Marcin Bielecki, Advanced Macroeconomics IE, Spring 2026
Endogenous Growth Models (first generation)

In the exogenous growth models, ongoing increases in technology are necessary to sustain long-run growth. However, in these models technological progress is assumed, and not explained within the model (that's why they are named exogenous growth models). In this lecture we will encounter models where long-run growth arises endogenously within the model, and thus are potentially more attractive to economists, as they allow to ask questions on which factors can potentially affect the rate of growth of economy in the long run.

1 AK model

One way to generate the possibility of sustained long-run growth is to eliminate the diminishing returns to capital. One way to do it is to assume that the production function is linear in capital, hence the name AK. In this model the concept of capital is broad, as it encompasses human capital, knowledge, public infrastructure, and so on. In the next section we will encounter a model with human capital that makes this interpretation explicit. For now, let us work with the simplified setup.

Households

As usual, households want to maximize their utility, subject to the budget constraint:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t + (1+n)\, a_{t+1} = w_t + (1+r_t)\, a_t \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{c_t}^{1-\sigma}}{1-\sigma} + \lambda_t \left[w_t + (1+r_t)\, \boldsymbol{a_t} - \boldsymbol{c_t} - (1+n)\, \boldsymbol{a_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} c_t: \quad & \beta^t \left\{ c_t^{-\sigma} + \lambda_t [-1] \right\} = 0 && \rightarrow && \lambda_t = c_t^{-\sigma} \\ a_{t+1}: \quad & \beta^t \lambda_t [-(1+n)] + \beta^{t+1} \lambda_{t+1} (1+r_{t+1}) = 0 && \rightarrow && \lambda_t = \beta \lambda_{t+1} \frac{(1+r_{t+1})}{(1+n)} \end{aligned}$$

Resulting Euler equation:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} \frac{(1+r_{t+1})}{(1+n)} = c_{t+1}^{-\sigma} \frac{(1+r_{t+1})}{(1+\rho)(1+n)} \quad \Big| \quad \cdot c_{t+1}^{\sigma} \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \frac{(1+r_{t+1})}{(1+\rho)(1+n)} \quad \Big| \quad (\cdot)^{1/\sigma} \\ \frac{c_{t+1}}{c_t} &= \left[\frac{(1+r_{t+1})}{(1+\rho)(1+n)} \right]^{1/\sigma} \end{aligned}$$

Rate of growth of consumption per worker:

$$\begin{aligned} & g_c = \frac{c_{t+1} - c_t}{c_t} = \frac{c_{t+1}}{c_t} - 1 \quad \rightarrow \quad 1 + g_c = \frac{c_{t+1}}{c_t} \\ & g_c \simeq \ln(1+g_c) = \ln\left(\frac{c_{t+1}}{c_t} \right) = \frac{1}{\sigma} \left[\ln(1+r_{t+1}) - \ln(1+\rho) - \ln(1+n) \right] \simeq \frac{r_{t+1} - \rho - n}{\sigma} \end{aligned}$$

For simpler notation going forward, let us assume $n = 0$. In that case we get $g_c = \dfrac{r_{t+1} - \rho}{\sigma}$.

Firms

The production function is now assumed to be of AK form, and the profit maximization problem of the firms is stated as:

$$\begin{aligned} \max_{Y_t,\,K_t} \quad & D_t = Y_t - (r_t + \delta) K_t \\ \text{subject to} \quad & Y_t = AK_t \end{aligned}$$

Plug in the production function into the objective function:

$$\max_{K_t} \quad D_t = AK_t - (r_t + \delta) K_t$$

First order condition:

$$K_t : \qquad A - (r_t + \delta) = 0 \qquad \rightarrow \qquad r_t = A - \delta$$

Note that since the production function is linear in capital, the real interest rate is a constant, independent on the level of capital stock. Also, since raw labor is useless in production process, wages are equal to 0. Since the production function exhibits constant returns to scale and all markets are perfectly competitive, dividends are also equal to 0.

General Equilibrium

Again, since the economy is closed and there is no government, $a = k$. Prices are given by $r = A - \delta$ and $w = 0$. We assumed that $n = 0$. We can transform the budget constraint into the resource constraint:

$$\begin{aligned} c_t + a_{t+1} &= w_t + (1 + r_t) a_t + d_t \\ k_{t+1} &= w_t + (1 + r_t) k_t + d_t - c_t \\ k_{t+1} &= (1 + A - \delta) k_t - c_t \\ k_{t+1} &= Ak_t + (1 - \delta) k_t - c_t \end{aligned}$$

And we can plug in the interest rate into the Euler equation:

$$g_c = \frac{r_{t+1} - \rho}{\sigma} = \frac{A - \delta - \rho}{\sigma}$$

As long as $A > \rho + \delta$, the growth rate of consumption per worker is positive and constant. The capital can continue accumulating forever, without diminishing returns:

$$g_k = \frac{k_{t+1} - k_t}{k_t} = \frac{Ak_t + (1 - \delta) k_t - c_t - k_t}{k_t} = A - \delta - \frac{c_t}{k_t}$$

Along the Balanced Growth Path (BGP) the growth rate of capital per capita is assumed to be constant. This requires the c/k ratio to be constant as well and thus $g_c = g_k = g$. The model has a closed-form solution, and there is no transitional dynamics, as the economy is always at its BGP:

$$\begin{aligned} & g = g_c = \frac{A - \delta - \rho}{\sigma} \\ & g = g_k = A - \delta - \frac{c_t}{k_t} \quad \rightarrow \quad c_t = (A - \delta - g) k_t = \left(A - \delta - \frac{A - \delta - \rho}{\sigma} \right) k_t \end{aligned}$$

Note that the model generates strong predictions about the determinants of the growth rate. For example, a decrease in households' impatience ρ or risk aversion σ permanently raises the economy's growth rate.

1.1 Externalities in AK models

Learning-by-doing and knowledge spillovers

Consider now an economy with many firms. Each of them hires capital and labor to produce final output. What is now different is that technology level A increases when any firm invests, although individual firms treat A as a number that they cannot influence. To be more specific, the production function of an i-th firm is given by:

$$Y_{it} = K_{it}^{\alpha} \left(A_t L_{it}\right)^{1-\alpha}$$

The aggregate capital stock is equal to the sum of capital across firms:

$$K_t = \sum_i K_{it}$$

and the level of technology is equal to the aggregate capital stock:

$$A_t = K_t$$

Such an economy experiences a positive externality from capital accumulation, but a decentralized, private economy will underinvest and the growth rate will be lower than is socially optimal.

Each firm solves the following profit maximization problem:

$$\max_{L_{it},\, K_{it}} \quad K_{it}^{\alpha} \left(A_t L_{it}\right)^{1-\alpha} - w_t L_{it} - \left(r_t + \delta\right) K_{it}$$

First order conditions:

$$\begin{array}{llcl} L_{it}: & (1-\alpha) K_{it}^{\alpha} A_t^{1-\alpha} L_{it}^{-\alpha} - w_t = 0 & \rightarrow & w_t = (1-\alpha) A_t^{1-\alpha} k_{it}^{\alpha} \\ K_{it}: & \alpha K_{it}^{\alpha-1} A_t^{1-\alpha} L_{it}^{1-\alpha} - (r_t + \delta) = 0 & \rightarrow & r_t = \alpha A_t^{1-\alpha} k_{it}^{\alpha-1} - \delta \end{array}$$

Since individual firms treat factor prices as given, they all choose the same capital to labor ratio, k. Therefore the aggregate production function can be written as:

$$Y_t = \sum_i Y_{it} = \sum_i K_{it}^{\alpha} \left(A_t L_{it}\right)^{1-\alpha} = A_t^{1-\alpha} \sum_i \left(\frac{K_{it}}{L_{it}}\right)^{\alpha} L_{it} = A_t^{1-\alpha} k_t^{\alpha} \sum_i L_{it} = A_t^{1-\alpha} k_t^{\alpha} L_t$$

Now we use the assumption that $A_t = K_t$ to rewrite the production function into the AK form:

$$Y_t = K_t^{1-\alpha} \left(\frac{K_t}{L_t}\right)^{\alpha} L_t = K_t L_t^{1-\alpha}$$

Rewrite the interest rate:

$$r_t = \alpha K_t^{1-\alpha} \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta = \alpha L_t^{1-\alpha} - \delta$$

Plug in the interest rate into the Euler equation:

$$g = g_c \simeq \frac{r_{t+1} - \rho - n}{\sigma} = \frac{\alpha L_{t+1}^{1-\alpha} - \delta - \rho - n}{\sigma}$$

If the population is constant, the economy's rate of growth is also constant:

$$g = g_C \simeq \frac{\alpha L^{1-\alpha} - \delta - \rho}{\sigma}$$

Social planner's solution

The social planner maximizes households' welfare given the resource constraints and is aware of the externality which the private sector ignores:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & K_{t+1} = K_t L^{1-\alpha} + (1-\delta) K_t - C_t \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{C_t}^{1-\sigma}}{1-\sigma} + \lambda_t \left[\boldsymbol{K_t} L^{1-\alpha} + (1-\delta) \boldsymbol{K_t} - \boldsymbol{C_t} - \boldsymbol{K_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} C_t: \quad & \beta^t \left\{ C_t^{-\sigma} + \lambda_t [-1] \right\} = 0 & \rightarrow \quad & \lambda_t = C_t^{-\sigma} \quad \rightarrow \quad C_t = \lambda_t^{-1/\sigma} \\ K_{t+1}: \quad & \beta^t \lambda_t [-1] + \beta^{t+1} \lambda_{t+1} \left[L^{1-\alpha} + (1-\delta) \right] = 0 & \rightarrow \quad & \lambda_t = \beta \lambda_{t+1} \left[1 + L^{1-\alpha} - \delta \right] \end{aligned}$$

The ratios of consumption and λ over time are:

$$\begin{aligned} \frac{C_{t+1}}{C_t} &= \frac{\lambda_{t+1}^{-1/\sigma}}{\lambda_t^{-1/\sigma}} = \left(\frac{\lambda_{t+1}}{\lambda_t} \right)^{-1/\sigma} = \left(\frac{\lambda_t}{\lambda_{t+1}} \right)^{1/\sigma} \\ \frac{\lambda_t}{\lambda_{t+1}} &= \beta \left[1 + L^{1-\alpha} - \delta \right] = \frac{1 + L^{1-\alpha} - \delta}{1+\rho} \end{aligned}$$

The rate of growth of the social planner's economy exceeds the rate of growth of the decentralized one:

$$g = g_C \simeq \ln \left(\frac{C_{t+1}}{C_t} \right) = \ln \left[\left(\frac{\lambda_t}{\lambda_{t+1}} \right)^{1/\sigma} \right] = \frac{1}{\sigma} \left[\ln \left(1 + L^{1-\alpha} - \delta \right) - \ln (1+\rho) \right] \simeq \frac{L^{1-\alpha} - \delta - \rho}{\sigma}$$

$$g^{sp} \simeq \frac{L^{1-\alpha} - \delta - \rho}{\sigma} > \frac{\alpha L^{1-\alpha} - \delta - \rho}{\sigma} \simeq g^{dec}$$

2 Human capital in a one sector economy

Assume now that the production requires the use of physical and human capital:

$$Y_t = AK_t^{\alpha} H_t^{1-\alpha}$$

Both the physical and human capital accumulate through investment and for simplicity we assume that they depreciate at the same rate δ:

$$\begin{aligned} K_{t+1} &= I_t^K + (1-\delta)\,K_t \\ H_{t+1} &= I_t^H + (1-\delta)\,H_t \end{aligned}$$

Continue assuming $n = 0$ for simplicity. Since population is constant, maximizing aggregate consumption is equivalent to maximizing consumption per worker:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & Y_t = AK_t^{\alpha} H_t^{1-\alpha} = C_t + I_t^K + I_t^H \\ & K_{t+1} = I_t^K + (1-\delta)\,K_t \\ & H_{t+1} = I_t^H + (1-\delta)\,H_t \end{aligned}$$

Lagrangian (where the capital accumulation equation was plugged into the national accounting constraint):

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{C_t}^{1-\sigma}}{1-\sigma} + \lambda_t \left[A\boldsymbol{K_t}^{\alpha} \boldsymbol{H_t}^{1-\alpha} - \boldsymbol{C_t} - \boldsymbol{K_{t+1}} + (1-\delta)\,\boldsymbol{K_t} - \boldsymbol{I_t^H} \right] + \mu_t \left[\boldsymbol{I_t^H} + (1-\delta)\,\boldsymbol{H_t} - \boldsymbol{H_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} C_t: &\quad \beta^t \left\{ C_t^{-\sigma} + \lambda_t\,[-1] \right\} = 0 && \rightarrow \quad \lambda_t = C_t^{-\sigma} \\ I_t^H: &\quad \beta^t \left\{ \lambda_t\,[-1] + \mu_t \right\} = 0 && \rightarrow \quad \lambda_t = \mu_t \\ K_{t+1}: &\quad \beta^t \lambda_t\,[-1] + \beta^{t+1} \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta) \right] = 0 && \rightarrow \quad \lambda_t = \beta \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta) \right] \\ H_{t+1}: &\quad \beta^t \mu_t\,[-1] + \beta^{t+1} \left\{ \lambda_{t+1} \left[(1-\alpha)\,A K_{t+1}^{\alpha} H_{t+1}^{-\alpha} \right] + \mu_{t+1}\,(1-\delta) \right\} = 0 && \\ &\quad \rightarrow \quad \mu_t = \beta \left[\lambda_{t+1}\,(1-\alpha)\,A K_{t+1}^{\alpha} H_{t+1}^{-\alpha} + \mu_{t+1}\,(1-\delta) \right] && \end{aligned}$$

Since $\lambda_t = \mu_t$, we have:

$$\alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta) = (1-\alpha)\,A K_{t+1}^{\alpha} H_{t+1}^{-\alpha} + (1-\delta) \quad \rightarrow \quad \frac{K_{t+1}}{H_{t+1}} = \frac{\alpha}{1-\alpha}$$

If we assume (for now) that investment in both types of capital can be negative, the ratio of physical to human capital is constant at all times and has the value derived above. We can then rewrite the production function in the AK form:

$$Y_t = AK_t^{\alpha} H_t^{1-\alpha} = A \left(\frac{K_t}{H_t} \right)^{\alpha-1} K_t = A \left(\frac{\alpha}{1-\alpha} \right)^{\alpha-1} K_t \equiv BK_t$$

We can now use e.g. the FOC for physical capital to determine the rate of growth of the economy, which is always on its BGP:

$$\lambda_t = \beta \lambda_{t+1} \left[1 + \alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} - \delta \right] = \beta \lambda_{t+1} \left[1 + \alpha A \left(\frac{\alpha}{1-\alpha} \right)^{\alpha-1} - \delta \right] = \beta \lambda_{t+1} \left[1 + \alpha B - \delta \right]$$

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{1 + \alpha B - \delta}{1+\rho}$$

$$g = g_C \simeq \ln \left(\frac{C_{t+1}}{C_t} \right) = \ln \left[\left(\frac{\lambda_t}{\lambda_{t+1}} \right)^{1/\sigma} \right] = \frac{1}{\sigma} \left[\ln(1 + \alpha B - \delta) - \ln(1+\rho) \right] \simeq \frac{\alpha B - \delta - \rho}{\sigma}$$

Non-negativity constraints on investment

Suppose now that we add a condition that gross investment cannot be negative, so you cannot transform one type of capital to the other after it has been already built: $I_t^K \geq 0$, $I_t^H \geq 0$. If an economy has imbalanced quantities of physical or human capital, it has to accumulate the less abundant one, and let the other depreciate over time. Consider now an economy with overabundance of human capital, but low levels of physical capital (the case of overabundance of physical capital is fully symmetric):

$$\frac{K_t}{H_t} < \left(\frac{K}{H}\right)^* = \frac{\alpha}{1-\alpha}$$

Although it seems that a problem with non-negativity constraints would be mathematically more difficult to tackle, in this case it is actually simpler, as it is optimal to set $I^H = 0$ and let it depreciate until the BGP ratio of K/H is reached. The constraints can now be reduced to a single capital accumulation equation:

$$K_{t+1} = AK_t^\alpha H_t^{1-\alpha} - C_t + (1-\delta) K_t$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{C_t}^{1-\sigma}}{1-\sigma} + \lambda_t \left[A\boldsymbol{K_t}^\alpha \boldsymbol{H_t}^{1-\alpha} - \boldsymbol{C_t} + (1-\delta)\boldsymbol{K_t} - \boldsymbol{K_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} C_t: &\quad \beta^t \left\{ C_t^{-\sigma} + \lambda_t [-1] \right\} = 0 &&\rightarrow \quad \lambda_t = C_t^{-\sigma} \\ K_{t+1}: &\quad \beta^t \lambda_t [-1] + \beta^{t+1} \left\{ \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta) \right] \right\} = 0 &&\rightarrow \quad \lambda_t = \beta \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta) \right] \end{aligned}$$

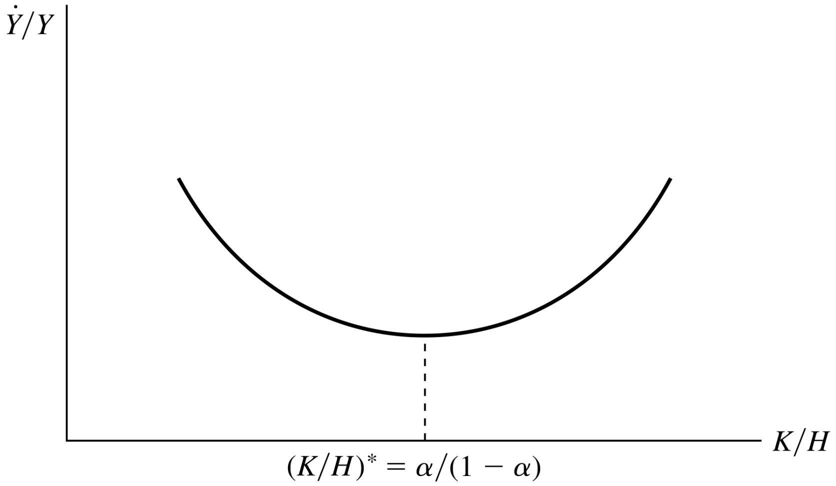
The economy now behaves as a Ramsey-Cass-Koopmans economy, and its growth rate depends on the relative abundance of different capital types:

$$\begin{aligned} g_C \simeq \ln\left(\frac{C_{t+1}}{C_t}\right) &= \ln\left[\left(\frac{\lambda_t}{\lambda_{t+1}}\right)^{1/\sigma}\right] = \frac{1}{\sigma}\left[\ln\left(1 + \alpha A K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} - \delta\right) - \ln(1+\rho)\right] \\ &\simeq \frac{\alpha A \left(K_{t+1}/H_{t+1}\right)^{\alpha-1} - \delta - \rho}{\sigma} = \frac{\alpha A \left(H_{t+1}/K_{t+1}\right)^{1-\alpha} - \delta - \rho}{\sigma} \end{aligned}$$

We can also demonstrate that such economy grows faster than along its BGP:

$$\frac{K_{t+1}}{H_{t+1}} < \left(\frac{K}{H}\right)^* \quad \rightarrow \quad \frac{H_{t+1}}{K_{t+1}} > \left(\frac{H}{K}\right)^* \quad \rightarrow \quad g_C > g_C^* \quad \text{and} \quad g_Y > g_Y^*$$

This faster-than-BGP growth is possible due to the imbalance effect: while one factor of production slowly depreciates, the other is accumulated faster than along the BGP and both consumption and output increase at a faster rate:



3 Human capital in a two sector economy (Uzawa-Lucas model)

In this model the economy consists of two sector. The first one produces physical goods. The second is responsible for human capital accumulation. Let u denote the share of human capital employed in the physical goods producing setor, and $(1-u)$ denote the share of human capital employed in the human capital producing sector. The problem can be stated as:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & K_{t+1} = AK_t^{\alpha} \left(u_t H_t\right)^{1-\alpha} + (1-\delta) K_t - C_t \\ & H_{t+1} = B(1-u_t) H_t + (1-\delta) H_t \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{C_t^{1-\sigma}}}{1-\sigma} + \lambda_t \left[A\boldsymbol{K_t^{\alpha}} \left(\boldsymbol{u_t H_t}\right)^{1-\alpha} + (1-\delta)\boldsymbol{K_t} - \boldsymbol{C_t} - \boldsymbol{K_{t+1}} \right] + \mu_t \left[B(1-\boldsymbol{u_t})\boldsymbol{H_t} + (1-\delta)\boldsymbol{H_t} - \boldsymbol{H_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} C_t: \quad & \beta^t \left\{ C_t^{-\sigma} - \lambda_t [-1] \right\} = 0 \\ u_t: \quad & \beta^t \left\{ \lambda_t \left[(1-\alpha) AK_t^{\alpha} u_t^{-\alpha} H_t^{1-\alpha} \right] + \mu_t [-BH_t] \right\} = 0 \\ K_{t+1}: \quad & \beta^t \left\{ \lambda_t [-1] \right\} + \beta^{t+1} \left\{ \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} \left(u_{t+1} H_{t+1}\right)^{1-\alpha} + (1-\delta) \right] \right\} = 0 \\ H_{t+1}: \quad & \beta^t \left\{ \mu_t [-1] \right\} + \beta^{t+1} \left\{ \lambda_{t+1} \left[(1-\alpha) AK_{t+1}^{\alpha} u_{t+1}^{1-\alpha} H_{t+1}^{-\alpha} \right] + \mu_{t+1} \left[B(1-u_{t+1}) + (1-\delta) \right] \right\} = 0 \end{aligned}$$

Simplified first order conditions:

$$\lambda_t = C_t^{-\sigma} \tag{1}$$

$$\mu_t B H_t = \lambda_t \left[(1-\alpha) AK_t^{\alpha} u_t^{-\alpha} H_t^{1-\alpha} \right] \tag{2}$$

$$\lambda_t = \beta \lambda_{t+1} \left[\alpha A K_{t+1}^{\alpha-1} \left(u_{t+1} H_{t+1}\right)^{1-\alpha} + (1-\delta) \right] \tag{3}$$

$$\mu_t = \beta \left\{ \lambda_{t+1} \left[(1-\alpha) AK_{t+1}^{\alpha} u_{t+1}^{1-\alpha} H_{t+1}^{-\alpha} \right] + \mu_{t+1} \left[B(1-u_{t+1}) + (1-\delta) \right] \right\} \tag{4}$$

The idea behind the solution procedure is as follows. To obtain the rate of growth of the economy, we need to know the rate of growth of λ. But it will be much easier to obtain the rate of growth of μ. If we aren able to show that λ and μ grow at the same rates, we will be done. Consider first (2):

$$\begin{aligned} \mu_t B H_t &= \lambda_t (1-\alpha) AK_t^{\alpha} u_t^{-\alpha} H_t^{1-\alpha} \quad \Big| \cdot \frac{u_t}{H_t} \\ \mu_t B u_t &= \lambda_t (1-\alpha) AK_t^{\alpha} u_t^{1-\alpha} H_t^{-\alpha} \end{aligned}$$

Then compare with (4):

$$\begin{aligned} \mu_t &= \beta \left\{ \lambda_{t+1} \left[(1-\alpha) AK_{t+1}^{\alpha} u_{t+1}^{1-\alpha} H_{t+1}^{-\alpha} \right] + \mu_{t+1} \left[B(1-u_{t+1}) + (1-\delta) \right] \right\} \\ \mu_t &= \beta \left\{ \mu_{t+1} B u_{t+1} + \mu_{t+1} \left[B(1-u_{t+1}) + (1-\delta) \right] \right\} \\ \mu_t &= \beta \mu_{t+1} \left[B + 1 - \delta \right] \\ \frac{\mu_t}{\mu_{t+1}} &= \frac{1+B-\delta}{1+\rho} \end{aligned}$$

Use (1):

$$g_C \simeq \ln\left(\frac{C_{t+1}}{C_t}\right) = \ln\left[\left(\frac{\lambda_t}{\lambda_{t+1}}\right)^{1/\sigma}\right] = \frac{\ln\left(\lambda_t/\lambda_{t+1}\right)}{\sigma}$$

If $\lambda_t/\lambda_{t+1} = \mu_t/\mu_{t+1}$ then:

$$g_C \simeq \frac{\ln\left(\mu_t/\mu_{t+1}\right)}{\sigma} = \frac{\ln(1+B-\delta) - \ln(1+\rho)}{\sigma} \simeq \frac{B-\delta-\rho}{\sigma}$$

To demonstrate that $\lambda_t/\lambda_{t+1} = \mu_t/\mu_{t+1}$, consider (3):

$$\begin{aligned}
\frac{\lambda_t}{\lambda_{t+1}} &= \frac{1+\alpha A K_{t+1}^{\alpha-1}\left(u_{t+1}H_{t+1}\right)^{1-\alpha}-\delta}{1+\rho} \\
g_C \simeq \frac{\ln\left(\lambda_t/\lambda_{t+1}\right)}{\sigma} &= \frac{\ln\left[1+\alpha A K_{t+1}^{\alpha-1}\left(u_{t+1}H_{t+1}\right)^{1-\alpha}-\delta\right]-\ln\left(1+\rho\right)}{\sigma} \simeq \frac{\alpha A\left(\frac{K_{t+1}}{u_{t+1}H_{t+1}}\right)^{\alpha-1}-\delta-\rho}{\sigma}
\end{aligned}$$

Along the BGP, the share of human capital employed in respective sectors has to be constant:

$$g_u^* = 0$$

As a consequence:

$$\left(\frac{K}{uH}\right)^* = const \quad \rightarrow \quad g_K^* = g_H^*$$

Use now the capital accumulation equation to show that along the BGP all main macroeconomic variables grow at the same rate:

$$\begin{aligned}
K_{t+1} &= A K_t^{\alpha}\left(u_t H_t\right)^{1-\alpha} + \left(1-\delta\right)K_t - C_t \\
\frac{K_{t+1}}{K_t} &= A\left(\frac{K_t}{u_t H_t}\right)^{\alpha-1} + \left(1-\delta\right) - \frac{C_t}{K_t} \\
g_K = \frac{K_{t+1}}{K_t} - 1 &= A\left(\frac{K_t}{u_t H_t}\right)^{\alpha-1} - \delta - \frac{C_t}{K_t}
\end{aligned}$$

$$g_K^* = g_C^* = g_H^* = g_Y^*$$

Consider again (2) and prove that along the BGP λ and μ grow at the same rate:

$$\begin{aligned}
\mu_t B H_t &= \lambda_t\left[\left(1-\alpha\right)A K_t^{\alpha} u_t^{-\alpha} H_t^{1-\alpha}\right] \\
\frac{\mu_{t+1} B H_{t+1}}{\mu_t B H_t} &= \frac{\lambda_{t+1}\left[\left(1-\alpha\right)A K_{t+1}^{\alpha} u_{t+1}^{-\alpha} H_{t+1}^{1-\alpha}\right]}{\lambda_t\left[\left(1-\alpha\right)A K_t^{\alpha} u_t^{-\alpha} H_t^{1-\alpha}\right]} \\
\frac{\mu_{t+1}}{\mu_t} &= \frac{\lambda_{t+1} K_{t+1}^{\alpha} u_{t+1}^{-\alpha} H_{t+1}^{-\alpha}}{\lambda_t K_t^{\alpha} u_t^{-\alpha} H_t^{-\alpha}} \\
\left(1+g_\mu\right) &= \left(1+g_\lambda\right)\left(1+g_K\right)^{\alpha}\left(1+g_u\right)^{-\alpha}\left(1+g_H\right)^{-\alpha} \\
\left(1+g_\mu^*\right) &= \left(1+g_\lambda^*\right)\left(1+g_K^*\right)^{\alpha}\left(1+g_u^*\right)^{-\alpha}\left(1+g_H^*\right)^{-\alpha} = \left(1+g_\lambda^*\right)
\end{aligned}$$

As a final step, find u^*:

$$\begin{aligned}
H_{t+1} &= B\left(1-u_t\right)H_t + \left(1-\delta\right)H_t \\
\frac{H_{t+1}}{H_t} &= B\left(1-u_t\right) + 1 - \delta \\
g_H = \frac{H_{t+1}}{H_t} - 1 &= B\left(1-u_t\right) - \delta
\end{aligned}$$

$$\begin{aligned}
1-u^* &= \frac{g^*+\delta}{B} \\
u^* &= 1 - \frac{g^*+\delta}{B}
\end{aligned}$$

So at the end we obtain the result that the growth rate of the economy is in the long run determined by the efficiency of the human capital accumulation sector:

$$g^* = \frac{B-\delta-\rho}{\sigma}$$