Marcin Bielecki, Advanced Macroeconomics IE, Spring 2026
Homework 3, deadline: March 24, 4:45 PM

Problem 1

Solve the following problem of a worker household:

$$\max_{c,\,h} \quad U = \ln c + \psi \ln (1 - h)$$
$$\text{subject to} \quad (1 + \tau^c)\, c = (1 - \tau^w)\, wh + x$$

where τ^c is a tax on consumption, τ^w is a tax on labor income and x is non-labor income of the household.

(a) Find optimal c and h as functions of preference parameter ψ, wage w and tax rates τ^c and τ^w.

(b) In the $(h,\, w)$ space draw the labor supply curve. Show how it reacts to changes in taxes.

(c) Find the reservation wage $\bar{w}$, that is such wage level for which $h = 0$.

(d) What is the effect of an increase in consumption tax τ^c on optimal c, h and reservation wage $\bar{w}$? Provide intuition for this result.

(e) What is the effect of an increase in wage income tax τ^w on optimal c, h and reservation wage $\bar{w}$? Provide intuition for this result.

Problem 2

Consider the following general equilibrium model. Assume that household's capital stock k^s is fixed and that the depreciation rate $\delta = 0$. The household solves the following utility maximization problem:

$$\max_{c^d,\, h^s} \quad U = \ln c^d + \psi \ln(1 - h^s)$$
$$\text{subject to} \quad c^d = wh^s + rk^s + d$$

while the firm solves the following profit maximization problem (in per worker terms):

$$\max_{y^s,\, h^d,\, k^d} \quad d = y^s - wh^d - (r + \delta)\, k^d$$
$$\text{subject to} \quad y^s = A(k^d)^\alpha (h^d)^{1-\alpha}$$

where real wage w and real interest rate r are taken as given for both the firm and the household and will adjust to balance supply and demand in the labor and capital market ($h^s = h^d$ and $k^s = k^d$).

(a) Derive the first order conditions of the household.

(b) Derive the first order conditions of the firm.

(c) In the $(h,\, w)$ space draw the labor supply curve derived from household's problem and the labor demand curve derived from firm's problem.

(d) Assume that goods market clears and that the entire output of the firm is consumed by the household, $y^s = c^d$. Find the level of hours worked by the household. Find the equilibrium level of wage w and output y.

(e) How do the hours worked, wage and output level change if A improves? How do they change if ψ goes up (households value leisure more)?

Problem 3

Consider the following two-period model with two assets (bonds and physical capital):

$$\begin{aligned}\max_{c_t, c_{t+1}, b_{t+1}, k_{t+1}} \quad & U = \ln c_t + \beta \ln c_{t+1} \\ \text{subject to} \quad & c_t + b_{t+1} + k_{t+1} = y_t \\ & c_{t+1} = y_{t+1} + (1+r)\, b_{t+1} + (1-\delta)\, k_{t+1} \\ & y_{t+1} = A k_{t+1}^{\alpha}\end{aligned}$$

where b denotes bonds, $\delta \in (0,1)$ stands for capital depreciation rate, $A > 0$ is the level of technology and $\alpha \in (0,1)$ is the elasticity of production to capital.

(a) Write down the problem in the form of a Lagrangian.

(b) Find the optimal values of c_t, c_{t+1}, b_{t+1} and k_{t+1}. *Hint: once you find the optimal k_{t+1}, you can treat it as a parameter.*

(c) Calculate the derivative of optimal k_{t+1} with respect to r. Provide intuition for this result. *Hint: you can assume $\alpha = 1/2$ to simplify the calculations.*

(d) What is the effect of an increase in A on c_t, c_{t+1}, b_{t+1} and k_{t+1}? Provide intuition for this result.

Problem 4

Consider a two-period problem of choosing the level of capital stock by two firms: A and B. For simplicity we will assume that production requires capital only. We will follow the convention that in time period t the level of capital K_t is predetermined, but the firm can choose its future level of capital, K_{t+1}.

Firm A does not own its capital stock, but instead rents it at price r^k, the rental cost of capital. Future profit flows are discounted with the real interest rate r. The problem of maximizing the value of firm A is given by:

$$\max_{K_{t+1}} \quad V^A = F(K_t) - r^k K_t + \frac{1}{1+r}\left[F(K_{t+1}) - r^k K_{t+1}\right]$$

Firm B owns its capital stock, and can adjust its level via investment. After production will take place in period $t+1$, the firm will sell any undepreciated capital left. The problem of maximizing the value of firm B is given by:

$$\begin{aligned}\max_{I_t,\, K_{t+1}} \quad & V^B = F(K_t) - I_t + \frac{1}{1+r}\left[F(K_{t+1}) + (1-\delta) K_{t+1}\right] \\ \text{subject to} \quad & K_{t+1} = (1-\delta) K_t + I_t\end{aligned}$$

(a) Derive the first order condition of firm A.

(b) Derive the first order conditions of firm B.

(c) What condition has to be satisfied for both firms to choose the same level of K_{t+1}?

(d) Imagine you are the owner of firm C, which rents capital goods to firm A. What would be the maximal level of r^k that you could charge this firm?

Problem 5

Consider the following problem of a manager maximizing the value of the firm:

$$\max_{\{L_t, I_t^n, K_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[(1-\tau)\left(K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - \delta K_t - I_t^n\right) - \frac{\chi}{2} \frac{(I_t^n)^2}{K_t} \right]$$

$$\text{subject to} \quad K_{t+1} = I_t^n + K_t \qquad \forall t = 0, 1, \ldots, \infty$$

where τ is a tax levied on firm's profits, K is firm's capital stock, L are firm's employees, $\alpha \in (0,1)$ is output elasticity w.r.t. capital, I^n is net investment and $\delta \in (0,1)$ stands for capital depreciation. Parameter χ describes the magnitude of capital installation costs. Note that the tax code does not treat installation costs as tax deductible.

(a) Write down the problem in the Lagrangian form and derive the first order conditions.

(b) Find the steady state level of q (the Lagrange multiplier). Is it equal to 1? *Hint: by definition in the steady state firm's capital stock is constant and net investment is 0.*

(c) Find the desired level of firm's capital stock per employee, $k \equiv K/L$, treating interest rate r as given.

(d) Suppose the tax on firm's profits is reduced. What happens with the firm's investment if its level of capital stock per employee was at the level from (c) prior to the tax change?

(e) What happens with the firm's investment if its level of capital stock per employee was lower than the level from (c) prior to the tax change?