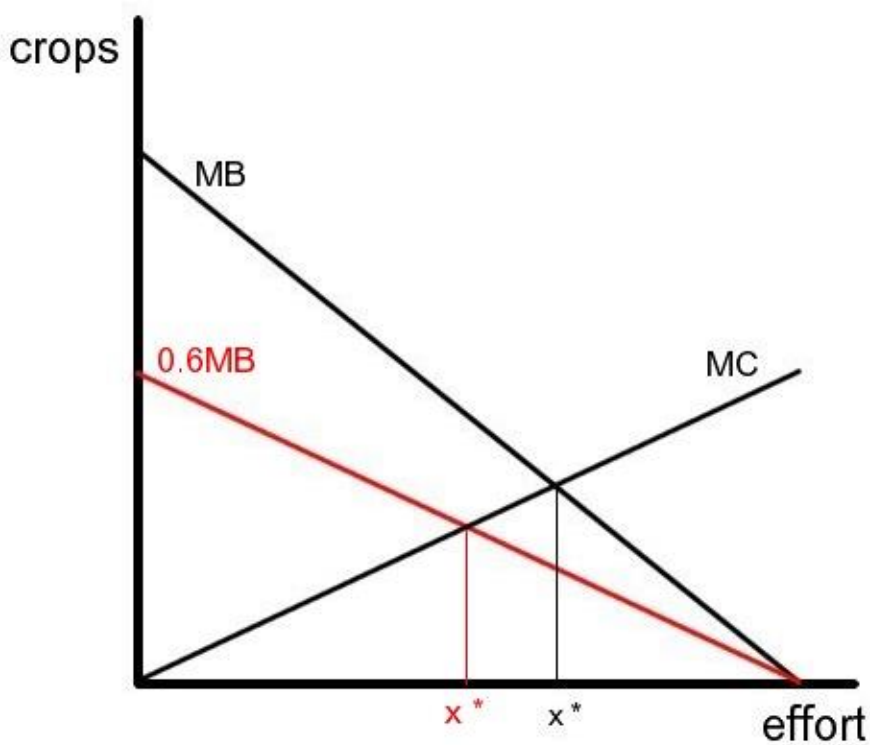
$$s(f(x^*))-c(x^*) \geq s(f(x))-c(x) \text{ for all } x, \text{ where}$$

- x^* maximizes $f(x)-s(f(x))$, i.e. $f(x)-c(x)-u^0$ that is (by conventional assumptions):
- $MP(x^*) = MC(x^*)$

Examples (of incentive-compatible contracts)

- Rental payment, R: $s(f(x)) = f(x)-R$, where R is derived from the participation constraint: $f(x^*)-c(x^*)-R = u^0$
- Hourly (daily) salary rate *w* plus flat rate *K* so that: $s(x) = wx+K$, where $w=MP(x^*)$, and K is derived from the participation constraint: $wx+K-c(x) = u^0$
- Threshold condition (*take-it-or-leave-it*), payment B, if $x \geq x^*$ (alternatively: if $y \geq f(x^*)$): the amount B is calculated from the participation constraint: $B-c(x^*) = u^0$ (assuming that $B \leq MP(x^*)$)

Example (of a not incentive-compatible contract)



(D.6) Moral hazard (hidden actions)

Specifications of the contract from D.5 are as follows. Let $\pi \in \Re$ be the (observable) profit from a project, and e – the level of effort spent by the agent (it can be non-observable by the principal). It is assumed that $e \in E = \{e_L, e_H\}$. The profit π is a realization of a random variable whose values are in the interval $\Pi = [\pi_{min}, \pi_{max}]$ with a conditional density function $f(\pi|e) > 0$ for all $\pi \in \Pi$ and $e \in E$ (i.e. no realization of π determines a specific level of effort e uniquely). However $\int_{\Pi} \pi f(\pi|e_H) d\pi > \int_{\Pi} \pi f(\pi|e_L) d\pi$.

Moral hazard (cont.)

The agent maximizes his/her utility function given by the formula $u(w,e)=v(w)-g(e)$, where $w(\pi)$ is the wage resulting from the contract, $v'>0$, $v''\leq 0$ and $g(e_H)>g(e_L)$. The agent accepts the contract only if its terms let him/her enjoy the reservation utility u^0. The principal maximizes the project's profit net of the wage paid to the agent: $\int_\Pi(\pi-w(\pi))f(\pi|e)d\pi$ (the net profit).

(T.8) If the effort e is observable by the principal then the contract from D.6 must determine that the agent spends the effort e^* which solves the optimization problem

$$\mathrm{Max}_e\{\int_\Pi \pi f(\pi|e)d\pi - v^{-1}(u^0+g(e))\}$$

and receives a fixed wage $w^*=v^{-1}(u^0+g(e^*))$. If $v''<0$, then this is the only optimal contract.

(T.9) By the assumptions of T.8, if $v(w)=w$, then the optimum amount of the effort e^* solves the problem $\mathrm{Max}_e\{\int_\Pi \pi f(\pi|e)d\pi-(u^0+g(e))\}$

(T.10) In model D.6, if the agent's effort is not observable, and the agent is neutral with respect to risk, then an optimum contract triggers the same level of effort e^*, wage w^* and the net profit for the principal as in T.9. The wage is given by the formula $w(\pi)=\pi-\alpha^*$, where $\alpha^*=\int_\Pi \pi f(\pi|e^*)d\pi-(u^0+g(e^*))$.

(T.11) In model D.6, if the agent's effort is not observable, and the agent is risk averse, then the optimum contract solves the problem $\text{Min}_{w(\pi)}\{\int_\Pi w(\pi)f(\pi|e^*)d\pi\text{: (i)} \wedge \text{(ii)}\}$, where the conditions (i) and (ii) are given by:

(i) $\int_\Pi v(w(\pi))f(\pi|e^*)d\pi - g(e^*) \geq u^0$

(ii) e^* solves the problem $\text{Max}_e\{\int_\Pi v(w(\pi))f(\pi|e)d\pi - g(e)\}$

Condition (ii) is called the incentive compatibility constraint

(T.12) By the assumptions of T.11 the expected wage for the effort e_H is higher than in T.8. In contrast, the expected wage for the effort e_L is fixed and equal to the amount from T.8. Hence, if the optimum effort $e^*=e_H$, then the non-observability of the effort causes welfare loss.

(T.13) In model D.6 the non-observability of effort leads to its lowering by the agent. A generalization of the model by assuming that E contains more two points implies that the effort is not optimal, but it does not allow to determine that it is lower than the optimal one.

(D.7) Auctions

Let us assume that the principal has an object for sale. It can be purchased by either of two agents which value the object v_1 and v_2, respectively. Each valuation is known only to the agent who has it. Nevertheless the principal and both agents know the distributions of both values treated as random variables: θ_1 for the first agent and θ_2 – for the second. Agents are supposed to indicate their valuations as sealed bids s_1 and s_2 (which do not have to coincide with v_1 and v_2). The object goes to the agent who offered the higher bid. If both bidders offer the same bid then the winner is selected randomly with probabilities 1/2 and 1/2.

Auctions vary with respect to what is the amount paid by the winner.

<u>(D.8) First-Price Auction</u>

The winner pays the amount from his/her own bid

<u>(D.9) Second-Price Auction</u>

The winner pays the amount from the loser's bid

<u>(T.14)</u>

Let us assume that θ_1 and θ_2 have the same distributions. In both variants of auctions the truthful revelation of preferences makes a Nash equilibrium in the 'bidding game' (in this game the payoff of the loser is 0, and the payoff of the winner *i* is v_i-s_i in the first-price auction or v_i-s_{-i} in the second-price auction).

Proof:

Let us consider the first-price auction first. The winner pays s_i, and gets the net benefit v_i-s_i. If $s_i>v_i$ then the net benefit would be negative and the winner would have a motivation to be the loser rather than the winner. If $s_i<v_i$ then the winner risks losing the auction and hence – s_i makes the preferred bid. Now let us consider the second-price auction. The motivation for the winner not to overstate the bid is even higher than before. A likely motivation to understate in order to pay less disappears, because the winner pays the bid of loser.

<u>(T.15)</u> Revenue Equivalence Theorem

In both types of auctions the expected revenue of the principal (who organizes an auction) is the same.

<u>Proof</u>:

The seller receives s_i of the winner (in the first-price auction) or s_i of the loser (in the second-price auction). By T.14, bidders indicate their true valuations (i.e. $s_i=v_i$, for i=1,2). As v_i are sampled from the same distributions, then $\mathit{E}v_1=\mathit{E}v_2$.

(T.16) Inefficiency Theorem

Let us assume that θ_1 and θ_2 have strictly positive distributions on intervals that overlap at least partially. Neither of the auctions can – in general – provide an outcome that satisfies three conditions:

- individual rationality,
- incentive compatibility, and
- budget balance (any payoffs for the principal need to be financed from agents' payments)

(difficult to prove)

Non-technical summary

A. In the lecture the asymmetry of information was explained by referring to the example of employment contracts. Similar problems arise in other transactions such as selling certain goods (e.g. used cars) or services (e.g. insurance policies).

B. Asymmetric information affects the stage of drafting a contract (it may lead to adverse selection) and the stage of carrying out a contract (it may lead then to moral hazard). In both cases market equilibrium may be not Pareto-optimal.

C. The adverse selection process can be modelled as a two-stage game: first employers offer wages; then potential employees decide whether to be hired or not. Making the information credible through "signalling" overcomes the negative selection, but it requires to bear a cost which lowers the welfare.

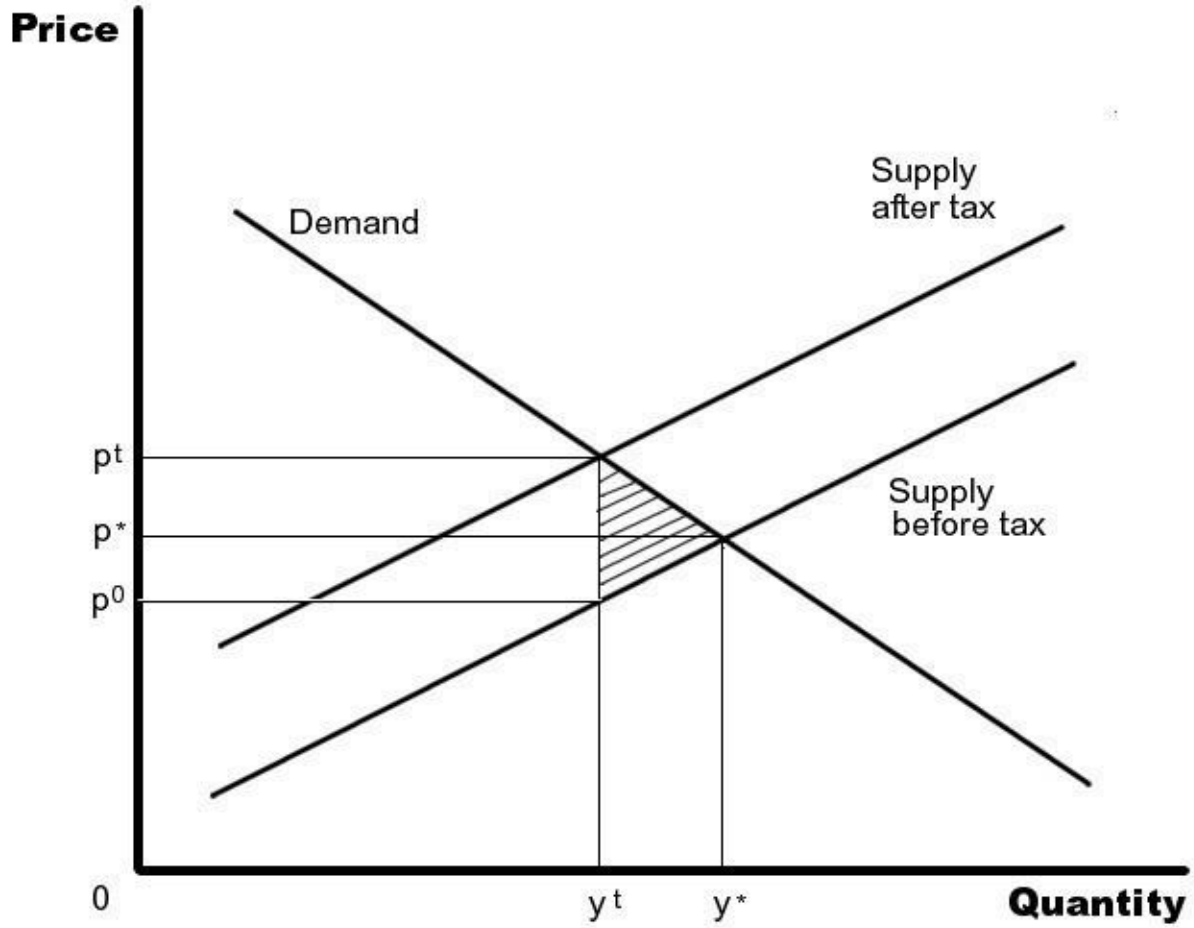
D. Elimination of the moral hazard may be helped by signing an incentive compatible contract. However, if the agent is risk-averse, the contract lowers the welfare with respect to the situation where his/her behaviour is observable.

E. Auctions provide an example of how to solicit information that is unknown to the principal who plans to sell an object. Ideas similar to those analysed in the class apply to buying an object or dealing with greater number of agents.

9. General equilibrium

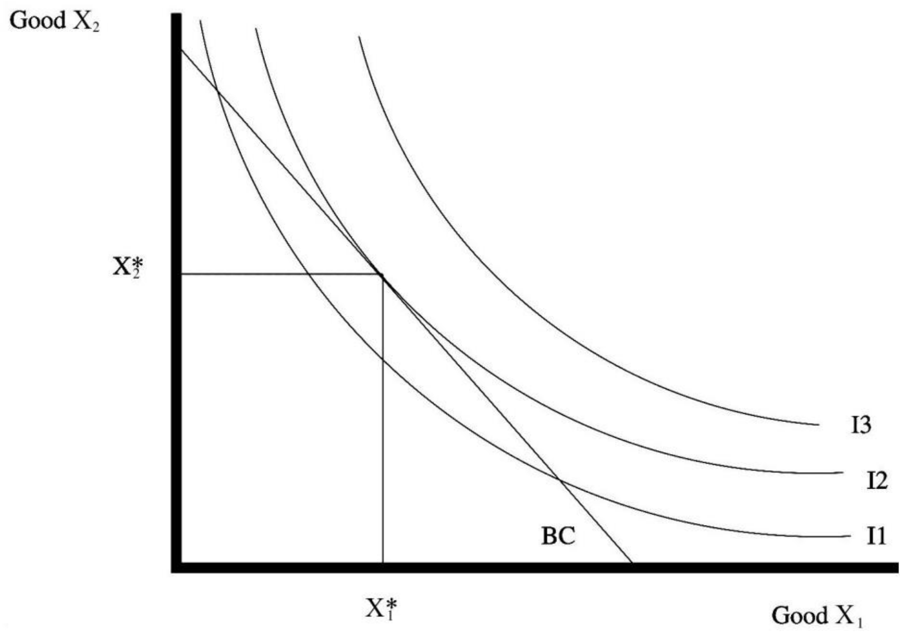
- I – the number of consumers
- L – the number of markets (products); one of the commodities can be labour – i.e. a resource owned by every consumer
- J – the number of firms

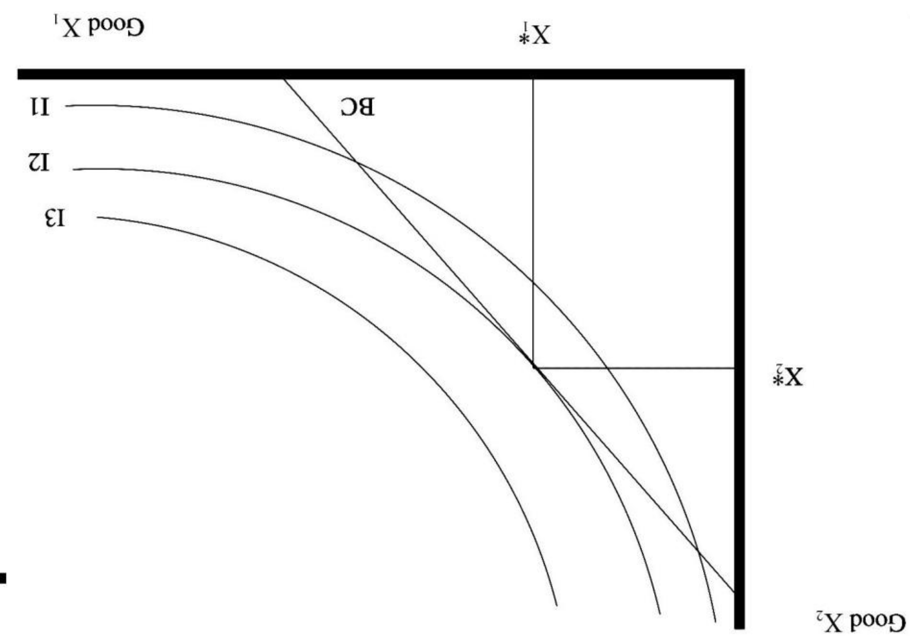
- Numbering of consumers: i=1,...,I
- Numbering of markets (products): ℓ=1,...,L
- Numbering of firms: r=1,...,J



Price
Demand
Supply
after tax
Supply
before tax
p^t
p^*
p^0
0
y^t
y^*
Quantity

The idea of the Edgeworth box





(D.1) Pure exchange in the Edgeworth box

The economy has two consumers, i=1,2 and two goods, ℓ=1,2 (one of the commodities can be labour – a resource owned by any consumer)

Initial allocation (endowment) of the consumer *i*:

$$\boldsymbol{\omega}^i=(\omega_{i1},\omega_{i2})$$

Gross demand (final allocation) of the consumer *i*:

$$\mathbf{x}^i=(x_{i1},x_{i2})$$

- Excess demand of the consumer *i*:

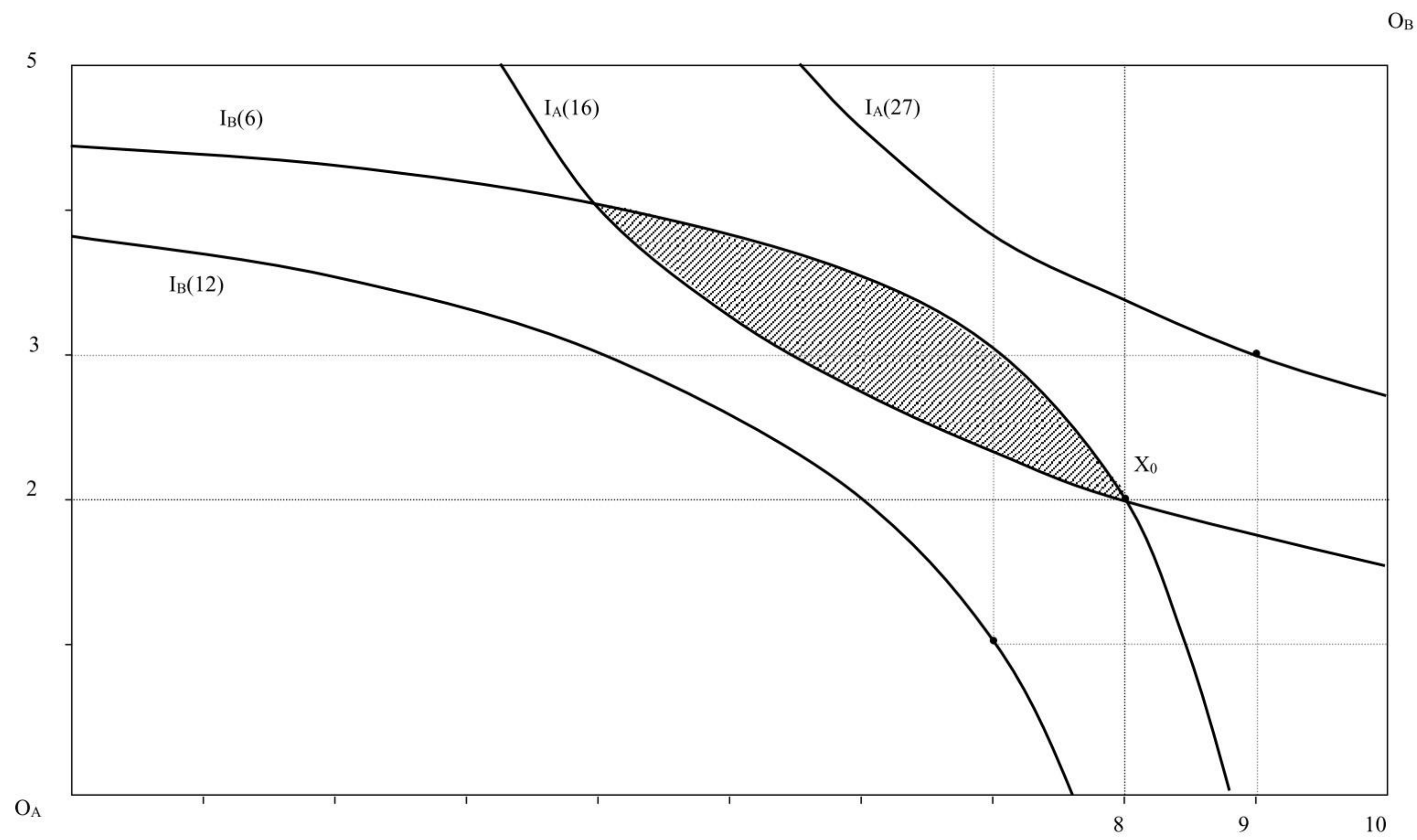
  $\mathbf{x}^i\text{-}\boldsymbol{\omega}^i = (x_{i1}-\omega_{i1}, x_{i2}-\omega_{i2})$

- Feasible allocation satisfies the following system of equations:
  - $x_1 = x_{11}+x_{21} = \omega_{11}+\omega_{21} = \omega_1$,
  - $x_2 = x_{12}+x_{22} = \omega_{12}+\omega_{22} = \omega_2$

Edgeworth box – an analysis of feasible allocations: a superposition of two systems of coordinates used for studying consumer choices; the width of the rectangle $\omega_1 = \omega_{11}+\omega_{21}$, its height $\omega_2 = \omega_{12}+\omega_{22}$; the second system is rotated by 180^o.

Both consumers are pricetakers (they reveal their respective demand and a supply in response to prices announced by an impartial auctioneer).

PhD-9-7



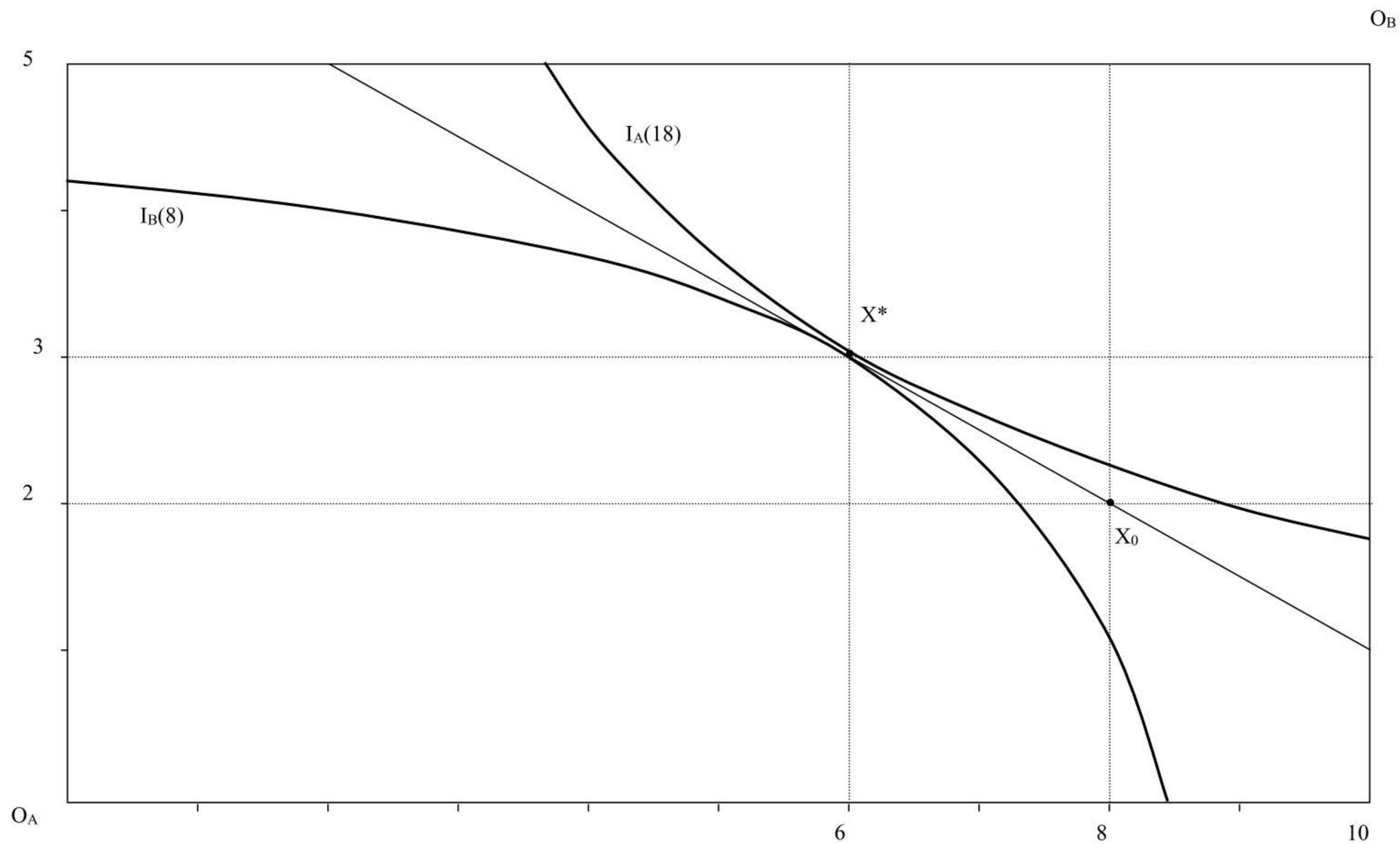
In the picture it is assumed that indifference curves $I_A(\alpha)$ of the agent A are given by formulae $x_{2A}=\alpha/x_{1A}$, and indifference curves $I_B(\beta)$ of the agent B – by $x_{2B}=\beta/x_{1B}$ ($\alpha,\beta>0$ – parameters). It is assumed additionally that the total amount of the good 1 (owned by both agents) is 10, and the total number of units of the good 2 is 5.

As a matter of example, it can be assumed further that the 10 units of the good 1 were distributed between agents A and B as 8:2, and the 5 units of the good 2 – as 3:2. These distributions (allocations) are illustrated by the point X_0 in the picture. Let us observe that by the construction of the Edgeworth box this single point corresponds to a quadruple of numbers 8,2,2,3 which characterize the situation. The point X_0 is located at the A's indifference curve given by the formula $x_{2A}=16/x_{1A}$ ($\alpha=16$) and at the B's indifference curve $x_{2B}=6/x_{1B}$ ($\beta=6$).

Agent A would prefer to be at a higher indifference curve, say, in the point (9,3), i.e. at the curve $x_{2A}=27/x_{1A}$ ($\alpha=27$). At the same time, agent B would also prefer to have more of everything, say, (3,4), i.e. to be located at the indifference curve $x_{2B}=12/x_{1B}$ ($\beta=12$). These aspirations cannot be satisfied jointly, since agent A would have to be given by B one unit of both goods, and – *vice versa* – agent B would have to be given by A one unit of both goods, while the transfer of a unit from one to another implies that the former has less rather than more.

However, the analysis of the picture suggests that a simultaneous improvement for A and B is possible, since there is some area located between indifference curves $I_A(16)$ and $I_B(6)$, where each of the agents finds himself (or herself) on an indifference curve higher than the original one. The analysis suggests also that an improvement is always possible whenever indifference curves are not tangent to each other, but they intersect.

Hence an improvement is impossible only when indifference curves the allocation is included in are tangent to each other, i.e. agent A's improvement is only possible when agent B gets worse off and *vice versa*. Finding themselves in X_0, the agents have an opportunity to improve their situation simultaneously. This requires that A exchanges some of the endowment in 1 for the good number 2 acquired from B. In this way the agents will move towards interior of the shaded area.



O_B
5
3
2
O_A
6
8
10
I_A(18)
I_B(8)
X*
X_0

It is easy to note that if the new allocation X^* was to be reached as a result of a voluntary exchange transaction, it must satisfy the condition that by selling a certain number of units of the good number 1 the agent A received exactly the same amount of money which is necessary to purchase additional number of units of the good # 2, and *vice versa*, by selling some units of the good # 2, the agent B received exactly what was necessary in order to purchase additional units of the good # 1.

Moreover, if the transaction is to be a final one, i.e. its parties exhaust all possibilities of improving their situation (with respect to the initial endowment), then their indifference curves must be tangent to each other in X^*. In addition, the tangent line must coincide with the straight line including all the points that the agents may move into from X_0 and applying non-negative prices p_1 i p_2 adopted in order to make a transaction.

<u>(T.1)</u> In the pure exchange model the wealth of the *i*th consumer is endogenously given by prices $\mathbf{p}=(p_1,p_2)^T$: $w_i = \mathbf{p}^T\cdot\boldsymbol{\omega}^i = p_1\omega_{i1}+p_2\omega_{i2}$; consequently the budget set is $B_i(\mathbf{p}) = \{\mathbf{x}^i\in\Re_+^2\text{: } \mathbf{p}^T\cdot\mathbf{x}^i\leq\mathbf{p}^T\cdot\boldsymbol{\omega}^i\}$

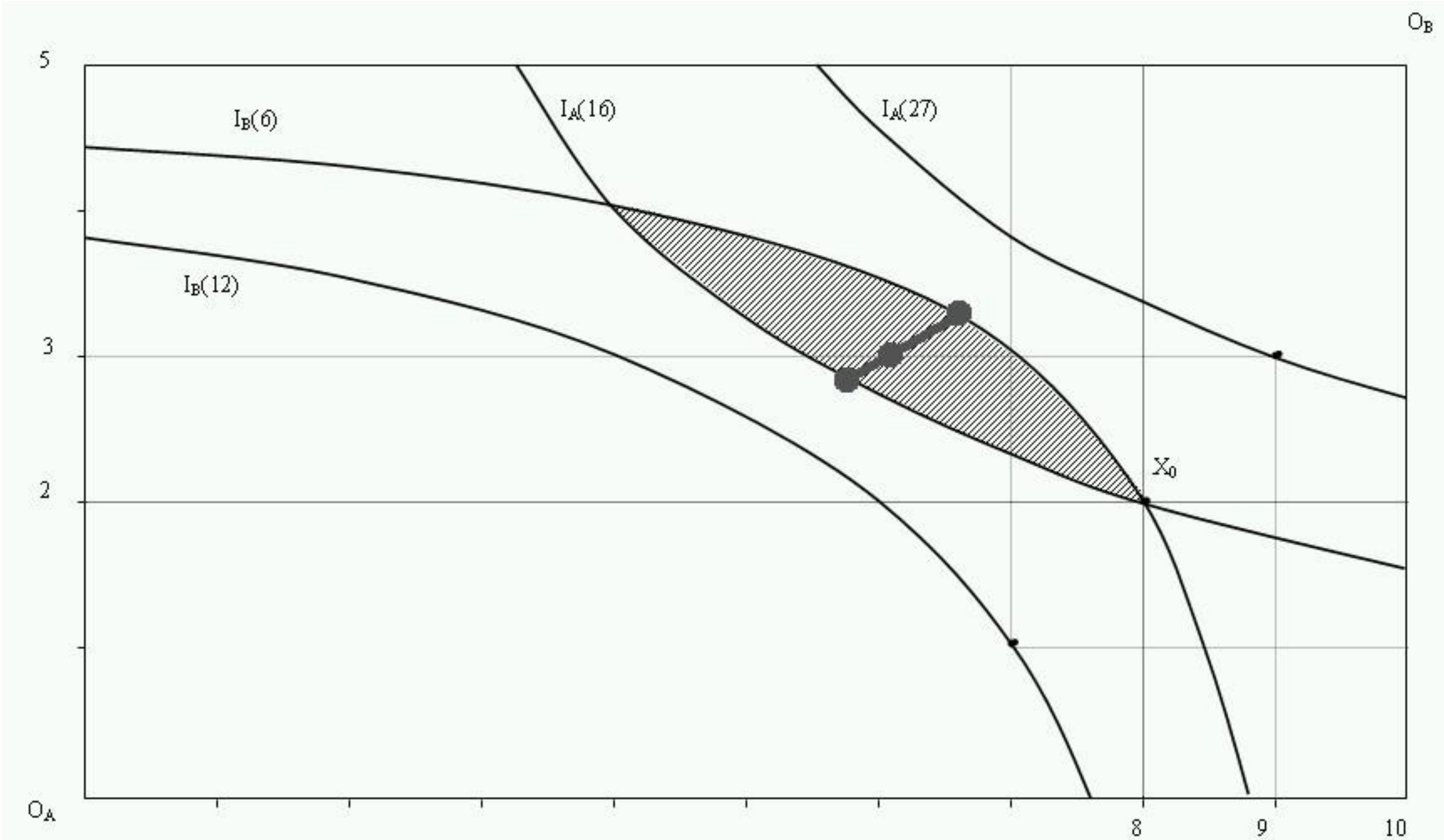
<u>(D.2) Walrasian (competitive) equilibrium in an Edgeworth box</u>

Any pair $(\mathbf{p}^*,\mathbf{x}^*)$, where $\mathbf{x}^*=(\mathbf{x}^{1*},\mathbf{x}^{2*})$, which satisfies:

$\forall i=1,2\ \forall \mathbf{x}'^i\in B_i(\mathbf{p}^*)\ [\mathbf{x}^{i*} \geq_i \mathbf{x}'^i]$

(D.3) Pareto optimum in an Edgeworth box

An allocation $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2)$ is Pareto optimal, if there is no other allocation $\mathbf{x}'$ in this box such that $\mathbf{x}'^i \geq_i \mathbf{x}^i$ for i=1,2 and $\mathbf{x}'^i >_i \mathbf{x}^i$ for some i. The set of all Pareto optima in an Edgeworth box is called the Pareto set. A part of the Pareto set including allocations preferred to the initial allocation $(\boldsymbol{\omega}^1,\boldsymbol{\omega}^2)$ is called a contract curve.



Contract curve

Welfare economics theorems

1. WE $\Rightarrow$ PO
2. PO $\Rightarrow$ WE

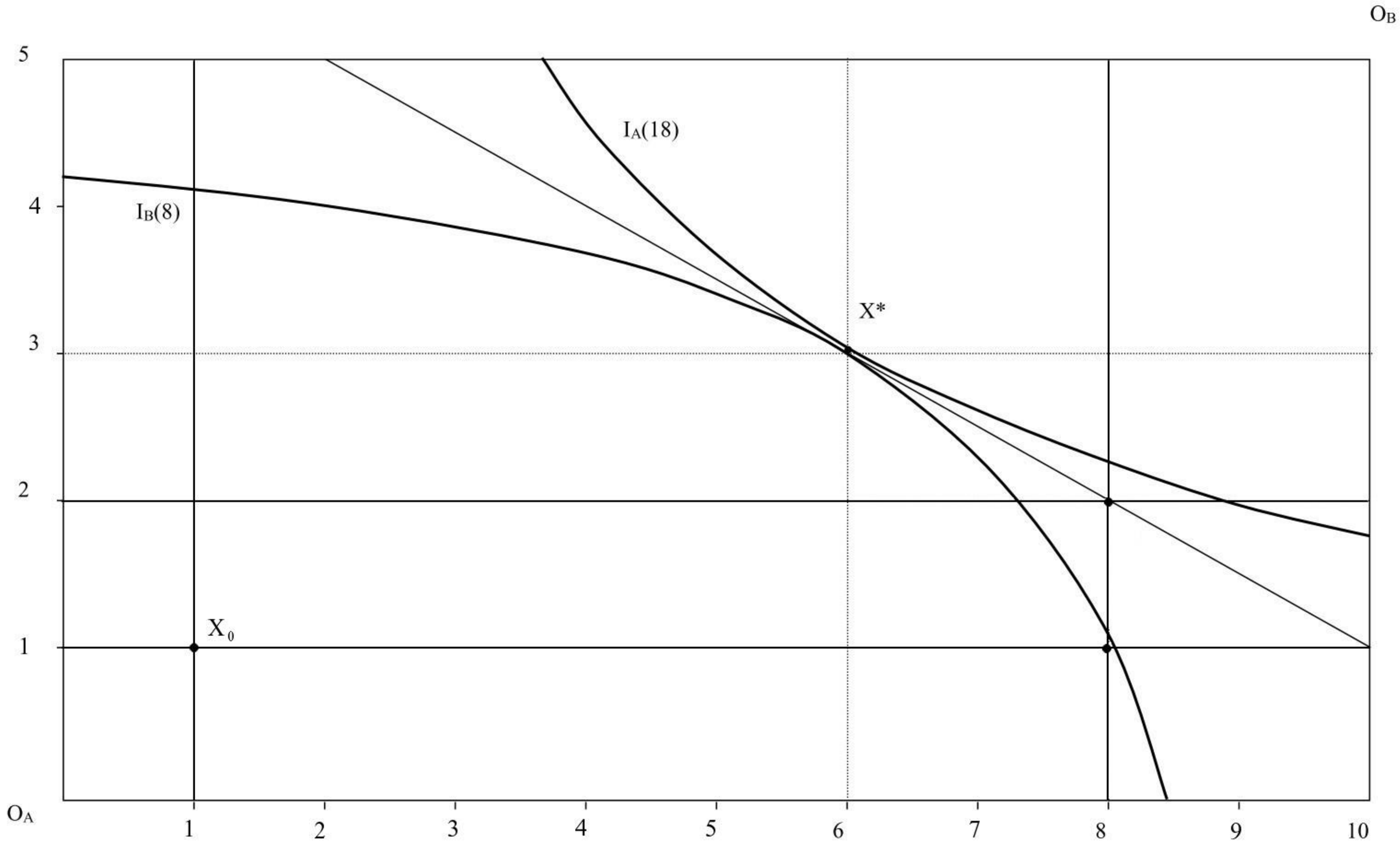
<u>(T.2) The first welfare economics theorem in an Edgeworth box</u>

Any Walrasian equilibrium is a Pareto optimum

<u>(D.4)</u> In an Edgeworth box an allocation $\mathbf{x}^*$ can be achieved as an equilibrium with transfers, if there is a price system $\mathbf{p}^*$ and wealth transfers T_1 and T_2 satisfying $T_1+T_2=0$ such that for every consumer *i* holds: $\mathbf{x}^{i*} \succeq_i \mathbf{x}'^{i}$ for all $\mathbf{x}'^{i} \in \Re_+^2$ which satisfy $\mathbf{p}^{*T} \cdot \mathbf{x}'^{i} \leq \mathbf{p}^{*T} \cdot \boldsymbol{\omega}^i + T_i$

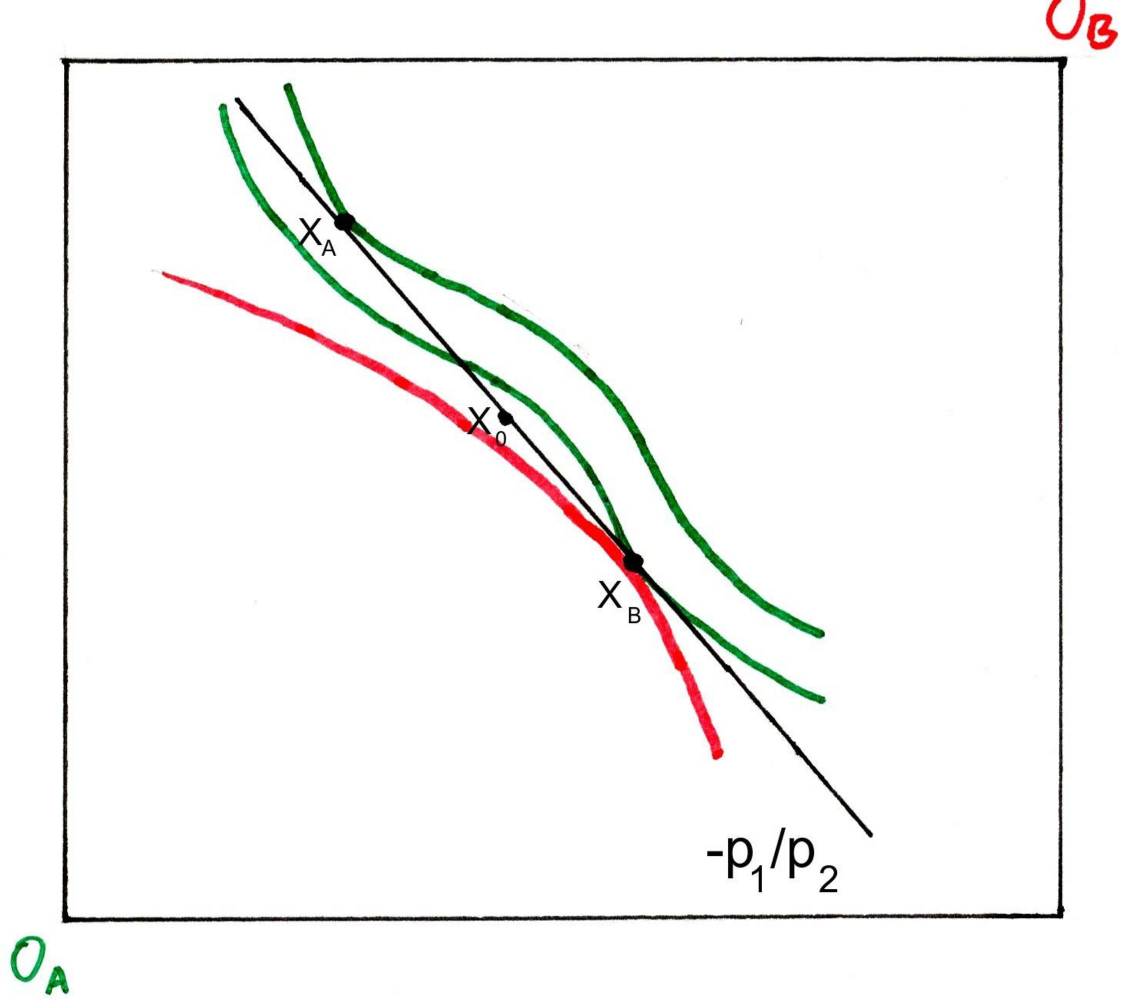
<u>(T.3) The second welfare economics theorem in an Edgeworth box</u>

If preferences are convex, then any Pareto optimum can be achieved as an equilibrium with transfers



O_B
5
4
3
2
1
O_A
1
2
3
4
5
6
7
8
9
10
I_A(18)
I_B(8)
X*
X_0

Non-convex indifference curves (of A)

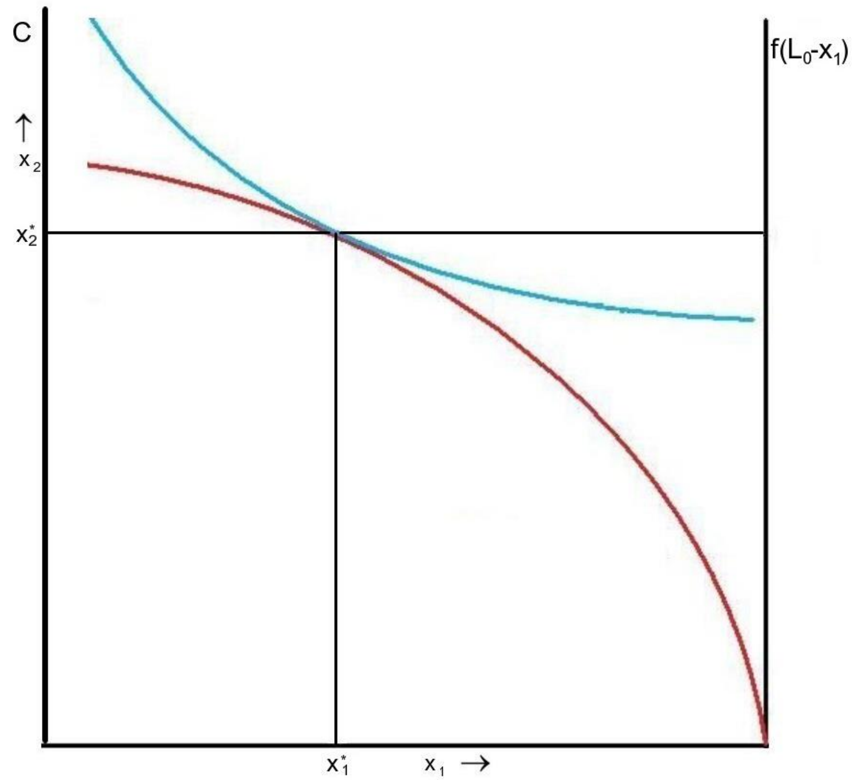


(D.5) Model of an economy with one consumer and one producer (The 'Robinson Crusoe Economy')

There is one consumer, one producer (firm), and two goods: leisure x_1 and a (material) consumption good x_2. The consumer has continuous, convex, and strictly monotonic preferences $\geq$. He/she has an endowment L^0 of leisure and a zero endowment of the consumption good. The firm uses labour in order to produce the consumption good according to a monotonically increasing and strictly concave production function f(z). The price of the consumption good is p, and the price of labour is w.

Both consumer and producer are price-takers. The firm maximizes its profit, $\mathrm{Max}_{z\geq0}\{pf(z)–wz\}$. As a result of the maximization problem, a (secondary) demand for labour is determined $z(p,w)$, the supply of the consumption good $q(p,w)$ and profit $\pi(p,w)$. The consumer maximizes the utility subject to a budget constraint given by selling labour and enjoying the profit of the firm he/she is the sole owner of: $\mathrm{Max}_{x1,x2\geq0}\{u(x_1,x_2): px_2\leq w(L^0–x_1)+\pi(p,w)\}$. The solution defines the consumer's (primary) demand $x_1(p,w),x_2(p,w)$. The Walrasian (competitive) equilibrium is given by the system of two equations:

$$x_2(p^*,w^*)=q(p^*,w^*) \text{ and } z(p^*,w^*)=L^0–x_1(p^*,w^*)$$


C
f(L0-x1)
x2
x2*
x1*
x1

(T.4) In the 'Robinson Crusoe economy' (D.5) the competitive equilibrium is Pareto optimal

Proof

The theorem is an equivalence, i.e. it establishes the equivalence WE $\Leftrightarrow$ PO. Consequently the proof will address two implications: (1) $\Rightarrow$ and (2) $\Leftarrow$.

Proof (cont.)

(1) We will prove that if $x_2(p^*,w^*)=q(p^*,w^*)$ and $z(p^*,w^*)=L^0-x_1(p^*,w^*)$ then $x_1^*=x_1(p^*,w^*)$ and $x_2^*= x_2(p^*,w^*)$ are a Pareto optimum. In the model with one consumer the Pareto optimum means maximizing the welfare of this consumer (there are no other consumers). (x_1^*,x_2^*) maximizes the consumer's utility given the budget determined by two sources: selling labour and 100% share in the firms profit. Selling labour is utility-neutral since it subtracts from the value of leisure which – by definition – is valued by w. Thus if profit is maximized then also the budget is maximized.

(2) If (x_1^*,x_2^*) is a Pareto optimum then it will be proved that there exist prices p^* and w^* such that (x_1^*,x_2^*) are solutions to optimization problems of the consumer and the producer. Let us assume that (x_1^*,x_2^*) solves the problem $\mathrm{Max}_{x1,x2\geq 0}\{u(x_1,x_2)$: $x_2\leq f(L^0-x_1)\}$, i.e. it is a Pareto optimum. As preferences are strictly monotonic, the constraint has to be satisfied as equality: $x_2^*=f(L^0-x_1^*)$. If f is differentiable at $L^0-x_1^*$ then $df(L^0-x_1^*)/dx_1$ is the proportion $–w^*/p^*$ we were looking for. Indeed, we will prove that, if faced with such prices, the producer maximizes the profit buying $L^0-x_1^*$ of labour, and the consumer maximizes utility by selling $L^0-x_1^*$ of labour. Let us check the producer first. The straight line going through $(x_2^*,f(L^0-x_1^*))$ with the coefficient $-w^*/p^*$ is the highest iso-profit curve. Thus the profit is highest there indeed.

Now the consumer. In the initial allocation (before the labour is sold to the producer) the consumer has L units of leisure and $\pi(p^*,w^*)$ units of money (as the only owner of the firm). The money lets buy $\pi(p^*,w^*)/p^*$ units of the consumption good. But the straight line going through this point and with the coefficient $-w^*/p^*$ is also going through the point (x_1^*,x_2^*) which completes the proof since this is where the consumer maximizes utility.

If f is not differentiable in L^0-x_1^* then – being monotonic and strictly concave – it has both one-sided derivatives, and its left derivative is larger than the right derivative. The proportion of equilibrium prices $-w^*/p^*$ is any number between the right and the left derivative. The rest of the proof is as in the differentiable case.

<u>(D.6) 2×2 Production model</u>

The economy consists of firms # 1 and # 2 which produce consumption goods q_1 and q_2, respectively, using two primary factors $\mathbf{z}_1=(z_{11},z_{21})^T\geq\mathbf{0}$, $\mathbf{z}_2=(z_{21},z_{22})^T\geq\mathbf{0}$. Production of each firm is determined by a convex, strictly increasing differentiable production function $f_j(\mathbf{z}_j)$. The initial endowments of factors $z^0{}_1,z^0{}_2>0$ are owned by consumers who do not derive any utility from them (directly).

The consumption good prices $\mathbf{p}=(p_1,p_2)^T$ are given exogenously, and factor prices $\mathbf{w}=(w_1,w_2)^T$ are to be determined endogenously by the model (small open economy model). The firms determine production levels by solving the following profit maximizing problems: $\mathrm{Max}_{z_j\geq 0}\{p_j f_j(\mathbf{z}_j)-\mathbf{w}^T\cdot\mathbf{z}_j\}$. Their solutions determine the (secondary) demands $z_j(\mathbf{p},\mathbf{w})\subset\Re_+^{\,2}$. It is assumed that equilibrium in the factor market will be an internal one, i.e. $\mathbf{w}^*=(w_1^{\,*},w_2^{\,*})^T\gg\mathbf{0}$. It satisfies the following conditions: $(z_{1j}^{\,*},z_{2j}^{\,*})\in z_j(\mathbf{p},\mathbf{w})$ for j=1,2 and $z_{\ell 1}^{\,*}+z_{\ell 2}^{\,*}=z^0_{\ell}$ for ℓ=1,2.

2×2 model example

- Firm 1 – farm (food producer)
- Firm 2 – hospital (medical care producer)

- Input 1 – land
- Input 2 – labour

<u>(T.5)</u> First order conditions (both necessary and sufficient) for the 2×2 production model are:

$p_j \partial f_j(\mathbf{z}_j^*)/\partial z_{\ell j} = w_\ell^*$ for $j=1,2$, $\ell=1,2$

$z_{\ell 1}^* + z_{\ell 2}^* = z^0_\ell$ for $\ell=1,2$

<u>(T.6)</u> First order conditions (both necessary and sufficient) for the 2×2 production model are:

$p_j = \partial c_j(\mathbf{w}^*, q_j^*)/\partial q_j$ for $j=1,2$

$\partial c_1(\mathbf{w}^*, q_1^*)/\partial w_\ell + \partial c_2(\mathbf{w}^*, q_2^*)/\partial w_\ell = z_\ell^0$ for $\ell=1,2$, where

c_1, c_2 – cost functions defined as in D.8 from lecture 3.

(T.7) First welfare economics theorem in the 2×2 production model

Equilibrium in factor market (D.6) solves the following optimization problem

$$\mathrm{Max}_{\mathbf{z}1,\mathbf{z}2\geq0}\{p_1f_1(\mathbf{z}_1)+p_2f_2(\mathbf{z}_2)\colon z_{11}^{*}+z_{12}^{*}=z^0_1,\ z_{21}^{*}+z_{22}^{*}=z^0_2\}$$

(D.7) 2×2 production model with constant scale effects

It is assumed that functions f_1 i f_2 in D.6 economy are homogeneous of degree 1. Let $c_j(\mathbf{w})=c_j(\mathbf{w},1)$ and let $a_j(\mathbf{w})=(a_{1j}(\mathbf{w}),a_{2j}(\mathbf{w}))$ for $j=1,2$ be factor inputs such that these costs are minimized (it is assumed that they are unique).

(T.8) In the 2×2 production model with constant scale effects $D_w c_j(\mathbf{w}) = a_j(\mathbf{w})$ for $j=1,2$

(D.8) Production of the good # 1 uses the factor # 1 relatively more intensely than the production of the good # 2, if for all **w**, $a_{11}(\mathbf{w})/a_{21}(\mathbf{w}) > a_{12}(\mathbf{w})/a_{22}(\mathbf{w})$

(T.9) The Stolper-Samuelson theorem

In the 2×2 production model with constant scale effects let D.8 be satisfied. Then, if the price p_1 increases, the price of the factor # 1 (w_1) increases too, and the price of factor # 2 (w_2) decreases; it is assumed that both the original prices and the new prices imply internal solutions.

(T.10) The Rybczynski theorem

In the 2×2 production model with constant scale effects let D.8 be satisfied. Then, if the endowment of the factor # 1 (z^0_1) increases, the production of the good # 1 (q_1) increases too, and the production of the good # 2 (q_2) decreases; it is assumed that both the original prices and the new prices imply internal solutions.

(D.9) General equilibrium model

- Consumers i=1,...,I have consumption sets $X_i \subset \Re^L$ and rational (complete and transitive) preference relations $\succeq_i$ defined over X_i.
- Firms j=1,...,J have production sets $Y_j \subset \Re^L$, and Y_j are non-empty and closed.
- Endowments of goods ℓ=1,...,L are $\boldsymbol{\omega}=(\omega_1,...,\omega_L)^T \in \Re^L$.
- Consumers and firms are price-takers.

(D.10) A D.9 economy is called a pure exchange economy if all the production sets $Y_j=-\mathfrak{R}_+^L$ for $j=1,...,J$.

(T.11) A pure exchange economy in Edgeworth box from D.1 is a pure exchange economy of D.10 for $L=2$, $I=2$, $X_1=X_2=\mathfrak{R}_+^2$, $J=1$, $Y_1=-\mathfrak{R}_+^2$

<u>(D.11)</u> In a D.9 model an allocation ($\mathbf{x}$,$\mathbf{y}$) is feasible if $x_{\ell 1}+...+x_{\ell I} = \omega_\ell+y_{\ell 1}+...+y_{\ell J}$ for all ℓ=1,...,L. The set of feasible allocations is

$$A=\{(\mathbf{x},\mathbf{y})\in X_1\times...\times X_I\times Y_1\times...\times Y_J\text{: } x_{\ell 1}+...+x_{\ell I} = \omega_\ell+y_{\ell 1}+...+y_{\ell J} \text{ for all } \ell=1,...,L\}\subset\Re^{L(I+J)}$$

<u>(D.12)</u> A feasible allocation ($\mathbf{x}$,$\mathbf{y}$) is a Pareto optimum if there is no allocation ($\mathbf{x}'$,$\mathbf{y}'$)$\in$A, such that

$$\forall i=1,...,I\ [x'^i \geq_i x^i] \wedge \exists i=1,...,I\ [x'^i >_i x^i]$$

(D.13) An economy with private ownership is any D.9 model where every consumer *i* has initial endowments $\boldsymbol{\omega}^i=(\omega_1^i,\ldots,\omega_L^i)^T\in\Re^L$ and shares $\theta_{ij}\in[0,1]$ in firms' profits for j=1,...J, with $\boldsymbol{\omega}^1+\ldots+\boldsymbol{\omega}^I = \boldsymbol{\omega}$ and $\theta_{1j}+\ldots+\theta_{Ij} = 1$ for any j=1,...,J.

(D.14) Competitive (Walrasian) equilibrium in an economy with private ownership

An allocation $(\mathbf{x}^*,\mathbf{y}^*)$ and prices $\mathbf{p}=(p_1,\ldots,p_L)^T$ are competitive equilibrium if:

i. for every $j=1,\ldots,J$ $\mathbf{y}_j^*$ maximizes the profit in Y_j, i.e. $\mathbf{p}^T\cdot\mathbf{y}_j\leq\mathbf{p}^T\cdot\mathbf{y}_j^*$ for all $\mathbf{y}_j\in Y_j$

ii. for every $i=1,\ldots,I$ $\mathbf{x}^{i*}$ is maximal with respect to $\succeq_i$ in a respective budget set, i.e. in
$$\{\mathbf{x}^i\in X_i\colon \mathbf{p}^T\cdot\mathbf{x}^i \leq \mathbf{p}^T\cdot\boldsymbol{\omega}^i+\theta_{i1}\mathbf{p}^T\cdot\mathbf{y}_1^*+\ldots+\theta_{iJ}\mathbf{p}^T\cdot\mathbf{y}_J^*\}$$

iii. $\mathbf{x}^{1*}+\ldots+\mathbf{x}^{I*} = \boldsymbol{\omega}+\mathbf{y}_1^*+\ldots+\mathbf{y}_J^*$

(D.15) Price equilibrium in an economy with transfers

An allocation $(\mathbf{x}^*,\mathbf{y}^*)$ and prices $\mathbf{p}=(p_1,...,p_L)^T$ are a price equilibrium in an economy with transfers, if there is a distribution of wealth $w_1,...,w_I$ satisfying $w_1+...+w_I = \mathbf{p}^T\cdot\boldsymbol{\omega}+\mathbf{p}\cdot(\mathbf{y}_1^*+...+\mathbf{y}_J^*)$ such that:

i. for every j=1,...,J $\mathbf{y}_j^*$ maximizes the profit in Y_j, i.e. $\mathbf{p}^T\cdot\mathbf{y}_j\leq\mathbf{p}^T\cdot\mathbf{y}_j^*$ for every $\mathbf{y}_j\in Y_j$

ii. for every i=1,...,I $\mathbf{x}^{i*}$ is maximal with respect to $\geq_i$ in a respective budget set, i.e. in

$$\{\mathbf{x}^i\in X_i\colon \mathbf{p}^T\cdot\mathbf{x}^i \leq w_i\}$$

iii. $\mathbf{x}^{1*}+...+\mathbf{x}^{I*} = \boldsymbol{\omega}+\mathbf{y}_1^*+...+\mathbf{y}_J^*$

(T.12) Any competitive equilibrium (D.14) is a price equilibrium in an economy with transfers (D.15)

(T.13) The first fundamental welfare economics theorem

If preferences are locally non-satiated and $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{p})$ is a price equilibrium in an economy with transfers then $(\mathbf{x}^*, \mathbf{y}^*)$ is a Pareto-optimum allocation. In particular, any Walrasian equilibrium is a Pareto optimum.

(D.16) Quasi-equilibrium in an economy with transfers

An allocation $(\mathbf{x}^*,\mathbf{y}^*)$ and prices $\mathbf{p}=(p_1,...,p_L)^T$ are a quasi-equilibrium in an economy with transfers if there is a distribution of wealth $w_1,...,w_I$ satisfying $w_1+...+w_I = \mathbf{p}^T\cdot\boldsymbol{\omega}+\mathbf{p}\cdot(\mathbf{y}_1+...+\mathbf{y}_J^*)$ such that:

i. for every $j=1,...,J$ $\mathbf{y}_j^*$ maximizes profit in Y_j, i.e. $\mathbf{p}^T\cdot\mathbf{y}_j\leq\mathbf{p}^T\cdot\mathbf{y}_j^*$ for every $\mathbf{y}_j\in Y_j$

ii. for every $i=1,...,I$ if $\mathbf{x}^i \geq_i \mathbf{x}^{i*}$, then $\mathbf{p}^T\cdot\mathbf{x}^i \geq w_i$

iii. $\mathbf{x}^{1*}+...+\mathbf{x}^{I*} = \boldsymbol{\omega}+\mathbf{y}_1^*+...+\mathbf{y}_J^*$

<u>(T.14)</u> The condition (ii) in D.16 is weaker than the condition (ii) in D.15

Statement "for every i=1,...,I if if $\mathbf{x}^i \geq_i \mathbf{x}^{i*}$, then $\mathbf{p}^T \cdot \mathbf{x}^i \geq w_i$" is weaker than "for every i=1,...,I $\mathbf{x}^{i*}$ is maximal with respect to $\geq_i$ in a respective budget set, i.e. in

$$\{\mathbf{x}^i \in X_i \colon \mathbf{p}^T \cdot \mathbf{x}^i \leq w_i\}".$$

(T.15) The second fundamental welfare economics theorem

Let all Y_j be convex sets and let all $\geq_i$ be convex, locally non-satiated preferences. Then for every Pareto-optimal allocation $(\mathbf{x}^*,\mathbf{y}^*)$ there exists a price system $\mathbf{p}=(p_1,...,p_L)^T\neq\mathbf{0}$, such that $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{p})$ is a quasi-equilibrium with transfers.

<u>(T.16)</u> Let us assume that for every $i=1,\dots,I$ X_i is a convex set with $\mathbf{0}\in X_i$, and $\geq_i$ is continuous. Then every quasi-equilibrium with transfers such that $w_1,\dots,w_I>0$ is also a price equilibrium with transfers.

<u>(T.17)</u> In a pure exchange economy (D.10), if preferences are continuous, strictly convex and locally non-satiated, then $\mathbf{p}\geq\mathbf{0}$ is a Walrasian equilibrium price system if and only if

$$(x^1(\mathbf{p},\mathbf{p}^T\cdot\boldsymbol{\omega}^1)-\boldsymbol{\omega}^1)+\dots+(x^I(\mathbf{p},\mathbf{p}^T\cdot\boldsymbol{\omega}^I)-\boldsymbol{\omega}^I)\leq\mathbf{0}$$

(D.17) Excess demand in a pure exchange economy model

Excess demand of the *i*th consumer is a function $\mathbf{z}^i(\mathbf{p})=x^i(\mathbf{p},\mathbf{p}^T\cdot\boldsymbol{\omega}^i)-\boldsymbol{\omega}^i$. Aggregate excess demand function is $\mathbf{z}(\mathbf{p}) = \mathbf{z}^1(\mathbf{p})+...+\mathbf{z}^I(\mathbf{p})$

(T.18) If for every consumer $i=1,...,I$, $X_i=\Re_+^L$, and preferences $\geq_i$ are continuous, strictly convex and strictly monotonic, then a price system $\mathbf{p}=(p_1,...,p_L)^T$ is a Walrasian equilibrium price system if and only if $\mathbf{z}(\mathbf{p})=\mathbf{0}$

<u>(T.19)</u> By the assumptions of T.18, if $\boldsymbol{\omega}^1+...+\boldsymbol{\omega}^I \gg \mathbf{0}$, then function $\mathbf{z}$ defined for all $\mathbf{p}\gg\mathbf{0}$ satisfies:

(i) $\mathbf{z}$ is continuous

(ii) $\mathbf{z}$ is homogeneous of degree 0

(iii) $\mathbf{p}^T\cdot\mathbf{z}(\mathbf{p})=0$ for all $\mathbf{p}$ (the 'Walras Law')

(iv) $\exists s>0\ \forall \ell=1,...,L\ \forall \mathbf{p}\ [z_\ell(\mathbf{p})>-s]$

(v) $(\lim_{n\to\infty}\mathbf{p}^n=\mathbf{p}\neq\mathbf{0} \wedge \exists \ell=1,...,L\ [p_\ell=0]) \Rightarrow \mathrm{Max}\{z_1(\mathbf{p}^n),...,z_L(\mathbf{p}^n)\}\to\infty$

<u>(D.17) Excess demand in an economy with production</u>

Let production sets be closed, strictly convex and bounded from above. Then an excess demand function $\mathbf{z}$ is defined for all $\mathbf{p} \gg \mathbf{0}$ by the following formula:

$$\mathbf{z}(\mathbf{p}) = \sum_i x^i(\mathbf{p}, \mathbf{p}^T \cdot \boldsymbol{\omega}^i + \sum_j \theta_{ij} \pi_j(\mathbf{p})) - \sum_i \boldsymbol{\omega}^i - \sum_j y_j(\mathbf{p})$$

<u>(T.20)</u> By the assumptions of T.18, a price system $\mathbf{p}=(p_1,...,p_L)^T$ is a Walrasian equilibrium price system in an economy with production if and only if $\mathbf{z}(\mathbf{p})=\mathbf{0}$

<u>(T.21)</u> By the assumptions of T.19, if the total endowment allows for producing a positive bundle of consumption goods, then excess demand function in an economy with production satisfies (i)–(v) from T.19

<u>(T.22)</u> Let $\mathbf{z}(\mathbf{p})$ be a function satisfying T.21. Then the system of equations $\mathbf{z}(\mathbf{p})=\mathbf{0}$ has a solution

<u>(T.23)</u> Let $\mathbf{z}(\mathbf{p})$ be a function satisfying (i)–(iii) from T.21. Then the system of inequalities $\mathbf{z}(\mathbf{p})\leq\mathbf{0}$ has a solution

Non-technical summary

A. Economic analysis of general equilibrium starts by abandoning an assumption that markets for specific goods are independent from one another.

B. Studying links between markets for specific goods makes the analysis much more complicated and suggests to look for simplified cases such as a pure exchange between two consumers, a production with one firm and one consumer, a supply from two firms using two factors (so-called '2×2 model') etc.

C. General equilibrium analysis culminates in welfare economics theorems which determine the links between competitive (Walrasian) equilibrium and Pareto optimality.

D. The first welfare economics theorem states that a market equilibrium can be attained only in a Pareto optimum. The second welfare economics theorem states that any Pareto optimum can be attained as a market equilibrium (perhaps with wealth transfers). The second theorem requires convexity assumptions for consumer preferences and production sets.

E. Assuming that preferences are continuous, strictly convex and locally non-satiated, the condition for market equilibrium is that excess demand is non-positive. If furthermore local non-satiation is substituted with strict monotonicity, then the non-positivity can be substituted with the condition that the excess demand is zero.

F. Characterizing the market equilibrium with the system of equations or inequalities allows for carrying out existence proofs.

10. Public choice theory

Condorcet paradox

- 1/3 prefer C>S>K,
- 1/3 prefer S>K>C, and
- 1/3 prefer K>C>S.

Thus

- 2/3 prefer C>S, and
- 2/3 prefer S>K.

Consequently the majority is for C>S>K.
But the same argument can be for S>K>C.

(D.1) Dichotomous choice model. Consumers i=1,...,I choose among two alternatives: x and y. If $\alpha_i=-1$, then consumer *i* prefers y over x, if $\alpha_i=0$, he or she is indifferent, and if $\alpha_i=1$, then he/she prefers x over y.

(D.2) Dichotomous social welfare functional (dichotomous social welfare aggregator) is a rule F which states a social preference for any system of individual preferences from D.1; i.e.

$$F: \{-1,0,1\}^I \rightarrow \{-1,0,1\}$$

(D.3) Dichtomous social welfare functional is Paretian (has a Pareto property), if

$$F(1,...,1)=1 \text{ and } F(-1,...,-1)=-1$$

(T.1) Dichotomous social welfare functional reflecting a majority voting $F(\alpha_1,...,\alpha_I)=\text{sign}(\alpha_1+...+\alpha_I)$ is Paretian

(D.4) Dichotomous social welfare functional is dictatorial, if there is a consumer *h*, called a dictator, who makes that $\alpha_h=-1$ implies $F(\alpha_1,...,\alpha_I)=-1$ and $\alpha_h=1$ implies $F(\alpha_1,...,\alpha_I)=1$

(T.2) Dichotomous dictatorial social welfare functional is Paretian

(D.5) Dichotomous social welfare functional is symmetric (anonymous) if for any permutation π of the set $\{1,...,I\}$, $F(\alpha_1,...,\alpha_I)=F(\alpha_{\pi(1)},...,\alpha_{\pi(I)})$

(D.6) Dichotomous social welfare functional is neutral between alternatives if

$$F(\alpha_1,...,\alpha_I) = -F(-\alpha_1,...,-\alpha_I)$$

(D.7) Dichotomous social welfare function is positively responsive if $(\alpha_1,\ldots,\alpha_I) \geq (\alpha'_1,\ldots,\alpha'_I)$, $(\alpha_1,\ldots,\alpha_I) \neq (\alpha'_1,\ldots,\alpha'_I)$ and $F(\alpha_1',\ldots,\alpha_I') \geq 0$ imply $F(\alpha_1,\ldots,\alpha_I)=1$

(T.3) May Theorem

Dichotomous social welfare function F is a majority voting function if and only if three conditions hold:

(i) F is symmetric,

(ii) F is neutral between alternatives, and

(iii) F is positively responsive

(D.8) General choice model. Consumers $i=1,\ldots,I$ have rational preferences $\geq_i$ over the same set X. Let $\mathcal{R}$ denote the set of all rational preference relations on X. Social welfare functional (social welfare aggregator) for $\mathcal{R}_0 \subset \mathcal{R}^I$ is a function $F: \mathcal{R}_0 \rightarrow \mathcal{R}$ defining a rational preference $F(\geq_1,\ldots,\geq_I)$ for any system of individual preferences in $\mathcal{R}_0$. F_p denotes a strict preference implied by F.

(D.9) A social welfare functional is Paretian if for any pair of alternatives $\{x,y\} \subset X$ and any system of preferences $(\geq_1,\ldots,\geq_I) \in \mathcal{R}_0$: $(\forall i=1,\ldots,I\ [x >_i y]) \Rightarrow xF_p(\geq_1,\ldots,\geq_I)y$

<u>(D.10)</u> A social welfare functional $F: \mathcal{R}_0 \to \mathcal{R}$ is <u>independent of irrelevant alternatives</u>, if the social preference between two alternatives $\{x,y\} \subset X$ depends only on individual preferences with respect to these very alternatives, i.e.:

$$\forall \{x,y\} \subset X \ \forall (\geq_1,\ldots,\geq_I) \in \mathcal{R}_0 \ \forall (\geq'_1,\ldots,\geq'_I) \in \mathcal{R}_0$$

$$[(\forall i=1,\ldots,I \ [(x \geq_i y \Leftrightarrow x \geq'_i y) \wedge (y \geq_i x \Leftrightarrow y \geq'_i x)]) \Rightarrow$$

$$(xF(\geq_1,\ldots,\geq_I)y \Leftrightarrow xF(\geq'_1,\ldots,\geq'_I)y)$$

$$\wedge (yF(\geq_1,\ldots,\geq_I)x \Leftrightarrow yF(\geq'_1,\ldots,\geq'_I)x)]$$

(D.11) A social welfare functional F is dictatorial if there exists a consumer *h*, called a dictator, which makes that $x >_h y$ implies that for any system of preferences $(\geq_1, \ldots, \geq_I) \in \mathcal{R}_0$:

$$x F_p(\geq_1, \ldots, \geq_I) y$$

(T.4) Arrow Theorem

Let X include at least 3 alternatives. Let $\mathcal{R}_0=\mathcal{R}^{\mathrm{I}}$ or $\mathcal{R}_0=\mathcal{P}^{\mathrm{I}}$, where $\mathcal{P}$ is the set of all rational preferences on X such that only identical alternatives are indifferent. Then every Paretian social welfare functional independent of irrelevant alternatives is dictatorial.

(D.12) A relation $\geq_L$ on a set X of alternatives is a linear order if it is complete and transitive (hence also reflexive – see T.1.1 from lecture 1). Moreover if $x \neq y$, then $x \geq_L y$ excludes $y \geq_L x$ and *vice versa*.

<u>(D.13)</u> A rational preference relation $\succsim$ is <u>single peaked</u> with respect to a linear order $\geq_L$ on X if there is an alternative $x \in X$ such that $\succsim$ is increasing with respect to $\geq_L$ on

$$\{y \in X\colon x \geq_L y\}$$

and decreasing on

$$\{y \in X\colon y \geq_L x\}$$

(i.e. $x \geq_L y >_L z \Rightarrow y \succ z$ and $z >_L y \geq_L x \Rightarrow z \prec y$)

<u>(D.14)</u> For a given linear order $\geq_L$ on X, $\mathcal{R}_{\geq} \subset \mathcal{R}$ denotes the set of all rational preferences on X which are single peaked with respect to $\geq_L$

<u>(D.15)</u> A <u>majority voting</u> function for $(\mathcal{R}_{\geq})^I$ is defined as $F^m: (\mathcal{R}_{\geq})^I \to \mathcal{R}_{\geq}$, such that for any $x,y\in X$ we have:

$$xF^m(\geq_1,\ldots,\geq_I)y$$

$$\Leftrightarrow$$

$$\mathrm{card}\{i\in\{1,\ldots,I\}: x>_i y\}\geq \mathrm{card}\{i\in\{1,\ldots,I\}: y>_i x\}$$

<u>(D.16)</u> Consumer $h\in\{1,\ldots,I\}$ is called a <u>median agent</u> for the system of preferences $(\geq_1,\ldots,\geq_I)\in(\mathcal{R}_{\geq})^I$ if

$$\mathrm{card}\{i\in\{1,\ldots,I\}: x_i\geq x_h\}\geq I/2 \text{ and}$$

$$\mathrm{card}\ \{i\in\{1,\ldots,I\}: x_h\geq x_i\}\geq I/2,$$

where x_i is the (only) alternative best preferred by *i*.

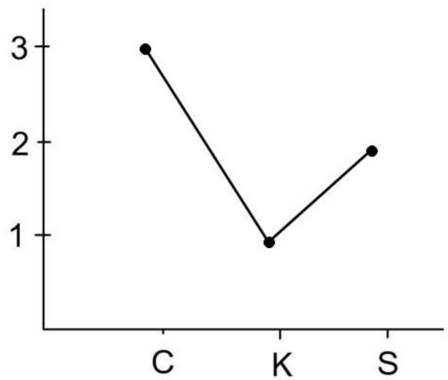
<u>(T.5)</u> For any system of preferences $(\geq_1,\ldots,\geq_I)\in(\mathcal{R}_{\geq})^I$ there exists a median agent; moreover, if I is an odd number and every consumer has a different peak, then there is a unique median agent.

<u>(D.17)</u> For a system of preferences $(\geq_1,\ldots,\geq_I)\in(\mathcal{R}_{\geq})^I$ an alternative $x\in X$ is called a <u>Condorcet winner</u>, if for every alternative $y\in Y$: $xF^m(\geq_1,\ldots,\geq_I)y$

<u>(T.6)</u> For any system of preferences $(\geq_1,\ldots,\geq_I)\in(\mathcal{R}_{\geq})^I$ the alternative best preferred by a median agent x_h satisfies for any alternative $y\in X$: $x_hF^m(\geq_1,\ldots,\geq_I)y$ (thus it is a Condorcet winner)

<u>(T.7)</u> Let I be an odd number and let $\geq_L$ be a linear ordering on X. Moreover let $\mathcal{P}_{\geq} \subset \mathcal{R}_{\geq}$ denote the set of all preference relations that are single peaked and where an indifference holds between identical alternatives only. Then the majority voting procedure makes a relation $F^m(\geq_1, \ldots, \geq_I)$ that is complete and transitive.

Condorcet preferences



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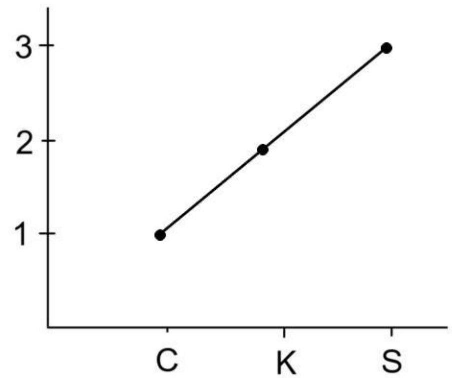

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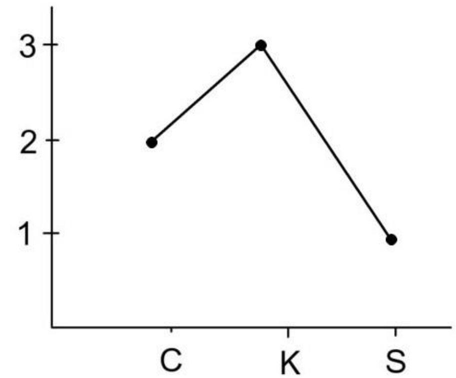

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<u>(D.18) Social choice function</u>, any f: $\mathcal{R}_0 \to X$ defined over any $\mathcal{R}_0 \subset \mathcal{R}^I$ consistent with a social welfare functional

<u>(T.8)</u> By the assumptions of T.7, let $\mathcal{R}_0 \subset (\mathcal{P}_{\geq})^I$. Then any social choice function $f(\geq_1,\ldots,\geq_I)$ returns a Condorcet winner

<u>(D.19)</u> A social choice function f: $\mathcal{R}_0 \to X$ defined on $\mathcal{R}_0 \subset \mathcal{R}^I$ is <u>weakly Paretian</u> if for any system of preferences $(\geq_1,\ldots,\geq_I) \subset \mathcal{R}_0$ the choice $f(\geq_1,\ldots,\geq_I) \in X$ is a <u>weak Pareto optimum</u>, i.e. for some $x,y \in X$ and for every $i=1,\ldots,I$: $x >_i y$ implies $y \neq f(\geq_1,\ldots,\geq_I)$

(D.20) An alternative $x \in X$ maintains its position from the system $(\geq_1,...,\geq_I) \in \mathcal{R}^I$ to $(\geq'_1,...,\geq'_I) \in \mathcal{R}^I$ if for every $i=1,...,I$ and every $y \in X$: $x \geq_i y \Rightarrow x \geq'_i y$

(D.21) A social choice function $f: \mathcal{R}_0 \to X$ defined on $\mathcal{R}_0 \subset \mathcal{R}^I$ is monotonic if for any two systems $(\geq_1,...,\geq_I) \in \mathcal{R}_0$ and $(\geq'_1,...,\geq'_I) \in \mathcal{R}_0$ such that the alternative chosen $x=f(\geq_1,...,\geq_I)$ maintains its position from $(\geq_1,...,\geq_I)$ to $(\geq'_1,...,\geq'_I)$ we have: $x=f(\geq'_1,...,\geq'_I)$

<u>(D.22)</u> Consumer $h \in \{1,...I\}$ is called a <u>dictator</u> for a social choice function $f: \mathcal{R}_0 \to X$, if for any system $(\geq_1,...,\geq_I) \in \mathcal{R}_0$, $f(\geq_1,...,\geq_I) \in \{x \in X: \forall y \in X\ [x \geq_h y]\}$. If there is a dictator for a given social choice function, it is called <u>dictatorial</u>.

<u>(T.9) Arrow theorem</u>

By the assumptions of T.4 every monotonic and weakly Paretian social choice function $f: \mathcal{R}_0 \to X$ is dictatorial

Non-technical summary

A. Public choice theory studies whether individual preferences can be aggregated into social preferences in a way consistent with certain obvious principles such as e.g. satisfying the will of a majority.

B. The Arrow theorem states that – in general – such an aggregation is not possible.

C. Aggregation consistent with certain obvious principles is possible in some special cases. For instance it is possible when there are only two alternatives to choose from, or when all the consumers have single peaked preferences.

D. The Arrow theorem can also be stated for social choice functions.