Marcin Bielecki, Advanced Macroeconomics IE, Spring 2026
Homework 4, deadline: April 21, 4:45 PM

Problem 1

Households have access to two financial instruments: equity shares in firms (stocks) s and corporate bonds b issued by firms, which are characterized (for simplicity of notation) by a constant real interest rate r. We adopt the convention that stocks are traded (without any transaction costs) at the end of each period at price p^s, and holding them within the period entitles the owner to the current dividend per share d^s. The problem of the households is (where one can define end-of-period assets per household as $a_{t+1} = b_{t+1} + p_t^s s_{t+1}$ and return on assets via $(1 + r_t^a) a_t = (d_t^s + p_t^s) s_t + (1 + r) b_t$).

$$\begin{aligned} \max_{\{c_t,\, b_{t+1},\, s_{t+1}\}_{t=0}^{\infty}} \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \\ \text{subject to} \quad & c_t + b_{t+1} + p_t^s s_{t+1} = w_t + (d_t^s + p_t^s) s_t + (1 + r) b_t \end{aligned}$$

(a) Derive the first-order conditions. Obtain the Euler equation for bonds.

(b) Combine the first-order conditions for bonds and stocks to obtain an expression for the stock price.

(c) Derive an expression for the fundamental market value of the firm sector as a function of future dividends (assuming a constant number of shares $\mathcal{S}$ in the economy, so that $V = p^s \mathcal{S}$ and $D = d^s \mathcal{S}$.

(d) Solve the following beginning-of-period 0 firm value maximization problem. The firm issues corporate debt B and is the direct owner of its capital stock K, increasing it via investment I. The firm pays out dividends D to its shareholders. Derive the first order conditions of the firm.

$$\begin{aligned} \max_{\{L_t,\, I_t,\, B_{t+1},\, K_{t+1}\}_{t=0}^{\infty}} \quad & V_0^D = D_0 + V_0 = \sum_{t=0}^{\infty} \frac{D_t}{(1 + r)^t} \\ \text{subject to} \quad & D_t = K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - I_t + B_{t+1} - (1 + r) B_t \\ & K_{t+1} = I_t + (1 - \delta) K_t \end{aligned}$$

(e) Starting from firm's FOC for capital, show that in this economy the following equivalence holds.

$$q_t K_{t+1} = B_{t+1} + V_t = Assets_{t+1}$$

Problem 2

Consider now the capital adjustment cost model with a constant returns to scale adjustment cost function[1] Φ, but in the variant where it is the households that directly own and invest in physical capital and rent it to firms at capital rental rate r^K. For notational simplicity maintain the assumption that interest rate r on bonds is fixed. The household problem is as follows:

$$
\begin{aligned}
\max_{\{C_t,\, I_t,\, B_{t+1},\, K_{t+1}\}_{t=0}^{\infty}} \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \cdot u\left(C_t\right) \\
\text{subject to} \quad & C_t + I_t + \Phi\left(I_t, K_t\right) + B_{t+1} = w_t L + r_t^K K_t + \left(1+r\right) B_t \\
& K_{t+1} = I_t + \left(1-\delta\right) K_t
\end{aligned}
$$

(a) Derive the first-order conditions with respect to C_t, I_t, B_{t+1} and K_{t+1}. It will be convenient to label the Lagrange multiplier for the budget constraint as λ_t and the multiplier for the capital accumulation as $\lambda_t q_t$.

(b) Using the FOC for capital and other FOCs, derive the expression for marginal benefit of investment q_t as a function of future capital rental rate, adjustment costs, and q_{t+1}.

(c) Solve the following firm value maximization problem of the firm that rents capital K. Treat L in the firm problem as fixed. Plug in the result from capital FOC into the equation for q_t in (b).

$$
\max_{K_{t+1}} \quad V\left(K_{t+1}\right) = \frac{1}{1+r}\left[F\left(K_{t+1}, L\right) - r_{t+1}^K K_{t+1}\right]
$$

(d) Switch now to the situation, where it is the firms that own capital stock. The firm problem is as follows. Use q to denote Lagrange multiplier for the capital accumulation constraint. Derive the firm first order conditions. Treat L in the firm problem as fixed.

$$
\begin{aligned}
\max_{\{I_t,\, K_{t+1}\}_{t=0}^{\infty}} \quad & V_0^D = D_0 + V_0 = \sum_{t=0}^{\infty} \frac{D_t}{\left(1+r\right)^t} \\
\text{subject to} \quad & D_t = F\left(K_t, L\right) - w_t L - I_t - \Phi\left(I_t, K_t\right) \\
& K_{t+1} = I_t + \left(1-\delta\right) K_t
\end{aligned}
$$

(e) Compare now the results from (d) and (c). Does it matter, who directly owns capital stock? Why or why not?

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[1] If you prefer, you can assume a particular functional form for Φ, like below, but this is not necessary for the results

$$
\Phi\left(I_t, K_t\right) = \frac{\chi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 \cdot K_t = \frac{\chi}{2}\left(\frac{I_t - \delta K_t}{K_t}\right)^2 \cdot K_t = \frac{\chi}{2}\frac{\left(I_t^n\right)^2}{K_t}
$$

Problem 3

Consider the following variant of the Solow-Swan model. This time we allow for firms to have market power, and denote the degree of said power with $\mu \geq 0$. This will allow the firms to sell their products at prices above the marginal cost, and generate positive dividends for the firm owners.

(a) Solve the following problem of the firms and obtain the modified prices of factors of production

$$\begin{aligned} \max_{Y_t, K_t, L_t} \quad & \frac{1}{1+\mu} Y_t - w_t L_t - r_t^K K_t \\ \text{subject to} \quad & Y_t = K_t^{\alpha} L_t^{1-\alpha} \end{aligned}$$

The dividend payout is equal to

$$D_t = Y_t - w_t L_t - r_t^K K_t$$

(b) Derive the expression for the labor share of income, wL/Y. How does the presence of market power μ modify this term?

(c) Plug in the information from (a) into the budget constraint of the household sector. Assume that firms pay their entire cashflow as dividend D on a period by period basis.

$$C_t + K_{t+1} = w_t L_t + \left(1 + r_t^K - \delta\right) K_t + D_t$$

(d) Use now the assumption that households consume a constant fraction $(1 - s)$ of their disposable income $Y^d = wL + r^K K + D$. Derive the fundamental equation of the Solow-Swan model in per-worker terms. For simplicity assume constant population, with $n = 0$.

(e) Find the steady state of the model. Does the presence of market power modify this result?

Problem 4

Robert Solow in his 1956 article "A Contribution to the Theory of Economic Growth" considered the behavior of economy when output was produced according to various production functions. One of them was the following Constant Elasticity of Substitution (CES) function:

$$Y_t = \left[aK_t^{\rho} + (1-a)\, L_t^{\rho}\right]^{1/\rho}$$

where $a \in (0,1)$, $\rho \leq 1$ and for simplicity technology growth rate is set to 0. The CES function reduces to the Cobb-Douglas function in the limit of $\rho \to 0$.

Saving and investment behaviour of the economy are described respectively as:

$$i_t = s \cdot y_t$$
$$(1+n)\, k_{t+1} = i_t + (1-\delta)\, k_t$$

where lower case letters i_t, y_t, k_t denote per worker quantities, n denotes population growth, and δ denotes the depreciation rate.

(a) Transform the production function into per worker form (divide it by L_t).

(b) Find the steady state value for capital per worker k^*.

(c) Show how an increase in the saving rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).

(d) Show how an increase in the population growth rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).

Problem 5

Consider an "endowment" version of the two-period OLG model. N_t^y agents are born in time t, where $N_t^y = (1+n)^t N_0^y$. Normalize $N_0^y = 1$ and let $n > 0$. Preferences of a young agent born in time t are:

$$U_t = \ln c_t^y + \beta \ln c_{t+1}^o$$

The initial old agents want to consume as much as possible. Each young agent is endowed with y units of the consumption good. The old have no endowment whatsoever. There is a storage technology that allows to convert one unit of period t goods into $1+r$ units of period $t+1$ goods. There is a social security system that is "pay-as-you-go". In each period t the government taxes the young and uses the receipts to make transfers to the old. We consider a per capita tax on the young that is constant over time, i.e. $\tau_t = \tau$ for all $t = 0, 1, 2, \ldots$

(a) What is the government budget constraint in period t? Write it both in aggregate and in per person terms.

(b) Solve for the competitive equilibrium consumption levels. Find the savings of the representative young agent a_{t+1}.

(c) What is the effect of an increase in τ on savings of the representative young agent?

(d) What is the optimal tax rate τ if $n < r$? Explain why.