Bundling or unbundling? Analysis of profit gains from bundling

MACIEJ SOBOLEWSKI [1][2]

IRENEUSZ MIERNIK

TOMASZ KOPCZEWSKI

ABSTRACT

So far bundling literature has mainly focused on conditions for marginal profit dominance. However this classical perspective has little practical applicability, as it does not tell under which conditions pure or mixed bundling can be expected to strictly outperform separate sales of goods and by how much. Therefore in this paper we shift to a different perspective and look at the magnitude of profit gains from bundling. We propose specific demand setting governed by a single parameter joint density function of reservation prices for two goods which has two convenient properties: (i) marginal distributions are uniform yielding linear market demands for each of bundle components and (ii) correlation coefficient between reservation prices has a closed-form. Our analysis, carried out for the whole possible range of marginal costs and degree of correlated valuations for bundle components, shows that in most cases bundling brings only small gain or no improvement. Profit gains magnitude exceeds 10% only for low marginal cost and sufficient negative correlation of valuations. Hence, our results conform with practical observations that bundling is applicable mainly to low marginal costs industries like information goods or ICT services.

HIGHLIGHTS

- We introduce bivariate density function with linear component demands and closed-form correlation.
- We analyze profit gains from bundling for any degree of correlation and marginal cost.
- Combination of strong negative correlation of reservation prices and low marginal costs is critical for the magnitude of profit gains.

KEYWORDS

correlated reservation prices, mixed bundling, pure bundling, uniform marginal distribution

JEL CLASSIFICATION

D21, L12

[1] Corresponding author. email: maciej.sobolewski@uw.edu.pl
[2] All authors work at Faculty of Economic Sciences, University of Warsaw, Poland.

INTRODUCTION

The question about conditions for optimality of bundling is a long studied issue in literature. The problem has been posted since the seminal works of Stigler (1963) and Adams and Yellen (1976) and still has not received a definite answer. In their famous study, Adams and Yellen have shown on discrete examples that monopolist will earn extra profit from bundling, compared to independent sales of products, if a distribution of reservation prices for the package has reduced variance compared to valuations of each component. Their second conclusion was that whether bundling is more profitable depends crucially on the level of unit costs.[1]

Since then numerous attempts has been made to derive conditions under which bundling generates more profit for possibly the least restrictive assumptions about consumer demand space and costs. Venkatesh and Mahajan (2009) provide an excellent overview of those efforts undertaken throughout more than three decades. They conclude that although mixed bundling is generally expected to weakly outperform pure components, the conditions for strict profit dominance are known only for special cases. Existing research identified at least several demand and supply side elements which determine profitability of bundling, such as: (non) additivity and correlation in reservation prices and (non) additivity of unit costs. Scope economics and demand complementarity, if present, are both additional rationale for bundling.

Bundling works through reduced heterogeneity in reservation prices across consumers, thus facilitating the capture of greater part of consumer surplus (Schmalensee 1984). On the other hand mixed bundling is essentially a price discriminating tool and might yield even higher profit if marginal production costs are substantial.[2] The idea of reduced heterogeneity of reservation prices for a bundle is linked to the issue of negative correlation of component reservation prices. McAfee, McMillan et al. (1989) establish general sufficient condition for mixed bundling to dominate separate sales for any atomless joint distribution of reservation prices and show that it holds also for independently distributed reservation prices. Their condition partially corresponds to, but does not imply, a negative correlation property of reservation prices. More recently, Chen and Riordan (2013) utilized copula approach to restate sufficient and necessary conditions of McAfee, McMillan et al. (1989) for bundles of size larger than two. Their

[1] Adams and Yellen considered two forms of bundling. Under pure bundling, monopolist sells only the package, while under mixed bundling consumers may choose to buy either a package (for the price that is different from the sum of single-good prices) or any of the component goods separately. This distinction is an established standard in literature until now.

[2] Mixed bundling enables monopolist to sell a bundle to a large mass of customers with at least moderate valuations for both goods and at the same time capture higher mark-up from consumers, who are mainly interested in one of the goods.

conditions use broader concepts of negative dependence or tail dependence which are properties of copula functions (Nelsen 1999). However bundling can be profitable also when reservation prices for two products are positively correlated, at least for specific distributions of individual demands such as bivariate normal (Schmalensee 1984).

All these studies have adopted local analysis approach, looking at improvement of profits when shifting to mixed bundling, from introducing package price with infinitesimal small discount. Our study takes different approach. We focus on by how much (mixed) bundling outperforms unbundling. It requires switching from local improvement analysis towards searching for global optima. Despite its obvious practical relevance, the problem of (global) profit gains magnitude has not received much attention in literature so far. Some existing studies conducted global optima analysis for specified stylizations of components reservation prices and dependency pattern, sometimes with supportive use of simulation techniques.[3] Eckalbar (2010) explores optimality of bundling under uniform distribution of reservation prices and gives analytical mixed and pure bundling solutions, but only for the three special cases: independence and perfect positive or negative correlation.[4] He also reflects on the profit gains issue and claims that mixed bundling offers up to 10% larger profits then separate sales while the gains from pure bundling do not exceed 8.8% - both results holding only for independence case and zero marginal costs. Despite this result, hardly anything can be found about how the magnitude of profit gains from bundling is affected by costs and correlation of valuations.

This paper aims to shed some light on this issue. To progress with analysis of profit gains, we introduce a new bivariate distribution of reservation prices which possesses advantages of both previously indicated. On one hand it has analytically tractable demand and profit functions for pure bundling and pure components strategies and on the other hand it allows to capture underlying correlation with a single parameter.

Our contribution to existing research is twofold. First, we propose joint density function of reservation prices for two goods f^a, with a single parameter controlling for the shape of demand space. We show that this joint distribution has two convenient properties for the analysis of

[3] For continuous case, the most frequently used families of distributions are uniform and normal. Gaussian demand is analytically intractable, but on the other hand, bivariate normal distribution captures underlying correlation through a single parameter.

[4] Uniform distribution yields linear demand curve which makes this distribution convenient for analytical profit maximization, especially for pure components strategy, but so far there have been no attempts to analytically model partial correlation between uniformly distributed valuations, perhaps in consent with a view expressed by Venkatesh and Mahajan (2009, p.236): "*This form is analytically quite tractable, can capture complementarity and substitutability, but is not convenient for modeling correlation (except perfect positive/negative correlation)*".

bundling problem: (i) marginal distributions derived from f^a are always uniform in the whole parameter space and (ii) correlation coefficient between marginal distributions is a closed-form function of distribution's parameter and can take any value in the whole range [-1,+1]. Under our formulation of reservation price space, we are able to analytically model pure bundling problem for any degree of correlation. Although closed-form solutions for mixed-bundling do exist in proposed setting, our treatment of mixed bundling relies on simulations, as this case has too many subcases that need to be considered analytically. Second, for the introduced joint density function we analyze optimality and profit gains from shifting to monopoly pure and mixed bundling strategies for a full range of correlation and marginal costs. Our results are valid for the specific type of distribution we have adopted, but nevertheless they support an astonishing conclusion that while bundling is never worse than unbundling, it is rarely better enough for practical implementation. This conclusion holds even for strong negative correlation between reservation prices, pointing to the critical role of costs in bundling analysis. We assess bundling gains with arbitrary assumption that 10% markup over operating surplus from pure components is sufficient to cover implementation costs.[5] The fact that mixed bundling rarely exceeds this rather modest threshold is the most astonishing finding from our study.

The paper is organized as follows. In section 2 we describe the general structure of bundling problem and set out our analytical framework. More specifically we introduce bivariate density function with 'belt-shaped' domain to describe reservation price space and derive formula for correlation of reservation prices. In section 3 we provide solutions to bundling problem within our framework and determine profit gains from pure and mixed bundling for different degrees of correlation and marginal costs. The last section provides summary and conclusions.

Model of Commodity Bundling

The focus of this section is on mathematical model of commodity bundling. Let $\mathbb{R}$ denote the ordinary real line $(-\infty,\infty)$, $\mathbb{R}^2$ denote the real plane $\mathbb{R}\times\mathbb{R}$, and $\overline{\mathbb{R}}$ denote the extended real line $[-\infty,\infty]$, $\overline{\mathbb{R}}^2$ denote the extended real plane $\overline{\mathbb{R}}\times\overline{\mathbb{R}}$. We also let $Domf$ and $Ranf$ denote the domain and the range of the f function. There are two goods in our model. We will use the capital letters $\mathbb{X}$ and $\mathbb{Y}$ to represent these goods and lowercase letters x, y to represent their reservation prices. Valuations of $\mathbb{X}$ and $\mathbb{Y}$ goods by consumers will be represented in our model by joint bivariate reservation price density function.

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[5] In reality there are always some additional fixed and variable implementation costs, which are usually ignored in bundling models as a consequence of simplifying assumptions about additivity of constant marginal costs.

Definition 1

The bivariate reservation price density function for goods $\mathbb{X}$ and $\mathbb{Y}$ is a real function f with the following properties:

1. $Domf = [0, \overline{x}] \times [0, \overline{y}] \subset \overline{\mathbb{R}}^2$ is the Cartesian product of two closed intervals, where $\overline{x} \in \overline{\mathbb{R}}$, $\overline{y} \in \overline{\mathbb{R}}$ and $\overline{x} \geq \overline{y} > 0$,
2. $Ranf \subset \mathbb{R}$ and $f(x, y) \geq 0$ for all (x, y) in $[0, \overline{x}] \times [0, \overline{y}]$,
3. $\iint_{[0,\overline{x}]\times[0,\overline{y}]} f(x, y) dy dx = 1$.

Note that for all $x_1, x_2 \in [0, \overline{x}]$, $x_1 \leq x_2$ and for all $y_1, y_2 \in [0, \overline{y}]$, $y_1 \leq y_2$ in $Domf$

$$0 \leq \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} f(x, y) dy \leq 1$$

is the fraction of a population of consumers with the reservation prices for good $\mathbb{X}$ and $\mathbb{Y}$ falling within rectangle $[x_1, x_2] \times [y_1, y_2]$

Definition 2

The univariate reservation price density function for good $\mathbb{X}$ is a real function f_x with the following properties:

1. $Domf_x = [0, \overline{x}] \subset \overline{\mathbb{R}}$ is the closed interval in $\overline{\mathbb{R}}$,
2. $Ranf_x \subset \mathbb{R}$ and $f_x(x) \geq 0$ for all x in $[0, \overline{x}]$,
3. $\int_0^{\overline{x}} f_x(x) dx = 1$.

Note that

$$f_x(x) = \int_0^{\overline{y}} f(x, y) dy$$

and for all $x_1, x_2 \in [0, \overline{x}]$, $x_1 \leq x_2$, in $Domf_x$

$$0 \leq \int_{x_1}^{x_2} f_x(x) dx \leq 1$$

is the fraction of a population of consumers with the reservation prices of good $\mathbb{X}$ falling within interval $[x_1, x_2]$.

The following properties hold: $f_y(y) = \int_0^{\overline{x}} f(x, y) dx$, $Domf_y = [0, \overline{y}] \subset \overline{\mathbb{R}}$ and for every interval $[y_1, y_2]$, where $y_1, y_2 \in [0, \overline{y}]$, $y_1 \leq y_2$, in $Domf_y$ $0 \leq \int_{y_1}^{y_2} f_y(y) dy \leq 1$.

Univariate reservation price density function for good $\mathbb{Y}$ is defined analogously. Following the framework of Adams and Yellen (1976) monopolist can choose between three following strategies:

1. Sell goods $\mathbb{X}$ and $\mathbb{Y}$ separately - **pure component** strategy (PC).
2. Offer goods $\mathbb{X}$ and $\mathbb{Y}$ for sale only in packages - **pure bundling** strategy (PB).
3. Combine PC and PB strategies by offering goods $\mathbb{X}$ and $\mathbb{Y}$ not only separately but also in a package. This is **mixed bundling** strategy (MB).

2.1 PURE COMPONENTS STRATEGY

A consumer with single demand will purchase a unit at price p if his reservation price of this good (weakly) exceeds this price. Therefore the market demand for any good at price p is a fraction of consumers with the reservation prices greater than or equal to p. Hence, we can define the demand function of good $\mathbb{X}$ for PC strategy as follows:

Definition 3

The demand function for good $\mathbb{X}$ under pure components strategy is a real function d_x^{PC} with domain $[0, \overline{x}]$ given by:

$$d_x^{PC}(p) = 1 - \int_0^p f_x(x)dx = \int_p^{\overline{x}} f_x(x)dx$$

Demand function for good $\mathbb{Y}$ is defined analogously.

2.2 PURE BUNDLING STRATEGY

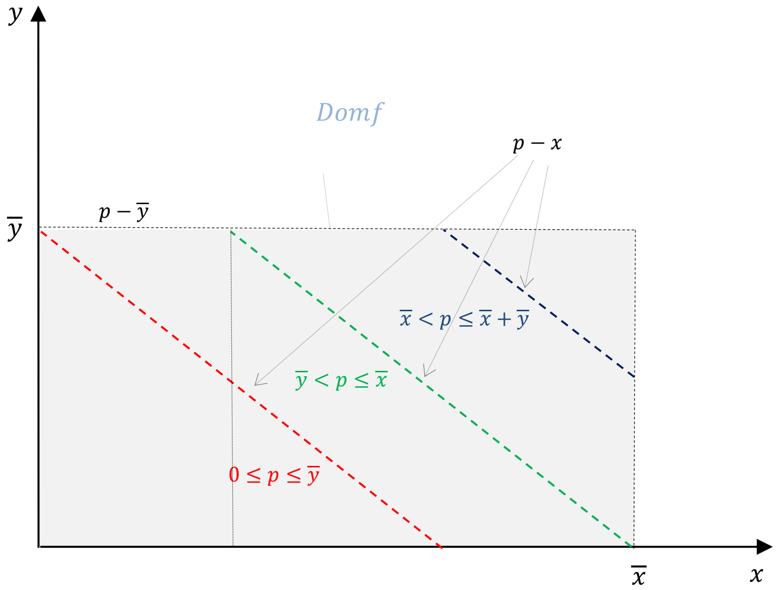
When a package consisting of the $\mathbb{X}$ and $\mathbb{Y}$ goods is offered at price p, a consumer will make a purchase if and only if the sum of his or her reservation prices for these goods will be greater than or equal to p. This additivity property is assumed in most studies and adopt it as well. Consumer space under pure bundling strategy is graphed in Figure 1. We need to define demand function for the bundle for three different intervals of a price p: $[0, \overline{y}]$, $(\overline{y}, \overline{x}]$, $(\overline{x}, \overline{x} + \overline{y}]$.

Definition 4

The demand function of the bundle the $\mathbb{X}$ and $\mathbb{Y}$ goods under pure bundling strategy is a real function d_b^{PB} with domain $[0, \overline{x} + \overline{y}]$ given by:

$$
d_b^{PB}(p) = \begin{cases} 1 - \int_0^p dx \int_0^{p-x} f(x,y)dy, & 0 \le p \le \overline{y} \\ 1 - \int_0^{p-\overline{y}} dx \int_0^{\overline{y}} f(x,y)dy - \int_{p-\overline{y}}^{p} dx \int_0^{p-x} f(x,y)dy, \quad where & \overline{y} < p \le \overline{x} \\ 1 - \int_0^{p-\overline{y}} dx \int_0^{\overline{y}} f(x,y)dy - \int_{p-\overline{y}}^{\overline{x}} dx \int_0^{p-x} f(x,y)dy, & \overline{x} < p \le \overline{x} + \overline{y} \end{cases}
$$

Figure 1. Pure bundling strategy.



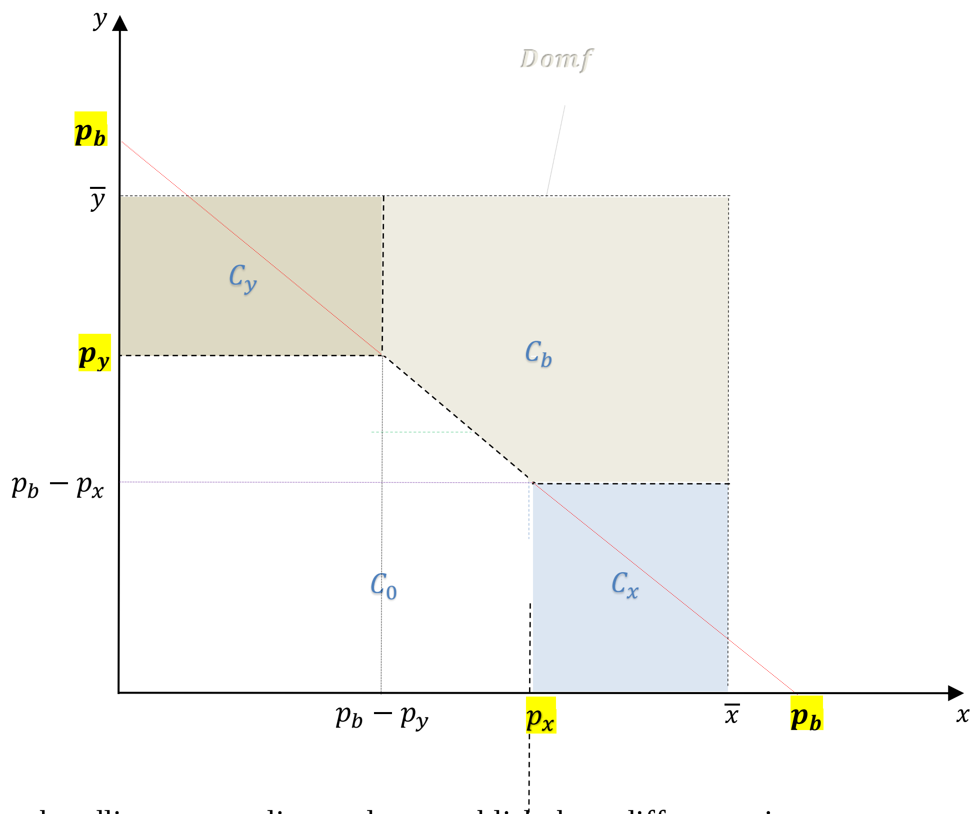
Note that for symmetric case: $\overline{x} = \overline{y} < +\infty$ demand for a bundle simplifies to:

$$
d_b^{PB}(p) = \begin{cases} 1 - \int_0^p dx \int_0^{p-x} f(x,y)dy, & 0 \le p \le \overline{x} \\ 1 - \int_0^{p-\overline{x}} dx \int_0^{\overline{x}} f(x,y)dy - \int_{p-\overline{x}}^{\overline{x}} dx \int_0^{p-x} f(x,y)dy, & \overline{x} < p \le 2\overline{x} \end{cases}
$$

2.3 MIXED BUNDLING STRATEGY

This is most interesting but also the most analytically demanding strategy, due to the fact that consumer can buy not only a package consisting of goods $\mathbb{X}$ and $\mathbb{Y}$ but also each good separately. Essentially mixed bundling strategy introduces discrimination between consumers who now have four options to choose from: buying only good $\mathbb{X}$, buying only good $\mathbb{Y}$, buying a package of both and at the end buying nothing. Consumer space under MB strategy is presented in Figure 2.

Figure 2. Mixed bundling strategy.



Under pure bundling monopolist needs to establish three different prices:

- p_x - a price for good $\mathbb{X}$;
- p_y - a price for good $\mathbb{Y}$;
- p_b - a price for the bundle consisting of $\mathbb{X}$ and $\mathbb{Y}$.

Note that if monopolist offers goods $\mathbb{X}$ and $\mathbb{Y}$ separately at prices p_x and p_y then it makes sense to offer a bundle provided that its price includes a discount:

$$p_b < p_x + p_y.$$

Based on this inequality, we get two conditions for implicit prices for $\mathbb{X}$ and $\mathbb{Y}$ which must hold if monopolist wants to implement mixed bundling strategy effectively:

$$p_b - p_x < p_y \text{ and } p_b - p_y < p_x.$$

The interpretation of those conditions is simple: $p_b - p_x > 0$ is the implicit price of good $\mathbb{Y}$ for a consumer who would buy only good $\mathbb{X}$ at a price p_x and $p_b - p_y > 0$ is the implicit price of good $\mathbb{X}$ for the consumer who would buy only good $\mathbb{Y}$ at a price p_y. Implicit prices for $\mathbb{X}$ and $\mathbb{Y}$ are also bounded from the above by the price of a bundle or the maximal reservation price in the population - whichever is smaller. Thus for every p_x in $[0,\overline{x}]$, p_y in $[0,\overline{y}]$, p_b in $[0,\overline{x}+\overline{y}]$ we have the following restrictions on (p_y, p_x, p_b):

$$0 < p_b - p_x < p_y < min(p_b, \overline{y}),$$

and

$$0 < p_b - p_y < p_x < min(p_b, \overline{x}).$$

In Figure 2, we can see the following four demand segments:

$$C_0 = \left\{(x,y) \in Domf \middle| x < p_x \; and \; y < p_y \; and \; x + y < p_b \right\}$$

is the area with consumers who buy nothing;

$$C_x = \{(x,y) \in Domf | x \geq p_x \; and \; y < p_b - p_x\}$$

is the area with consumers who buy only good $\mathbb{X}$;

$$C_y = \left\{(x,y) \in Domf \middle| y \geq p_y \; and \; x < p_b - p_y\right\}$$

is the area with consumers who buy only good $\mathbb{Y}$;

$$C_b = \left\{(x,y) \in Domf \middle| x \geq p_b - p_y \; and \; y \geq p_b - p_x \; and \; x + y \geq p_b \right\}$$

is the area with consumers who buy only the bundle. Based on the above, for mixed bundling strategy we need to define:

- demand for good $\mathbb{X}$,
- demand for good $\mathbb{Y}$,
- demand for the bundle consisting of both goods.

Definition 5

The demand function for good $\mathbb{X}$ under mixed bundling strategy is a real function d_x^{MB} with domain $\left(p_b - p_y, min(p_b, \overline{x})\right) \times [0, \overline{x}+\overline{y}]$ given by

$$d_x^{MB}(p_x, p_b) = \int_0^{p_b - p_x} dy \int_{p_x}^{\overline{x}} f(x,y)dx.$$

Definition 6

The demand function for good $\mathbb{Y}$ under mixed bundling strategy is a real function d_y^{MB} with domain $\left(p_b - p_x, min(p_b, \overline{y})\right) \times [0, \overline{x}+\overline{y}]$ given by

$$d_y^{MB}(p_y, p_b) = \int_0^{p_b - p_y} dx \int_{p_y}^{\overline{y}} f(x,y)dy.$$

Definition 7

The demand function for the bundle under mixed bundling strategy is a real function d_b^{MB} with domain $\left(p_b - p_y, min(p_b, \overline{x})\right) \times \left(p_b - p_x, min(p_b, \overline{y})\right) \times [0, \overline{x} + \overline{y}]$ given by $d_b^{MB}(p_x, p_y, p_b) = \int_{p_b - p_y}^{p_x} dx \int_{p_b - x}^{\overline{y}} f(x,y)dy + \int_{p_b - p_x}^{\overline{y}} dy \int_{p_x}^{\overline{x}} f(x,y)dy$. independent uniform marginal densities

Now we apply the general structure of bundling problem to the case when reservation prices for both goods are independent and uniformly distributed. Under these assumptions bivariate joint reservation price distribution takes the form $f(x,y) = \frac{1}{\overline{x}\,\overline{y}}$ with $Domf = [0, \overline{x}] \times [0, \overline{y}] \subset \overline{\mathbb{R}}^2$, where $\overline{x} \in \overline{\mathbb{R}}$, $\overline{y} \in \overline{\mathbb{R}}$ and $\overline{x} \geq \overline{y} > 0$, the demand functions evaluate to:

1. For PC strategy (Definition 3)

$$d_x^{PC}(p) = 1 - \frac{1}{\overline{x}}p, \qquad d_y^{PC}(p) = 1 - \frac{1}{\overline{y}}p,$$

2. For PB strategy (Definition 4)

$$d_b^{PB}(p) = \begin{cases} 1 - \frac{1}{2\overline{x}\,\overline{y}}p^2, & 0 \leq p \leq \overline{y} \\ 1 - \frac{1}{2\overline{x}\,\overline{y}}(p^2 - (p - \overline{y})^2), \qquad for & \overline{y} < p \leq \overline{x} \\ 1 - \frac{1}{2\overline{x}\,\overline{y}}(p^2 - (p - \overline{x})^2 - (p - \overline{y})^2), & \overline{x} < p \leq \overline{x} + \overline{y} \end{cases}$$

3. For MB strategy (Definitions 5, 6 and 7)

$$d_x^{MB}(p_x, p_b) = \frac{1}{\overline{x}\,\overline{y}}(\overline{x} - p_x)(p_b - p_x), \qquad d_y^{MB}(p_y, p_b) = \frac{1}{\overline{x}\,\overline{y}}(\overline{y} - p_y)(p_b - p_y),$$

$$d_b^{MB}(p_x, p_y, p_b) = 1 - \frac{1}{\overline{x}\,\overline{y}}\left(\overline{x}(p_b - p_x) + \overline{y}(p_b - p_y)\right) - \frac{1}{2\overline{x}\,\overline{y}}\left(({p_x}^2 + {p_y}^2) - {p_b}^2\right).$$

Eckalbar (2010) has analyzed bundling problem under the above assumptions. He shows that for all relevant values of marginal cost mixed bundling strategy generates the highest profits but reduces consumer surplus compared to separate sales of both goods. His second conclusion states that pure bundling will be inferior to pure components if marginal costs exceeds certain level. Our analysis extends Eckalbar's results beyond independence case.

2.4 Profit optimization

We can now formulate monopoly profit optimization. With the three strategies at hand, generic profit formula for monopolist is given by:

$$\Pi_{max} = \max_{(p_x^{PC}, p_y^{PC}, p_b^{PB}, p_x^{MB}, p_y^{MB}, p_b^{MB})} \{\Pi^{PC}(p_x^{PC}, p_y^{PC}), \Pi^{PB}(p_b^{PB}), \Pi^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB})\} \quad (1)$$

where:

- $\Pi^{PC}(p_x^{PC}, p_y^{PC})$ is profit from PC strategy,
- $\Pi^{PB}(p_b^{PB})$ is profit from PB strategy,
- $\Pi^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB})$ is profit from MB strategy,
- p_x^{PC}, p_y^{PC} are prices of the $\mathbb{X}$ and $\mathbb{Y}$ goods respectively for PC strategy,
- p_b^{PB} is price of the bundle for PB strategy,
- $p_x^{MB}, p_y^{MB}, p_b^{MB}$ are prices of the $\mathbb{X}$, $\mathbb{Y}$ and the bundle both goods respectively for MB strategy.

To evaluate expression (1) we need to specify the cost structure. We will follow the main stream approach adopted in bundling literature and assume additive cost function with constant marginal costs and no fixed costs. It should be noted that this assumption rather disfavors bundling as it rules out economies of scale and scope which likely increase gains from selling product in packages. Profit functions are now given by:

1. For PC strategy

$$\Pi^{PC}(p_x^{PC}, p_y^{PC}) = \Pi_x^{PC}(p_x^{PC}) + \Pi_y^{PC}(p_y^{PC}) \quad (3)$$

where:

- $\Pi_x^{PC}(p_x^{PC}) = d_x^{PC}(p_x^{PC})(p_x^{PC} - c_x)$ is profit from selling good $\mathbb{X}$ under PC,
- d_x^{PC} is demand function for good $\mathbb{X}$ under PC (Definition 3),
- c_x is marginal costs of good $\mathbb{X}$.

Profit formula for good $\mathbb{Y}$ is analogous.

2. For PB strategy

$$\Pi^{PB}(p_b^{PB}) = d_b^{PB}(p_b^{PB})(p_b^{PB} - c_b) \quad (4)$$

where:

- d_b^{PB} is demand function of the bundle with $\mathbb{X}$ and $\mathbb{Y}$ goods under PB (definition 4)
- $c_b = c_x + c_y$ is marginal cost of the bundle.

3. For MB strategy

$$\Pi^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB}) = \Pi_x^{MB}(p_x^{MB}, p_b^{MB}) + \Pi_y^{MB}(p_y^{MB}, p_b^{MB}) + \Pi_b^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB}) \quad (5)$$

where:

- $\Pi_x^{MB}(p_x^{MB}, p_b^{MB}) = d_x^{MB}(p_x^{MB}, p_b^{MB})(p_x^{MB} - c_x)$ is profit from selling $\mathbb{X}$ under MB,
- $\Pi_y^{MB}(p_y^{MB}, p_b^{MB}) = d_y^{MB}(p_y^{MB}, p_b^{MB})(p_y^{MB} - c_y)$ is profit from selling $\mathbb{Y}$ under MB,
- $\Pi_b^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB}) = d_b^{MB}(p_x^{MB}, p_y^{MB}, p_b^{MB})(p_b^{MB} - c_b)$ is profit from selling the bundle under MB,
- $d_x^{MB}, d_y^{MB} d_b^{MB}$ are demand functions for goods $\mathbb{X}$, $\mathbb{Y}$ and the bundle respectively, under MB (see, Definitions 5, 6, 7).

2.5 SETTING THE FRAMEWORK: JOINT DENSITY f^a

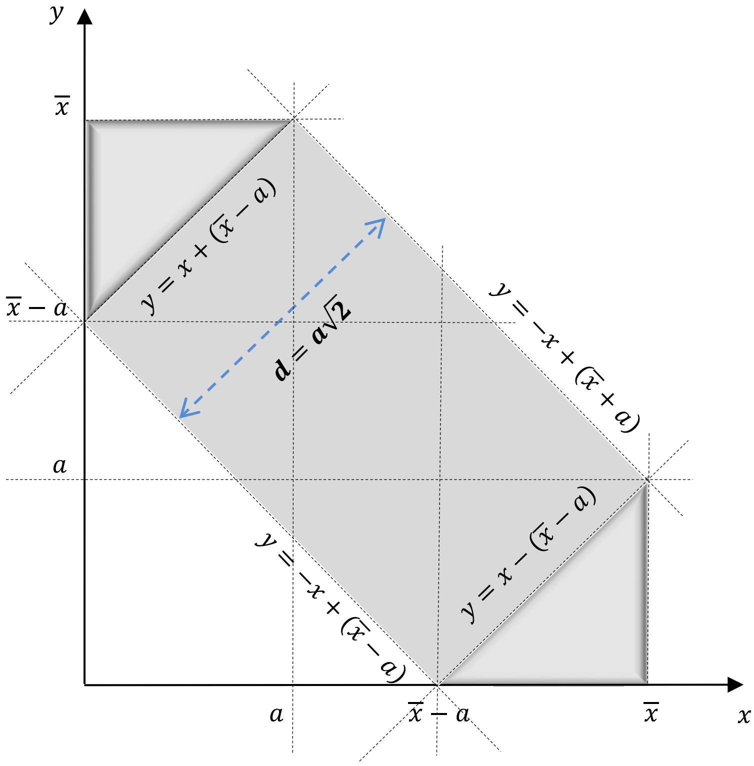
With a basic model at hand we can introduce reservation price space which has desirable properties for analysis of bundling problem, namely uniform marginal densities and a closed-form formula for correlation coefficient. Let f^a denote a joint bivariate density function of reservation prices for two goods, with a single shape parameter a. The domain of f^a where density is positive forms a 'belt', which is symmetric around the diagonal of square $[0, \overline{\mathrm{x}}] \times [0, \overline{\mathrm{x}}]$ as illustrated in Figure 3. Parameter a measures width of the 'belt', which in this case equals $d = a\sqrt{2}$. Depending on the value of a the 'belt' shrinks towards a diagonal line or expands to the whole square. The key feature of f^a is that density in both triangular areas is twice larger than in rectangular area Cooke (2009).[6] In what follows we show that f^a : (i) is indeed a proper density function; (ii) always has uniform margins and (iii) has a closed form expression for correlation coefficient which covers full range of values in [-1,+1].

We introduce two variants of f^a density. By f^{a-} we denote a function with belt-shaped domain directed towards top-left corner of $[0, \overline{\mathrm{x}}] \times [0, \overline{\mathrm{x}}]$ square and by f^{a+} a function with belt-shaped domain directed towards top-right corner.[7] Both variants cover respectively positive and negative correlation of reservation prices for goods $\mathbb{X}$ and $\mathbb{Y}$. In Figure 3, we show a negative belt: $Domf^{a-}$ for $a \in \left(0, \frac{\overline{x}}{2}\right]$ and in Figure 4 for $a \in \left(\frac{\overline{x}}{2}, \overline{x}\right]$. We distinguish both subranges for parameter a because the definition of f^{a-} slightly differs in each case, as is shown in Propositions 1a and 1b.

[6] Andrew Cooke proposed this density in a short technical note outside bundling context. We credit him here.

[7] Throughout the text we use the following naming convention: we refer to domain of f^{a-} as negative belt and f^{a+} as positive belt.

Figure 3. Negative belt: $\boldsymbol{Dom f^{a-}}$ for $\boldsymbol{a} \in \left(\mathbf{0}, \frac{\bar{x}}{2}\right]$.



In the following two propositions we formally define f^{a-} and prove uniformity of its margins.

Proposition 1a. *Function f^{a-} for $a \in \left(0, \frac{\bar{x}}{2}\right]$ given by:*

$$\bigwedge_{0 \le x \le a} f^{a-}(x,y) = \begin{cases} \frac{1}{a\bar{x}} & x + (\bar{x} - a) \le y \le \bar{x} \\ \frac{1}{2a\bar{x}} & -x + (\bar{x} + a) \le y < x + (\bar{x} - a) \\ 0 & otherwise \end{cases}$$

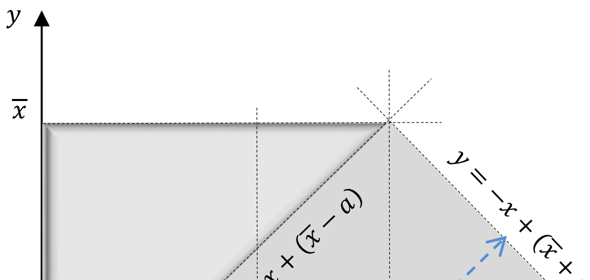
$$\bigwedge_{a < x \le \bar{x} - a} f^{a-}(x,y) = \begin{cases} \frac{1}{2a\bar{x}} & -x + (\bar{x} - a) \le y \le -x + (\bar{x} + a) \\ 0 & otherwise \end{cases} \tag{6}$$

$$\bigwedge_{\bar{x} - a < x \le \bar{x}} f^{a-}(x,y) = \begin{cases} \frac{1}{2a\bar{x}} & x - (\bar{x} - a) < y \le -x + (\bar{x} + a) \\ \frac{1}{a\bar{x}} & 0 \le y \le x - (\bar{x} - a) \\ 0 & otherwise \end{cases}$$

is a reservation price joint density function for goods $\mathbb{X}$ and $\mathbb{Y}$ with uniform marginal densities.

For proof, see Appendix. Next proposition defines f^{a-} in a range $a \in \left(\frac{\bar{x}}{2}, \bar{x}\right]$.

Figure 4. Negative belt: $\boldsymbol{Dom f^{a-}}$ for $\boldsymbol{a} \in \left(\frac{\bar{x}}{2}, \bar{x}\right]$.



Proposition 1b. *Function f^{a-} for $a \in \left(\frac{\overline{x}}{2}, \overline{x}\right]$ given by:*

$$\bigwedge_{0 \le x \le \overline{x}-a} f^{a-}(x,y) = \begin{cases} \frac{1}{a\overline{x}} & x + (\overline{x} - a) \le y \le \overline{x} \\ \frac{1}{2a\overline{x}} & -x + (\overline{x} + a) \le y < x + (\overline{x} - a) \\ 0 & otherwise \end{cases}$$

$$\bigwedge_{\overline{x}-a < x \le a} f^{a-}(x,y) = \begin{cases} \frac{1}{a\overline{x}} & 0 \le y \le x - (\overline{x} - a) \\ \frac{1}{2a\overline{x}} & x - (\overline{x} - a) \le y \le x + (\overline{x} - a) \\ \frac{1}{a\overline{x}} & x + (\overline{x} - a) \le y \le \overline{x} \end{cases} \tag{7}$$

$$\bigwedge_{a < x \le \overline{x}} f^{a-}(x,y) = \begin{cases} \frac{1}{2a\overline{x}} & x - (\overline{x} - a) < y \le -x + (\overline{x} + a) \\ \frac{1}{a\overline{x}} & 0 \le y \le x - (\overline{x} - a) \\ 0 & otherwise \end{cases}$$

is a reservation price joint density function for goods $\mathbb{X}$ and $\mathbb{Y}$ with uniform marginal densities. Proof of Proposition 1b is analogous to Proposition 1a and is therefore omitted. Having established negative belt density function for the whole range of parameter a, in Propositions 2a and 2b we introduce its positive belt variant.

Proposition 2a. *Function f^{a+} for $a \in \left(0, \frac{\overline{x}}{2}\right]$ given by:*

$$\bigwedge_{0\le x\le a} f^{a+}(x,y) = \begin{cases} \frac{1}{a\overline{x}} & 0 \le y < -x + a \\ \frac{1}{2a\overline{x}} & -x + a \le y \le x + a \\ 0 & otherwise \end{cases}$$

$$\bigwedge_{a<x\le\overline{x}-a} f^{a+}(x,y) = \begin{cases} \frac{1}{2a\overline{x}} & x - a \le y \le x + a \\ 0 & otherwise \end{cases} \tag{8}$$

$$\bigwedge_{\overline{x}-a<x\le\overline{x}} f^{a+}(x,y) = \begin{cases} \frac{1}{2a\overline{x}} & x - a \le y < -x + (2\overline{x} - a) \\ \frac{1}{a\overline{x}} & -x + (2\overline{x} - a) \le y \le \overline{x} \\ 0 & otherwise \end{cases}$$

is a reservation price joint density function for goods $\mathbb{X}$ *and* $\mathbb{Y}$ *with uniform marginal densities.*

Proposition 2b. *Function* f^{a+} *for* $a \in \left(\frac{\overline{x}}{2}, \overline{x}\right]$ *given by:*

$$\bigwedge_{0\le x\le\overline{x}-a} f^{a+}(x,y) = \begin{cases} \frac{1}{a\overline{x}} & 0 \le y \le -x + a \\ \frac{1}{2a\overline{x}} & -x + a \le y < x + a \\ 0 & otherwise \end{cases}$$

$$\bigwedge_{\overline{x}-a<x\le a} f^{a+}(x,y) = \begin{cases} \frac{1}{a\overline{x}} & 0 \le y \le -x + a \\ \frac{1}{2a\overline{x}} & -x + a \le y \le -x + (2\overline{x} - a) \\ \frac{1}{a\overline{x}} & -x + (2\overline{x} - a) \le y \le \overline{x} \end{cases} \tag{9}$$

$$\bigwedge_{a<x\le\overline{x}} f^{a+}(x,y) = \begin{cases} \frac{1}{2a\overline{x}} & x - a \le y \le -x + (2\overline{x} - a) \\ \frac{1}{a\overline{x}} & -x + (2\overline{x} - a) < y \le \overline{x} \\ 0 & otherwise \end{cases}$$

is a reservation price joint density function for goods $\mathbb{X}$ *and* $\mathbb{Y}$ *with uniform marginal densities.*

Proofs of Propositions 2a and 2b are analogous to Proposition 1a and hence are omitted. In Figure 5, we graph $Domf^{a+}$ for $a \in \left(0, \frac{\overline{x}}{2}\right]$. We skip the illustration of positive belt-shaped domain for $a \in \left(\frac{\overline{x}}{2}, \overline{x}\right]$.

The reason why f^a has uniform margins is that the joint density in both triangular areas is twice bigger then density in rectangular area. From managerial perspective heavier triangular corners represent explicit market segments of consumers having either diverging valuations for both goods - that is low for the first and high for the second, or converging valuations - that is low (negative belt) or high (positive belt) reservation prices for both goods. An interesting possibility follows from the construction of f^a. Using summation of $f^{a(\pm)}(x,y)$, one should be able to construct a joint density function over $Domf^{a(\pm)} \cup\ Domf^{b(\pm)}$ which also maintains uniformity of marginal distributions. Hence our approach to bundling analysis can be applied to a wider class of demand structures. For example one could model a situation where there are four segments of consumers having converging/diverging preferences for both goods 𝕏 or 𝕐 or a preference for only one good, preserving linear demand functions for both components.

2.6 CORRELATION OF RESERVATION PRICES UNDER JOINT DENSITY f^a

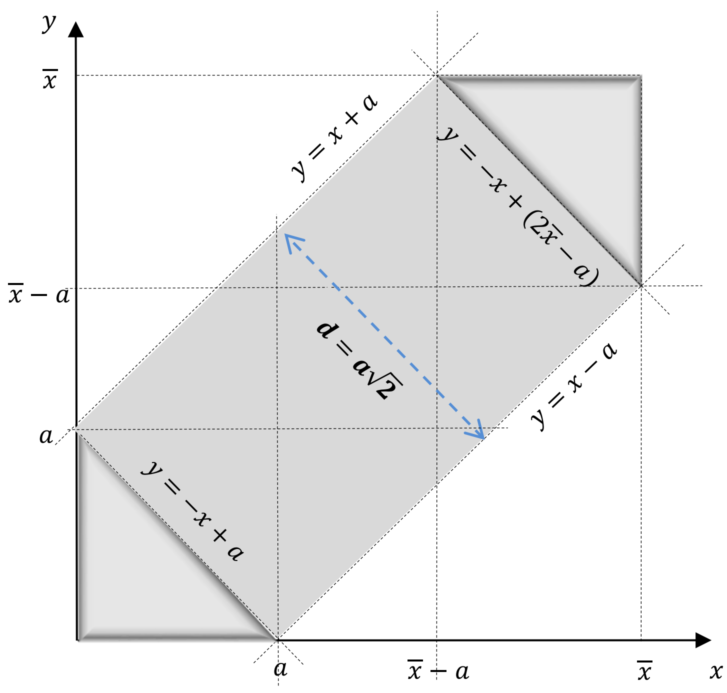
In the previous section we have proven that $\mathrm{f^a(x,y)}$ has uniform margins. In this section we derive correlation coefficient between its marginal distributions.

Proposition 3a. *For reservation price density function f^{a-} given by (6) the correlation coefficient between the consumer's reservation prices goods 𝕏 and 𝕐 is given by the following formula:*

$$\rho_{x,y} = -\frac{a^3 - 2a^2\overline{x} + \overline{x}^3}{\overline{x}^3} \quad (10)$$

For proof see Appendix. Note that for $a = 0$, we obtain perfect negative correlation $\rho_{x,y} = -1$ and for maximum value of parameter $(a = \overline{x})$ we obtain $\rho_{x,y} = 0$ (independence).

Figure 5. Positive belt: $\boldsymbol{Domf^{a+}}$ for $\boldsymbol{a} \in \left(\mathbf{0}, \frac{\overline{x}}{2}\right]$.



Proposition 3b. *For the reservation prices density function* f^{a+} *given by (8) the correlation coefficient between the consumer's reservation prices goods* $\mathbb{X}$ *and* $\mathbb{Y}$ *is given by the following formula:*

$$\rho_{x,y} = \frac{a^3 - 2a^2\overline{x} + \overline{x}^3}{\overline{x}^3} \quad (11)$$

Derivation of this formula is analogous to (10) so we leave it without a proof. For positive belt, perfect positive correlation $\rho_{x,y} = 1$ holds for $a = 0$ and $\rho_{x,y} = 0$ for $a = \overline{x}$ (independence). For both variants of joint density with belt-shaped domain, the absolute value of correlation between individual reservation prices is nonlinearly increasing in a. With formulas (10) and (11) we are equipped to analyze optimality of monopoly pure and mixed bundling strategies for the full range of correlation and marginal costs.

SOLUTION TO BUNDLING PROBLEM

1.1. PURE COMPONENTS STRATEGY

For both reservation price density functions ($f^{a-}$and f^{a+}) marginal density functions of good $\mathbb{X}$ ($f_x^{a-}$and f_x^{a+}) and good $\mathbb{Y}$ ($f_y^{a-}$and f_y^{a+}) do not depend on parameter a and are given by:

$$\bigwedge_{0\leq x\leq \overline{x}} f_x^{a-}(x) = f_x^{a+}(x) = f_y^{a-}(x) = f_y^{a+}(x) = \frac{1}{\overline{x}},$$

So, the demand functions for each good under pure components are given by:

$$d_x^{PCa-}(p) = d_x^{PCa+}(p) = d_y^{PCa-}(p) = d_y^{PCa+}(p) = 1 - \frac{1}{\overline{x}}p$$

and a profit of monopolist writes:

$$\Pi_x^{PCa-}(p) = \Pi_x^{PCa+}(p) = (p - c_x)\left(1 - \frac{1}{\overline{x}}p\right),$$

$$\Pi_y^{PCa-}(p) = \Pi_y^{PCa+}(p) = \left(p - c_y\right)\left(1 - \frac{1}{\overline{x}}p\right)$$

In this case first order condition are sufficient for derivation of optimal prices. Hence monopolist will choose the following prices and earn the following profits under PC strategy:

$$p_x^{PCa-} = p_x^{PCa+} = \frac{\overline{x} + c_x}{2}, \qquad p_y^{PCa-} = p_y^{PCa+} = \frac{\overline{x} + c_y}{2}.$$

$$\Pi_x^{PCa-}(p_x^{PCa-}) = \Pi_x^{PCa+}(p_x^{PCa+}) = \frac{(\overline{x} - c_x)^2}{4\overline{x}}$$

$$\Pi_y^{PCa-}\left(p_y^{PCa-}\right) = \Pi_y^{PCa+}\left(p_y^{PCa+}\right) = \frac{\left(\overline{x} - c_y\right)^2}{4\overline{x}}$$

$$\Pi^{PCa-}\left(p_x^{PCa-}, p_y^{PCa-}\right) = \Pi^{PCa+}\left(p_x^{PCa+}, p_y^{PCa+}\right) = \frac{(\overline{x} - c_x)^2 + \left(\overline{x} - c_y\right)^2}{4\overline{x}}$$

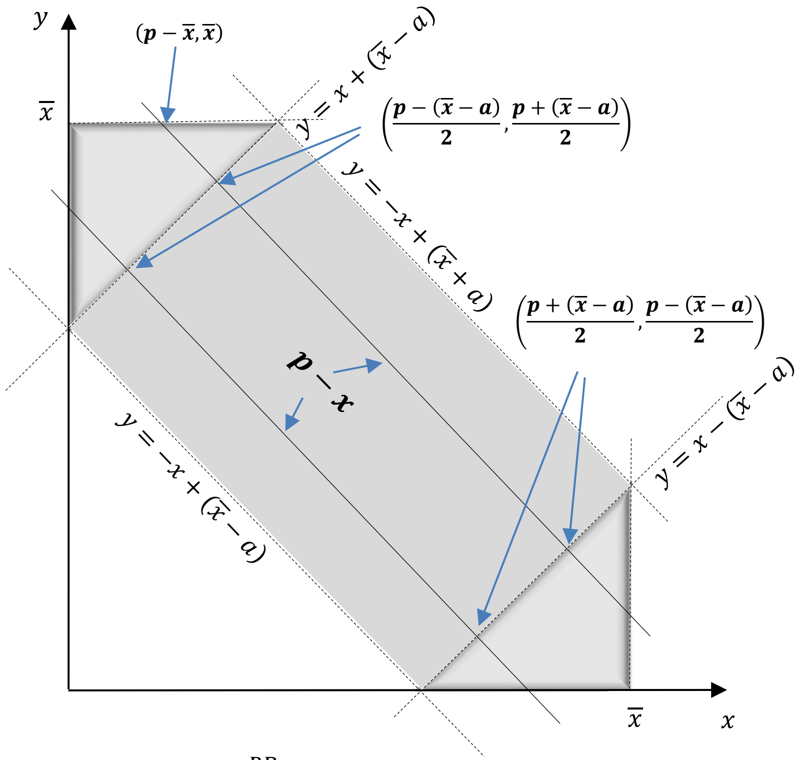
1.2. PURE BUNDLING

Profitability of pure bundling for both the cases of positive and negative correlation needs to be analyzed separately.

1.2.1. NEGATIVE CORRELATION: f^{a-} JOINT DENSITY

The graph of the demand space f^{a-} under PB strategy is presented in Figure 7. Depending on the price for the bundle, monopolist might sell to all or only to a subset of consumers located in rectangular and triangular areas of the belt-shaped domain. Thus several cases must be considered to locate buyers depending on the price.

Figure 7. PB strategy under negative belt: $\boldsymbol{Domf^{a-}}$.



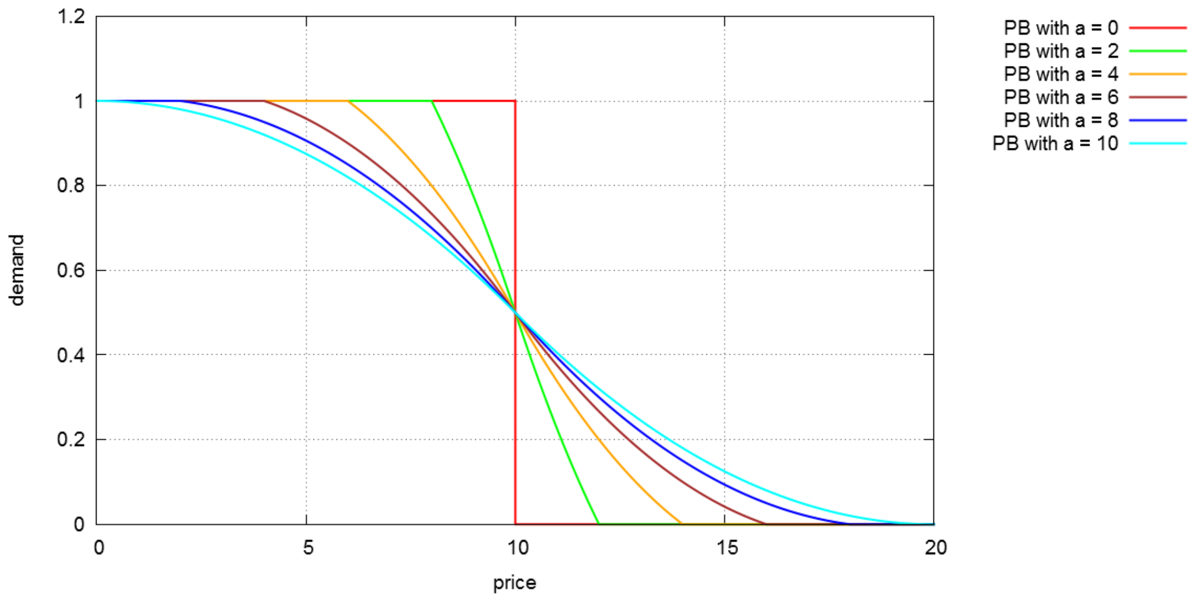
Demand function for the bundle d_b^{PBa-} is given by:

$$d_b^{PBa-}(p) = \begin{cases} 1 & 0 \leq p \leq \overline{x} - a \\ 1 - \dfrac{1}{2a\overline{x}} p\big(p - (\overline{x} - a)\big) & \overline{x} - a < p \leq \overline{x} \\ \dfrac{1}{2a\overline{x}} \big((\overline{x} + a) - p\big)(2\overline{x} - p) & \overline{x} < p \leq \overline{x} + a \\ 0 & p > \overline{x} + a \end{cases}$$

Demand for bundles is nonlinear in price. In Figure 8 we graph demand function for $\overline{x} = 10$ and different values of parameter $a \in (0,10)$.

Figure 8 shows that there is a strong dependency between parameter a and curvature of demand function under pure bundling. For the case of perfect negative correlation $(a = 0)$, demand curve is inelastic and monopolist can capture the whole consumer surplus. Consequently for higher values of parameter a, demand becomes more elastic. All demand curves intersect at price $p = \overline{x}$ for which 50% consumers buy the bundle. This is a consequence of the symmetry of $Domf^{a-}$ over diagonal of the square $[0,\overline{x}] \times [0,\overline{x}]$.

Figure 8. Demand function under PB, d_b^{PBa-} for $\overline{x} = 10$ and $a \in (0,10)$.



The profit for this demand function is given by:

$$\Pi^{PBa-}(p) = \begin{cases} (p - c_b) & 0 \le p \le \overline{x} - a \\ (p - c_b)\left(1 - \dfrac{1}{2a\overline{x}} p\big(p - (\overline{x} - a)\big)\right) & \overline{x} - a < p \le \overline{x} \\ (p - c_b)\left(\dfrac{1}{2a\overline{x}}\big((\overline{x} + a) - p\big)(2\overline{x} - p)\right) & \overline{x} < p \le \overline{x} + a \\ 0 & p > \overline{x} + a \end{cases}$$

First order conditions yield a closed form expression for optimal price of monopolist under PB strategy:

$$p^{opt} = \frac{\sqrt{\overline{x}^2 + (4a - c_b)\overline{x} + c_b^2 + ac_b + a^2} + \overline{x} + c_b - a}{3}.$$

Optimal PB profit is given by:

$$\Pi^{PBa-}(p^{opt}) = (p^{opt} - c_b)\left(1 - \frac{1}{2a\overline{x}} p^{opt}\big(p^{opt} - (\overline{x} - a)\big)\right).$$

When consumer valuations are negatively correlated, monopolist using pure bundling always optimizes price on the same segment of demand curve, but his profit obviously depends on degree of correlation as marked by parameter a. In Figure 9 we graph profit function Π^{PBa-} for $\overline{x} = 10, c_b = 0$ and different values of correlation parameter $a \in (0{,}10)$. As can be seen there, profit function behaves in close correspondence with the demand function shown in Figure 8. Profit takes the highest value for $a = 0$ (perfect negative correlation) and the lowest for $a = \overline{x}$ (independence). Marginal costs of a bundle c_b also have a substantial impact on the profitability of pure bundling. When costs are positive, those customers who have strong preference for only one good but buy a bundle are in fact subsidized to do so. Thus pure components might generate higher profit then pure bundling for certain level of marginal costs. In Figure 10 below, we graph profit functions Π^{PBa-} and Π^{PCa-} for $\overline{x} = 10, a = 2$ and different values of cost parameter $c_b \in (0{,}8)$. Note that $a = 2$ corresponds to strong negative correlation $\rho_{x,y} = -0{,}92$. Two immediate observations follow: First, optimal prices (corresponding to maximal values of profit functions) are increasing with marginal costs of component goods and the bundle which obviously restricts total demand and reduces total profits of monopolist. Next, it looks that for low cost levels $c_b = 0$ and $c_b = 2$ profit from pure bundling is greater than from pure components as shown by dashed red and green lines. However even for very strong negative correlation (note that $a = 2$ corresponds to strong negative correlation $\rho_{x,y} = -0{,}92$), this relation becomes reversed for sufficiently large marginal cost. Starting from $c_b = 4$ pure components strategy yields higher profit then pure bundling. Apparently subsidization becomes too costly and cannot be recovered by the profits from central segment of the demand function.

Later on we show that for any degree of correlation, including positive, there exist marginal cost c_b^* for which both strategies generate the same level of profit. Moreover, this critical value of marginal costs for which monopolist is indifferent between both strategies is <u>decreasing</u> with parameter a. This means that the range of costs for which pure bundling outperforms pure components shrinks with correlation of consumer valuations to zero. In fact our model shows that for $\rho > 0.2$ bundling is never better then pure components, but still can be worse off if marginal costs are large enough.

Figure 9. Profit function under PB, Π^{PBa-} for $\bar{x} = 10, c_b = 0$ and a $\in (0, 10)$.

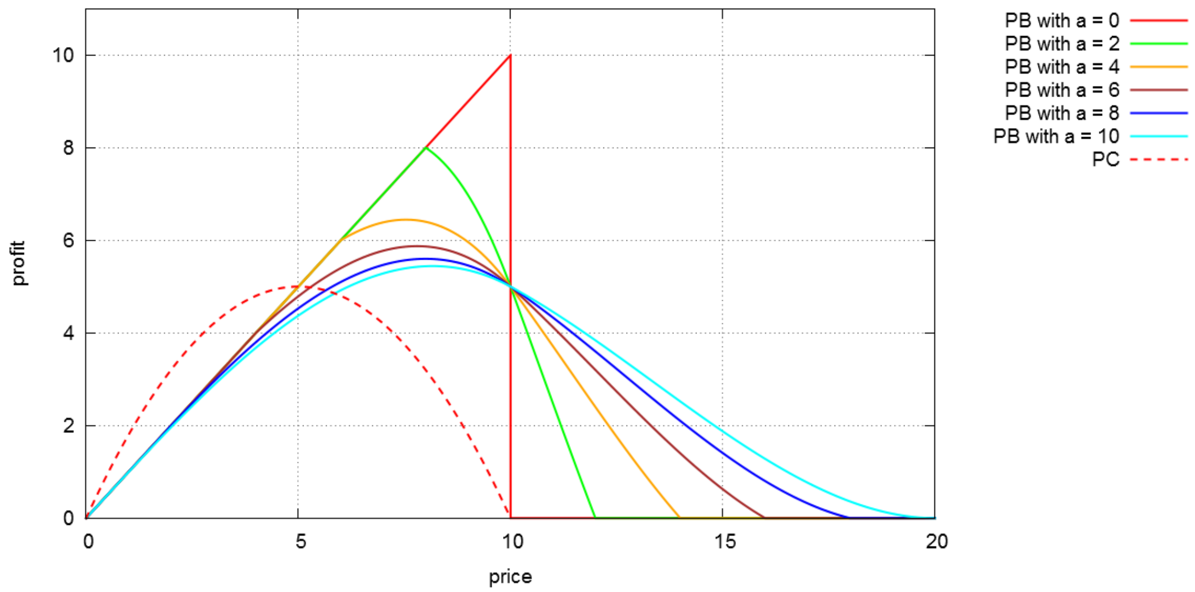
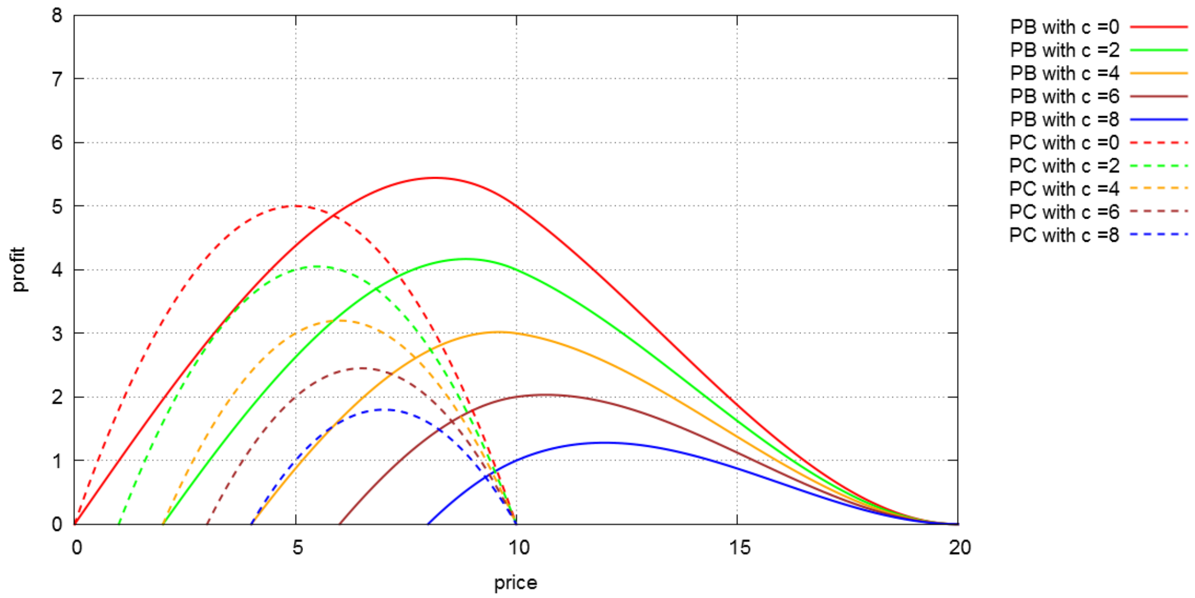


Figure 10. Profit functions Π^{PBa-}, Π^{PCa-} for $\bar{x} = 10$, a $= 2$ and $c_b \in (0, 8)$.

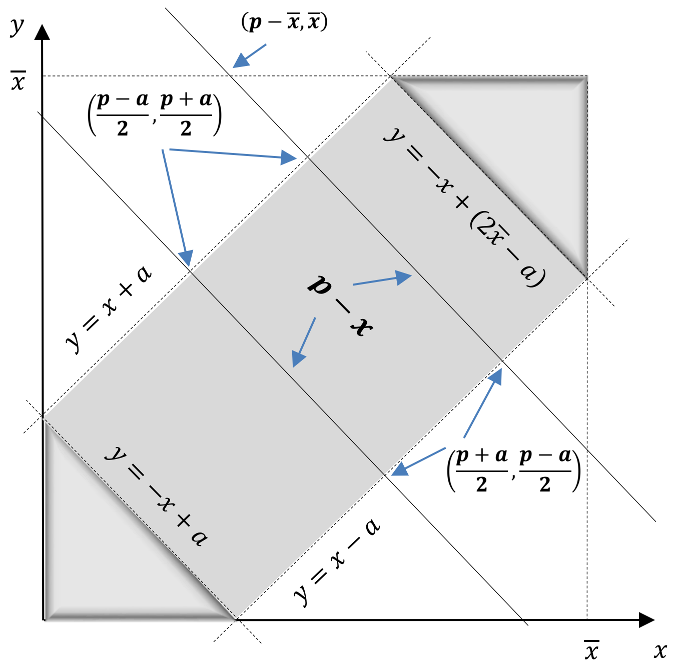


In the next subsection, we analyze a pattern of bundling optimality for the case of positively correlated demands.

1.2.2. Positive correlation: the f^{a+} joint density

The graph of the demand space f^{a+} under PB strategy is presented in Figure 11. Depending on the price for the bundle, monopolist might sell only to consumers located in the upper triangle or also to some or all consumers located in the rectangular area of the right belt-shaped domain. Eventually, if he sets very low price, also consumers from the lower tringle will buy.

Figure 11. PB strategy under left belt-shaped $\boldsymbol{Domf^{a+}}$.



Thus again several cases must be considered to locate buyers depending on the price. In this case the demand function of the bundle d_b^{PBa+} is given by:

$$d_b^{PBa+}(p)=\begin{cases} 1-\frac{1}{2a\bar{x}}p^2 & 0\le p\le a \\ 1-\frac{1}{2a\bar{x}}ap & a<p\le 2\bar{x}-a \\ \frac{1}{2a\bar{x}}(2\bar{x}-p)^2 & 2\bar{x}-a<p\le 2\bar{x} \\ 0 & p>2\bar{x} \end{cases}$$

For the right belt-shaped domain demand function is composed of 3 segments out of which the middle one is linear.

The profit function from pure bundling for positive belt: Π^{PBa+} is given by:

$$\Pi^{PBa+}(p)=\begin{cases} (p-c_b)\left(1-\frac{1}{2a\bar{x}}p^2\right) & 0\le p\le a \\ (p-c_b)\left(1-\frac{1}{2a\bar{x}}ap\right) & a<p\le 2\bar{x}-a \\ (p-c_b)\left(\frac{1}{2a\bar{x}}(2\bar{x}-p)^2\right) & 2\bar{x}-a<p\le 2\bar{x} \\ 0 & p>2\bar{x} \end{cases}$$

Analytical optimization for positive correlation case is more lengthy and hence will be avoided here.[8] Instead we show graphically how optimal profits from both strategies behave in response to changes in degree of correlation and level of marginal costs. In Figure 12 we draw profit function Π^{PBa+} for $\overline{x} = 10, c_b = 0$ and different values of parameter $a \in (0{,}10)$. As can be seen from there, monopolist maximizes profit from pure bundling always on the linear (least elastic) segment of demand curve. The optimal profit takes the highest value for $a = \overline{x} = 10$ (independence) and then for $a < 8$ becomes irresponsive to correlation of reservation prices. In case of positive correlation our model predicts that pure bundling can hardly generate higher profits. Even for moderate levels of correlation pure components yield equal profit as pure bundling.

Figure 12. Profit function under PB, Π^{PBa+} for $\overline{x} = 10, c_b = 0$ and $a \in (0, 10)$.

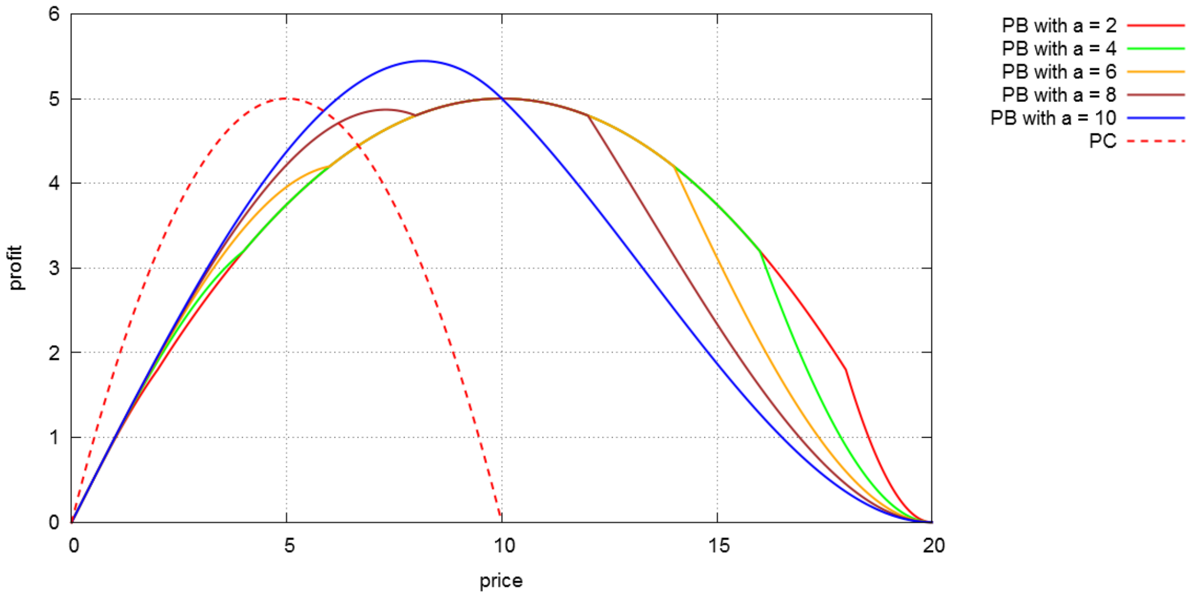
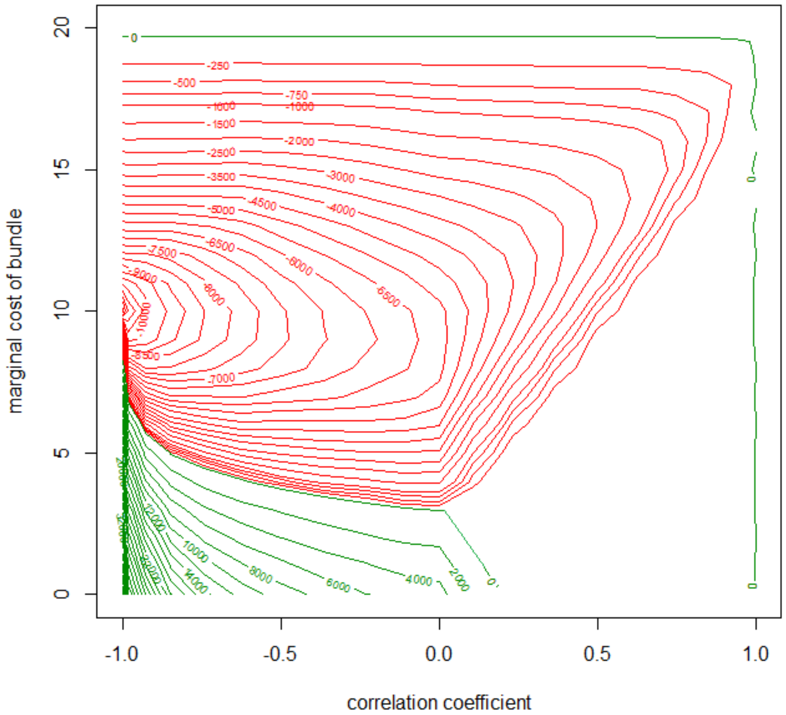


Figure 13 which compares performance of pure components and pure bundling using iso-difference curves of optimal profits $(\Pi^{PB} - \Pi^{PC})$ graphed in correlation-cost space. Full independence case is marked in the middle of the graph. Left part of the Figure 13 confirms what has been already noted in the previous subsection for negative correlation case. Now we focus attention on the right part which represents positive belt: f^{a+} with $corr \in (0{,}1]$. The zero difference curve marks correlation-cost combinations which yield equal profit from pure bundling and pure components. Red isocurves mark the area where pure bundling is profit-

[8] Note that depending on the value of correlation monopolist will optimize profits on the first or on the second segment of the demand curve. The intuition behind this observation is that for sufficiently low correlation the width of the belt get large enough and monopolist will choose to set lower price sell to more customers including those from more dense lower triangular area. However if the belt becomes narrower (or the costs increase) the seller will rise price of the bundle and move to the linear segment, where optimal price does not depend on correlation (see Figure 12).

inferior to pure components and the green ones mark the space where pure bundling yields greater profits. For sufficiently positive correlation (approximately $\rho > 0.2$) pure bundling is indeed never better from separate sales and more importantly becomes strictly worse for sufficiently high marginal costs. Intuition which stands behind this result is the following: With growing correlation the positive belt shrinks in width implying that individual demands become more and more homogeneous. As a result monopolist stops benefitting from the package, because of limited possibility to reduce heterogeneity of consumer demands. On the other hand if marginal costs are large enough, monopolist using pure bundling engages in implicit subsidization of consumers. Pure components can avoid this problem while reaching effectively the same people from the profitable central segment of pure bundling demand curve. This explains why this strategy outperforms pure bundling.[9]

Figure 13. Iso-difference curves for optimal profits from PB and PC strategies in correlation-cost space.



1.3. MIXED BUNDLING

[9] In a perfect correlation case there is no subsidization at all, hence both strategies converge in profits for in the whole range of marginal costs

Unlike pure bundling, analytical derivation of demand and profit functions under f^a density is complicated due to large number of cases which need to be considered for the three price parameters. For this reason we switch to simulation techniques to obtain a complete picture.[10]

We use the following algorithm. First we generate a fixed sample of $k = 10000$ customers with uniformly distributed reservation prices: $r \sim U[0,10]$ for each good according to the joint density f^a. Then for each variant of width parameter $a \in [-10,10]$ and marginal cost $c \in [0,20]$ both taken with a step 1 we find optimal prices under MB and PB. We take a smaller search step for price (0.1) to ensure sufficient level of precision – at the expense of considerable computation burden. For mixed bundling our algorithm checks 100x100x200 price combinations for each of 21x21 a,c variants in the sample of 10000 customers. We did robustness checks with regards to alternative parameters of simulation and are confident with the precision and reliability of results obtained for this set of parameters. In what follows we present the profit performance of mixed bundling against pure bundling and pure components in relative terms (see Figure14) as well as compare simulated optimal prices with the use of iso-difference curves (in Figure 15).

Figure 14 indicates mixed bundling weakly dominates both separate sales and pure bundling with respect to profit levels. This result is expected because MB contains in itself the remaining two pricing instruments. Hence with mixed bundling monopolist can always replicate any PC or PB strategy if it is optimal. What is remarkable and striking about Figure 14 is that mixed bundling offers very small profit gains as shown in relative terms in the left panel. Only in the bottom-left segment of the correlation-cost space advantage of mixed bundling can be considered attractive from managerial perspective.[11] This segment marks either (i) a combination of high to medium negative correlation (between -1 and -0,5) and medium to low levels of marginal cost (between 50% and 25% of a mean bundle valuation) or (ii) a combination of medium to low negative correlation and costs close to zero (less than 25% of a mean bundle valuation). For close to independent or positively correlated reservation prices the profit gains from mixed bundling are close to zero regardless of the level of costs. The same is true for cost exceeding 50% of a mean valuation, regardless of degree of correlation

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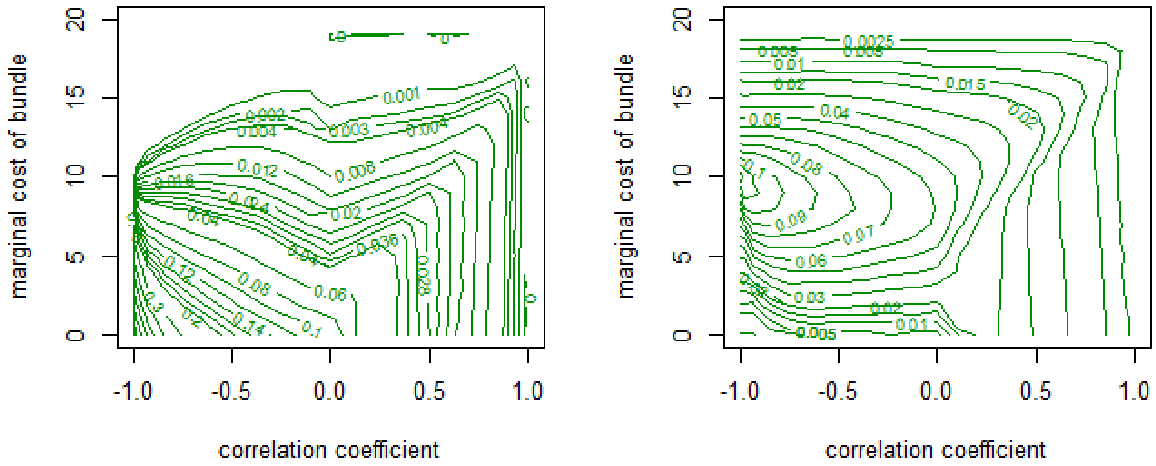
[10] Simulations have been done in R. The code is available on request from the authors under CC-BY 4.0 license.

[11] There is always a problem about what magnitude of gains is attractive. We acknowledge that there is no single answer to this question. The answer depends differs across industries and depends on the magnitude of additional fixed and variable costs resulting from implementation of bundling strategy. Therefore we have arbitrarily set the threshold for attractive gains at rather modest 10% level relative to operating surplus from pure components.

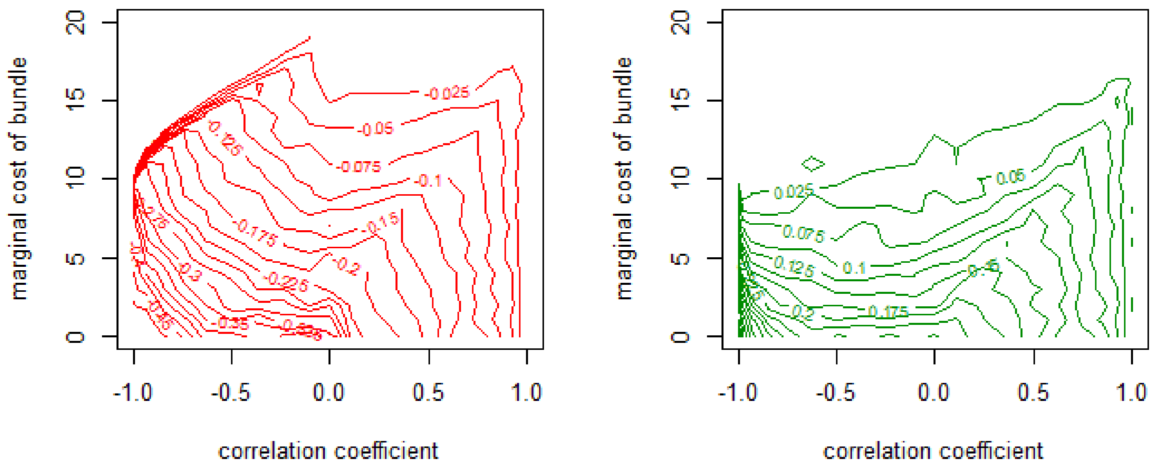
components valuations. Comparing optimal prices in mixed bundling and pure components which explains why the gains from mixed bundling diminish so rapidly.

Figure 14. Simulated isocurves for the relative difference in optimal profits in correlation-cost space. Left panel presents $(\Pi^{MB} - \Pi^{PC})/\Pi^{PC}$, right panel presents $(\Pi^{MB} - \Pi^{PB})/\ \Pi^{PB}$.



This insight is provided in Figure 15, which shows the magnitude of discount for buying a bundle offered by mixed bundling strategy (left panel) and also compares the sum of individual prices under mixed bundling and pure components (right panel).

Figure 15. Simulated isocurves for the relative difference in optimal prices in correlation-cost space. Left panel presents $(p_b^{MB} - p_1^{MB} - p_2^{MB})/(p_1^{MB} + p_2^{MB})$, right panel presents $(p_1^{MB} + p_2^{MB} - p_1^{PC} - p_2^{PC})/(p_1^{PC} + p_2^{PC})$,



Two interesting observations follow from Figure 15. First, in the segment of greatest relatives gains mixed bundling works through a classical pattern, namely a combination of higher prices for component products relative to pure components and considerable discount on price for the bundle. In this way mixed bundling can capture greater portion of a surplus from those customers who have largely heterogeneous valuations and at the same time realize profits from those customers who have moderate valuations for both goods. Second, if cost are sufficiently high or correlation coefficient is largely positive mixed bundling converges in prices to pure components.[12] Although in both cases convergence effectively drives profit gains from mixed bundling to zero it is governed by two distinct mechanisms. Under growing costs, selling a bundle at a discounted price will require implicit subsidization and monopolist will gradually increase price for a bundle relative to component prices. Eventually mixed bundling will converge to pure components and monopolist will stop selling bundles at all. Interestingly this result holds even for large negative correlation of reservation prices stressing a critical role of costs in solving bundling problem. For example, for $\rho \rightarrow -1$ all consumer will value package for 10, so if costs exceed this level, monopolist starts to generate loss from selling package. On the other hand under large positive correlation there are limited possibilities for reducing demand heterogeneity and hence the gains from selling bundle become very limited. Although monopolist can still earn extra profit from mixed bundling by extracting little bit more surplus from the most divergent customers at the expense of lower revenues from those buying a bundle, eventually for perfect positive correlation mixed bundling converges in prices to pure components.

We note two more things about mixed bundling from Figure 15. First, the zero difference isocurve in prices shown in the left panel indicates that monopolist will choose the same levels of individuals prices under pure components and mixed bundling. However, naturally those prices will rise with costs as indicated by optimal prices formula for pure bundling (see Section 3.1). Second, for $\rho > 0$ the threshold level of costs for which gains from mixed bundling are zero increases with degree of correlation. This happens because with growing ρ some customers are willing to pay high price for the bundle and monopolist will be able to make positive (albeit small) profit thanks to discriminatory power of mixed bundling strategy.

Taken as whole our results indicate that in our model mixed bundling can rise profits maximum by 70% compared to pure components and 11% compared to pure bundling. The distribution of gains in correlation-cost space reveals that the sufficient gains from mixed bundling can be realized only under specific combination of <u>both</u> very low marginal costs and relatively high negative correlation of valuations. In all other cases the gains from mixed bundling are

[12] We use the term convergence intuitively, meaning that individual prices are the same and a discount for buying a package is zero, which makes price for the bundle ineffective.

negligible, undermining rationale for implementation of this strategy. Our model reveals also the critical role of costs for bundling problem. This is best seen for large negative correlation, where profit gains from bundling are the most sensitive to cost increases because of the largest reduction in demand heterogeneity. We believe that both conclusions from our linear model are interesting from managerial perspective as they indicate that bundling will better enough to justify its implementation only for low cost services and only under specific pattern of component demands dependency.

SUMMARY AND CONCLUSIONS

In this paper we look at the magnitude of profit gains from mixed bundling. We introduce a model of reservation price space which yields linear market demands for both component goods. Our analysis of bundling gains is carried out for the whole possible range of marginal costs and degree of correlation of valuations of bundle components. Our results in principle show that bundling can generate sufficient gains only in low marginal cost industries provided that demands are sufficiently negatively correlated. In this way our results conform with practical observations that bundling is applicable mainly to ICT services or more broadly to information goods.[13]

Our study contributes to the bundling problem in two ways. First, we focus on profit gains rather than simply on profit dominance. We believe that this perspective is more insightful from managerial perspective. We argue that weakly dominance result has no practical importance for managers, as it does not tell under which conditions mixed bundling can be expected to strictly outperform the two remaining sales formats and by how much. Our perspective requires shifting from local improvement considerations towards global optimum search. Secondly, we propose analytical framework that has some desired properties which enable us to carry out global analysis. More specifically, we introduce a joint reservation price space which on one hand preserves linearity of market demand for individual goods – a feature which often is assumed not only in theoretical models but serves as starting point in practical analyses. Our so called 'belt shaped' density function allows to capture underlying correlation with a single parameter and yields analytically tractable demand and profit functions under pure bundling strategy. Approaches that have been explored so far in normative bundling literature, like uniform and

[13] Fixed line services typically meet both criteria. Marginal costs of broadband and telephony are close to zero and market demand is segmented in a way which ensures negative correlation. Younger consumers do not value fixed line telephony as they use IP or mobile telephony. On the other hand older customers are not interested in broadband but are attached to traditional telephony service.

Gaussian densities of customer valuations, miss one or the other element. Our model of reservation price space encompasses several situations so far analyzed in literature as special cases, like perfect correlation or full independence of valuations.

Two results follow from our study. First, mixed bundling is weakly dominant in payoff over pure bundling and pure components for all levels of marginal costs and any degree of correlation of consumers' reservation prices. This common sense result is just a confirmation of earlier findings. More importantly, we show that benefits from implementation of mixed bundling strategy vary significantly and are negligible for the wide range of correlation-cost combinations, as appears in our model. We assess bundling gains with arbitrary assumption that 10% markup over profits from pure components is sufficient to cover its implementation. The fact that mixed bundling rarely exceeds this rather modest threshold is the most astonishing finding from our study.

Generally our results point to the critical role of marginal costs for the optimality of bundling. Earlier bundling literature has argued that under strong negative correlation bundling reduces demand heterogeneity and makes market demand less elastic. We show that with growing marginal costs pure bundling sooner starts to generate losses because of homogeneity of bundling valuations.

Our results are not encouraging for implementation of bundling. Sufficient profit gains from mixed bundling can be expected only under specific combination of <u>both</u> very low marginal costs and relatively high negative correlation of valuations. Given that usually firms have limited knowledge about dependency of individual demands, they might choose prefer to use a safer pure components strategy to avoid risk of uncontrolled subsidization of its customers. Moreover, with incomplete information about the demand it might be difficult to find optimal vector of mixed bundling prices.

Our model is limited and could be extended several ways. By construction our 'belt-shaped' reservation price space corresponds to market with two distinct segments (marked by triangular areas with double density) and also a mass of consumers in between. One can try to use mixtures of bivariate distributions to cover demand structures, which are closer to reality. Four segments of consumers with divergent and convergent valuations for both goods would be an interesting case to study. Another limitations of current study is that we have focused solely on profit gains and do not consider welfare changes. Our analysis is also restricted to the usual simplifying assumptions adopted in bundling models such as additivity of reservation prices and unit costs.

Relaxing each of them would bring more in depth insights about potential of bundling in real market applications.

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APPENDIX

Proof of Proposition 2a.

First, we check that function f^{a-}has properties of the reservation prices density function requested in Definition 1:

1. $\mathrm{Dom} f = [0,\bar{x}] \times [0,\bar{x}] \subset \overline{\mathbb{R}}^2$, where $\bar{x} > 0$,
2. $\mathrm{Ran} f \subset \mathbb{R}$ and $f^{a-}(x,y) \geq 0$ for all (x,y) in $[0,\bar{x}] \times [0,\bar{x}]$,
3. $\iint_{[0,\bar{x}]\times[0,\bar{x}]} f(x,y)dydx$ can be calculated in the following way:

$$= \int_0^a dx \int_{-x+(\bar{x}-a)}^{x+(\bar{x}-a)} f^{a-}(x,y)dy + \int_0^a dx \int_{x+(\bar{x}-a)}^{\bar{x}} f^{a-}(x,y)dy +$$

$$+\int_{a}^{\overline{x}-a} dx \int_{-x+(\overline{x}-a)}^{-x+(\overline{x}+a)} f^{a-}(x,y)dy + \int_{\overline{x}-a}^{\overline{x}} dx \int_{0}^{x-(\overline{x}-a)} f^{a-}(x,y)dy +$$

$$+\int_{\overline{x}-a}^{\overline{x}} dx \int_{x-(\overline{x}-a)}^{-x+(\overline{x}+a)} f^{a-}(x,y)\, dy = 1.$$

Since $f^{a-}$is indeed a proper density function we calculate marginal densities. The reservation prices density function of good $\mathbb{X}$: f_x^{a-} is given by:

$$\bigwedge_{0\le x\le a} f_x^{a-}(x) = \int_{-x+(\overline{x}-a)}^{x+(\overline{x}-a)} f^{a-}(x,y)dy + \int_{x+(\overline{x}-a)}^{\overline{x}} f^{a-}(x,y)dy = \frac{1}{\overline{x}}$$

$$\bigwedge_{a<x\le\overline{x}-a} f_x^{a-}(x) = \int_{-x+(\overline{x}-a)}^{-x+(\overline{x}+a)} f^{a-}(x,y)dy = \frac{1}{\overline{x}}$$

$$\bigwedge_{\overline{x}-a<x\le\overline{x}} f_x^{a-}(x) = \int_{0}^{x-(\overline{x}-a)} f^{a-}(x,y)dy + \int_{x-(\overline{x}-a)}^{-x+(\overline{x}+a)} f^{a-}(x,y)\, dy = \frac{1}{\overline{x}}.$$

The reservation prices density function of good $\mathbb{Y}$: f_y^{a-} is given by:

$$\bigwedge_{0\le y\le a} f_y^{a-}(y) = \int_{-y+(\overline{x}-a)}^{y+(\overline{x}-a)} f^{a-}(x,y)dx + \int_{y+(\overline{x}-a)}^{\overline{x}} f^{a-}(x,y)dx = \frac{1}{\overline{x}}$$

$$\bigwedge_{a<y\le\overline{x}-a} f_y^{a-}(y) = \int_{-y+(\overline{x}-a)}^{-y+(\overline{x}+a)} f^{a-}(x,y)dx = \frac{1}{\overline{x}}$$

$$\bigwedge_{\overline{x}-a<y\le\overline{x}} f_y^{a-}(y) = \int_{0}^{y-(\overline{x}-a)} f^{a-}(x,y)dx + \int_{y-(\overline{x}-a)}^{-y+(\overline{x}+a)} f^{a-}(x,y)\, dy = \frac{1}{\overline{x}}.$$

Thus both marginal densities: f_x^{a-} and : $f_y^{a-}$are uniform for $a \in \left(0, \frac{\overline{x}}{2}\right]$.

Q.E.D

Proof of Proposition 3a.

For the case of only two random variables X, Y generic formula for correlation writes:

$$\rho(X,Y) = \frac{COV(X,Y)}{VAR(X)\cdot VAR(Y)} = \frac{E(XY) - E(X)\cdot E(Y)}{VAR(X)\cdot VAR(Y)} \qquad (1a)$$

where the nominator of (1a) stands for covariance between variables and denominator denotes a product of their variances. Because $f_x^{a-}(x) = \frac{1}{\overline{x}}$ for $x \in [0, \overline{x}\,]$

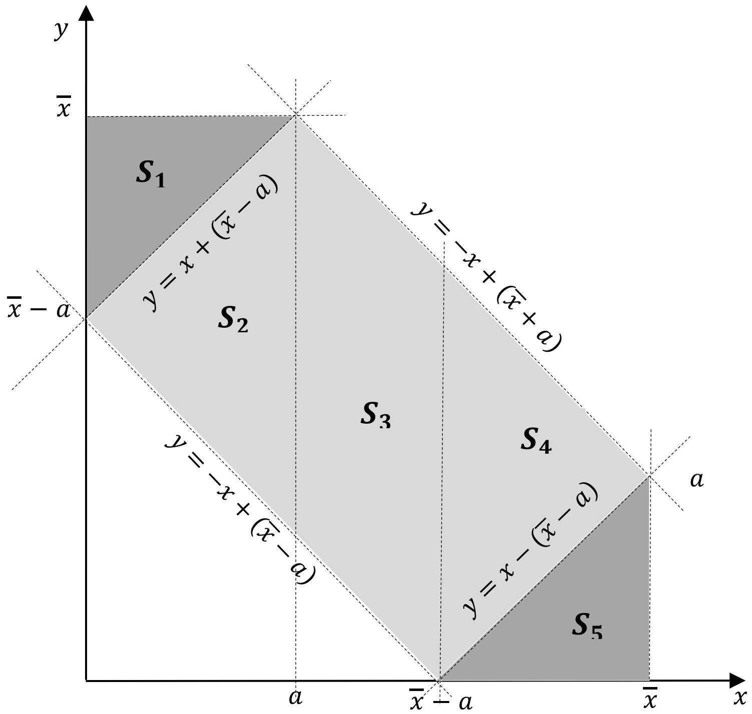
$$E(X) = E(Y) = \frac{\overline{x}}{2} \text{ and } VAR(X) = VAR(Y) = \frac{\overline{x}^2}{12}$$

So it remains to calculate $E(XY)$ which by definition is

$$E(XY) = \iint_{(x,y)\in Domf^{a-}} xyf^{a-}(x,y)\, dxdy \qquad (2a)$$

To conduct this integration we divided $Domf^{a-}$ for $a \in \left(0, \frac{\overline{x}}{2}\right]$ as shown in Fig. A.1.

Figure A.1. Graph of the subsets of Domf^{a-} **for** $a \in \left(0, \frac{\bar{x}}{2}\right]$.



Where:

$$S_1 = \{(x, y) \in \mathrm{Domf}^{a-} | 0 \le x \le a \text{ and } x + (\bar{x} - a) \le y \le \bar{x}\}$$
$$S_2 = \{(x, y) \in \mathrm{Domf}^{a-} | 0 \le x \le a \text{ and } -x + (\bar{x} - a) \le y < x + (\bar{x} - a)\}$$
$$S_3 = \{(x, y) \in \mathrm{Domf}^{a-} | a < x \le \bar{x} - a \text{ and } -x + (\bar{x} - a) \le y \le -x + (\bar{x} + a)\}$$
$$S_4 = \{(x, y) \in \mathrm{Domf}^{a-} | \bar{x} - a < x \le \bar{x} \text{ and } x - (\bar{x} - a) < y \le -x + (\bar{x} + a)\}$$
$$S_5 = \{(x, y) \in \mathrm{Domf}^{a-} | \bar{x} - a < x \le \bar{x} \text{ and } 0 \le y \le x - (\bar{x} - a)\}$$

Expected value in (2a) is the sum of five integrals

$$E(XY) = \sum_{i=1}^{5} \left[\int_{x \in S_i} x \left(\int_{y \in S_i} y f^{a-}(x, y)\, dy \right) dx \right]$$

which can be easily evaluated for each S_i:

For $i = 1$

$$\int_0^a x \left(\int_{x+(\bar{x}-a)}^{\bar{x}} y \frac{1}{a\bar{x}}\, dy \right) dx = \frac{a^2(4\bar{x} - a)}{24\bar{x}}$$

For $i = 2$

$$\int_{x=0}^{a} x \left(\int_{-x+(\bar{x}-a)}^{x+(\bar{x}-a)} y \frac{1}{2a\bar{x}}\, dy \right) dx = \frac{a^2(\bar{x} - a)}{3\bar{x}}$$

For $i = 3$

$$\int_{a}^{\bar{x}-a} x\left(\int_{-x+(\bar{x}-a)}^{-x+(\bar{x}+a)} y\frac{1}{2a\bar{x}}\,dy\right)dx = \frac{4a^3 - 6a^2\bar{x} + \bar{x}^3}{6\bar{x}}$$

For $i = 4$

$$\int_{\bar{x}-a}^{\bar{x}} x\left(\int_{x-(\bar{x}-a)}^{-x+(\bar{x}+a)} y\frac{1}{2a\bar{x}}\,dy\right)dx = \frac{a^2(3\bar{x} - 2a)}{6\bar{x}}$$

For $i = 5$

$$\int_{\bar{x}-a}^{R} x\left(\int_{0}^{x-(\bar{x}-a)} y\frac{1}{a\bar{x}}\,dy\right)dx = \frac{a^2(4\bar{x} - a)}{24\bar{x}}$$

Finally, given $E(XY) = -\frac{a^3-2a^2\bar{x}-2\bar{x}^3}{12\bar{x}}$ we obtain correlation coefficient as in formula (10) Q.E.D.