Advanced Game Theory.
Solutions to selected problems

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Problem 2

An entrepreneur is considering three possible actions with respect to his business, yielding different returns in USD10^6 depending on market conditions, happening with prob. p_l, p_m, p_h respectively, see Table 1.

a) Assume utility function $u(x) = x$. For each strategy specify the conditions under which it is optimal. Which decisions are rationalizable and why?

b) Suppose that all states of the world are equally likely. Which strategy would an expected value maximizer choose?

c) What about a decision maker with utility function $u(x) = x^{0.5}$?

Solution Write down the expression for EU of each option. Then comparing them to obtain inequalities showing for what p_l/p_h ratio "invest" is the optimal choice, for what it is "stay" and for what "withdraw" (note that the prob. of medium doesn't matter at all, as payoffs are identical for all options in this case). Because each option is optimal for some probs., all are rationalizable. If all options are equally likely, the entrepreneur should invest. If his utility function changed to $u(x) = x^{0.5}$, "stay" would turn out to be optimal because $1/3(1 + 4 + 7) < 1/3(3 + 4 + 6)$.

Problem 5

In each of the following two-person games find

a) any dominant strategies

Table 1:

strategy; market demand	low	medium	high
invest	1	16	49
stay	9	16	36
withdraw	16	16	16

Table 2:

	L	M	R
U	0;2	2;0	3;6
I	4;1	0;2	2;0
D	1;2	3;0	5;1

Table 3:

	L	M	R
U	4;4	3;2	2;0
I	2;3	5;5	-3;1
D	-1;4	0;3	1;6

b) safest strategies

c) all Nash Equilibria

Solution In the first game U is dominated by D. Then R becomes dominated by L. In the remaining 2x2 game we find a mixed equilibrium – $p_I, p_D = 1 - p_I$ can be found by equating $EU(L)$ and $EU(M)$. $q_L, q_M = 1 - q_L$ can by found by equating $EU(I)$ and $EU(D)$. Similarly for the second game.

Problem 9 Two business partners independently and simultaneously decide on the level of effort they put in their joint project. With effort levels e_1 and e_2 the total gross proceeds are $\Pi(e_1, e_2) = 4(e_1 + e_2 + \frac{e_1 e_2}{4})$, to be shared equally by the partners. The cost associated with effort e_i of any player i is e_i^2. Assume that effort levels are real numbers on $[0, 4]$. Each player seeks to maximize his share of proceeds minus his individual cost.

a) Find the pure strategy NE of this game. What is each player's net gain?

b) What is the maximum possible net gain per player. What effort levels must be chosen to obtain it?

c) Show that strategies $e_1 = 0.5$ and $e_2 = 3$ are dominated.

Solution Player's 1 goal function is $2(e_1 + e_2 + \frac{e_1 e_2}{4}) - e_1^2$. Taking FOC we get $2(1 + \frac{e_2}{4}) - 2e_1 = 0$ or $e_1 = 1 + \frac{e_2}{4}$. SOC is satisfied. By symmetry we find $e_1 = e_2 = \frac{4}{3}$. We also see that when $e_1 = 0.5$ then we have that $2(1 + \frac{e_2}{4}) - 2e_1 \geq 2 - 1 > 0$ so it always pays to exert more effort. Hence $e_1 = 0.5$ is dominated. Similar reasoning for $e_2 = 3$. To find maximum possible net gain we maximize $4(e_1 + e_2 + \frac{e_1 e_2}{4}) - e_1^2 - e_2^2$ wrt. e_1 and e_2 finding $e_1 = e_2 = 4$.

Bonus (Malawski) Two firms are producing goods A and B which are imperfect substitutes. Demand for firm A's output is given by $q_A = 24 - 5p_A + 2p_B$ whereas it is $q_B = 24 - 5p_B + 2p_A$ for firm B. There are no production costs. Firms simultaneously choose their prices.

1. Find the Nash Equilibrium of the game

   Solution Firm A maximizes profit given as $p_A q_A = p_A(24 - 5p_A + 2p_B) = 24p_A - 5p_A^2 + 2p_B p_A$. Taking first derivative we have $24 - 10p_A + 2p_B = 0$ and analogously for Firm B: $24 - 10p_B + 2p_A = 0$. This set of equations is easily solved for $p_A = p_B = 3$.

2. Find prices that would maximize the total profit

   Solution To maximize total profit add profits of both firms, $\Pi_{total} = p_A(24 - 5p_A + 2p_B) + p_B(24 - 5p_B + 2p_A)$ and maximize wrt both prices, the two FOCs give a set of equations, yielding $p_A = p_B = 4$.

3. show that strategy $p_A = 2$ is dominated (by which strategy/ies)?

   Solution $p_A = 2$ is dominated because the reaction curve of Firm A is given by $\frac{\partial \Pi_A}{\partial p_A} = 24 - 10p_A + 2p_B = 0$, as mentioned before. This is never satisfied by $p_A = 2$. Indeed, $\frac{\partial \Pi_A}{\partial p_A}$ is always positive when $p_A = 2$ and $p_B \geq 0$. It means that a slightly higher price is always better for Firm A. Thus $p_A = 2$ is strictly dominated by $2 + \epsilon$. To determine which strategies dominate $p_A = 2$, consider the difference between profit from $p_A = 2$ and some other $p'_A > 2$: $48 - 20 + 4p_B - 24p'_A + 5(p'_A)^2 - 2p_B p'_A$. This difference is possibly greatest when p_B is smallest, i.e. 0. To find precisely which strategies dominate $p_A = 2$ we solve the quadratic equation $28 - 24p'_A + 5(p'_A)^2 = 0$.

Problem 13 Consider a market with 2 firms which produce the same good. Firms 1 and 2 simultaneously choose price p_1 and p_2 from the set of nonnegative real numbers (thus: not only integers!). Assume that there are no costs of production. If $p_i < p_j$ then firm i gets demand $D(p_i)$ and firm j gets demand 0. The demand function takes value of $D(p) = \frac{1}{\sqrt{1+p}}$. If the firms charge equal prices, assume that demand is split evenly.

1. Find the pure strategy Nash Equilibrium

   Solution In pure strategies, undercutting the other firm (charging ϵ less than the other firm) is always a good idea, for it lets you capture the entire market. Thus the unique pure strategy NE is $p_1 = p_2 = 0$.

2. Find the symmetric mixed strategy Nash Equilibrium where players choose strategies from a distribution with full support on $[10, \infty)$ (means: players never choose anything below 10 but sometimes choose any number equal or higher than 10). HINT: What is the expected payoff of a player choosing $p = 10$? What is thus the expected payoff of a player choosing any other p? What is thus the probability of capturing the market with this price? Won't this now suffice to determine the distribution used by the

other player – his mixed strategy? Check whether this is indeed a correct probability distribution.

Solution This was more difficult. First note that probability of choosing exactly $p = 10$ must be zero, even though it belongs to the support of the distribution (i.e. it is sometimes played). Otherwise, the undercutting argument applies it's better to charge $10 - \epsilon$ instead. Thus, when you charge $p = 10$, you will capture the entire market with probability one, thus your expected payoff is $pD(p) = p\frac{1}{\sqrt{1+p}} = \frac{10}{\sqrt{11}}$. The expected payoff upon choosing any other price in the support, i.e. any $p > 10$ must be exactly the same – all strategies in the support of a mixed equilibrium must yield the same payoff in expectation (otherwise you would drop the ones that yield lower payoff). Thus whatever $p > 10$ you choose, you will get $\frac{10}{\sqrt{11}}$ in expectation. However, we also know that you will get the $pD(p)$ if and only if the other player's price is higher (again, because of the undercutting argument, ties happen with probability zero because no particular price is chosen with positive probability–an "atomless distribution). Thus, from the perspective of player i, for any $p_i > 10$ we have that

$$\Pr(p_j > p_i)pD(p) = \frac{10}{\sqrt{11}}$$

From this we have that

$$\Pr(p_j > p_i) = \frac{10\sqrt{1+p}}{p\sqrt{11}}$$

and by symmetry, we have the same for player i, so the mixed strategy equilibrium is governed by the cumulative distribution function (CDF):

$$F(p) = 1 - \frac{10\sqrt{1+p}}{\sqrt{11}p}$$

on $[10, \infty)$, which luckily starts with 0 and goes to 1 as p goes to infinity as a proper CDF should.