Chapter 4

Solow-Swan model

The Solow–Swan model was proposed independently by Robert Solow and Trevor Swan in 1956. It became the foundational framework for studying long-run economic growth and is the starting point for modern growth theory.

Key assumptions (essential):

- **Two types of agents:** The economy consists of firms and households.
- **Constant savings / investment rate:** Households always save a fixed fraction of their income.
- **One-sector economy:** Firms produce a single good, which can either be consumed or invested.
- **Production uses capital, labor and technology** through a neoclassical production function.

Simplifying assumptions (non-essential):

- **Perfectly competitive markets** in the final goods market and factor markets.
- **Household ownership of capital:** Households own the capital stock and rent it to firms.
- **No government, closed economy:** no taxes, government expenditures, or international trade.

Three key equations of the Solow-Swan model:

1. **Capital accumulation equation** (accounting identity): $K_{t+1} = I_t + (1 - \delta) K_t$
2. **Neoclassical production function** (functional restriction): $Y_t = F(K_t, L_t, A_t)$
3. **Savings rate is independent of income level** (behavioral assumption): $S_t = sY_t$

A consequence of the (No government, closed economy) assumption is that private savings are equal to private investment. **This is an accounting identity and says nothing about any causal relationships between savings and investment.** Start from the national accounting identity and assume no government ($G = 0$) and no international trade ($NX = 0$):

$$Y = C + I + (G = 0) + (NX = 0) \quad \rightarrow \quad Y = C + I$$

Then consider the allocation of households' disposable income Y^d between consumption and savings. If there is no government, taxes T are 0 and disposable income is equal to the economy's output.

$$Y^d = C + S \quad \text{and} \quad Y^d = Y - (T = 0) \quad \rightarrow \quad Y = C + S$$

Together those two accounting exercises yield the result that savings are equal to investment:

$$Y = C + I \quad \text{and} \quad Y = C + S \quad \rightarrow \quad S = I$$

4.1 Neoclassical production function

A production function describes how capital K and labor L are transformed into output Y using technology A, and is written in its general form as $Y_t = F(K_t, L_t, A_t)$. A neoclassical production function satisfies the following properties:

- Continuous and at least twice differentiable
- Constant returns to scale in K and L. Increasing both capital and labor inputs by a certain proportion translates to an increase of output produced by that same proportion:

$$F(\gamma K,\, \gamma L,\, A) = \gamma F(K, L, A) = \gamma Y \quad \text{for all } \gamma > 0 \tag{4.1}$$

- Positive but diminishing marginal products of K and L:

$$\begin{aligned} \frac{\partial F(K,L,A)}{\partial K} &\equiv F_K(K,L,A) > 0 \qquad \text{and} \qquad & \frac{\partial F(K,L,A)}{\partial K} &\equiv F_K(K,L,A) > 0 \\ \frac{\partial^2 F(K,L,A)}{\partial K^2} &\equiv F_{KK}(K,L,A) < 0 \qquad \text{and} \qquad & \frac{\partial^2 F(K,L,A)}{\partial L^2} &\equiv F_{LL}(K,L,A) < 0 \end{aligned} \tag{4.2}$$

- Often (but not always) we also impose the following essentiality and Inada's conditions:

$$\begin{aligned} F(0,L,A) &= 0 \qquad \text{and} \qquad & F(K,0,A) &= 0 \\ \lim_{K\to 0} F_K(K,L,A) &= \infty \qquad \text{and} \qquad & \lim_{L\to 0} F_K(K,L,A) &= \infty \\ \lim_{K\to \infty} F_K(K,L,A) &= 0 \qquad \text{and} \qquad & \lim_{L\to \infty} F_K(K,L,A) &= 0 \end{aligned} \tag{4.3}$$

The assumption of constant returns to scale has important implications. For example, it makes the number of firms indeterminate, as the economy behaves identically in the case of one firm that employs all resources and in the case of M firms, each employing a $1/M$ fraction of all resources. Therefore we often assume that the firms' sector is represented by a single representative firm.

Definition: function homogenous of degree m

Let $X \in \mathbb{N}$. A function $f : \mathbb{R}^{X+2} \to \mathbb{R}$ is homogenous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if

$$f(\gamma x,\, \gamma y,\, z) = \gamma^m \cdot f(x,y,z) \quad \text{for all } \gamma \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^X$$

According to the above definition, a neoclassical production function is homogenous of degree one in capital K and labor L.

Theorem: Euler's homogeneous function theorem

Suppose that $f : \mathbb{R}^{X+2} \to \mathbb{R}$ is differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by f_x and f_y, and is homogeneous of degree m in x and y. Then:

$$m \cdot f(x,y,z) = f_x(x,y,z) \cdot x + f_y(x,y,z) \cdot y \quad \text{for all } x \in \mathbb{R},\, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^X$$

Moreover, $f_x(x,y,z)$ and $f_x(x,y,z)$ are themselves homogeneous of degree $m-1$ in x and y.

Proof: Differentiate the definition for function homogenous of degree m with respect to γ:

$$m\gamma^{m-1} \cdot f(x,y,z) = f_x(\gamma x,\, \gamma y,\, z) \cdot x + f_y(\gamma x,\, \gamma y,\, z) \cdot y$$

Setting $\gamma = 1$ proves the first result of the theorem. To obtain the second result, differentiate the definition for function homogenous of degree m with respect to x:

$$\begin{aligned} f_x(\gamma x,\, \gamma y,\, z) \cdot \gamma &= \gamma^m \cdot f_x(x,y,z) \quad \Big| \quad : \gamma \\ f_x(\gamma x,\, \gamma y,\, z) &= \gamma^{m-1} \cdot f_x(x,y,z) \end{aligned}$$

In our context, the theorem translates to the following results:

$$\begin{aligned} F(K,L,A) &= F_K(K,L,A) \cdot K + F_L(K,L,A) \cdot L \\ F_K(\gamma K,\, \gamma L,\, A) = F_K(K,L,A) &\quad \text{and} \quad F_L(\gamma K,\, \gamma L,\, A) = F_L(K,L,A) \end{aligned} \tag{4.4}$$

4.2 Solow-Swan model: the case of no technological progress

Consider first the case of no technological progress. We will allow for population level N to change at a constant (but not necessarily positive) rate n, and assume that the labor force L evolves proportionally to population, and therefore also grows at rate n:

$$N_{t+1} = (1+n)\, N_t \quad \rightarrow \quad L_{t+1} = (1+n)\, L_t$$

Since aggregate variables will trend together with population, it is convenient to write down the model in terms of variables per worker:

- capital per worker $k_t \equiv K_t/L_t$
- output per worker $y_t \equiv Y_t/L_t$
- consumption per worker $c_t \equiv C_t/L_t$

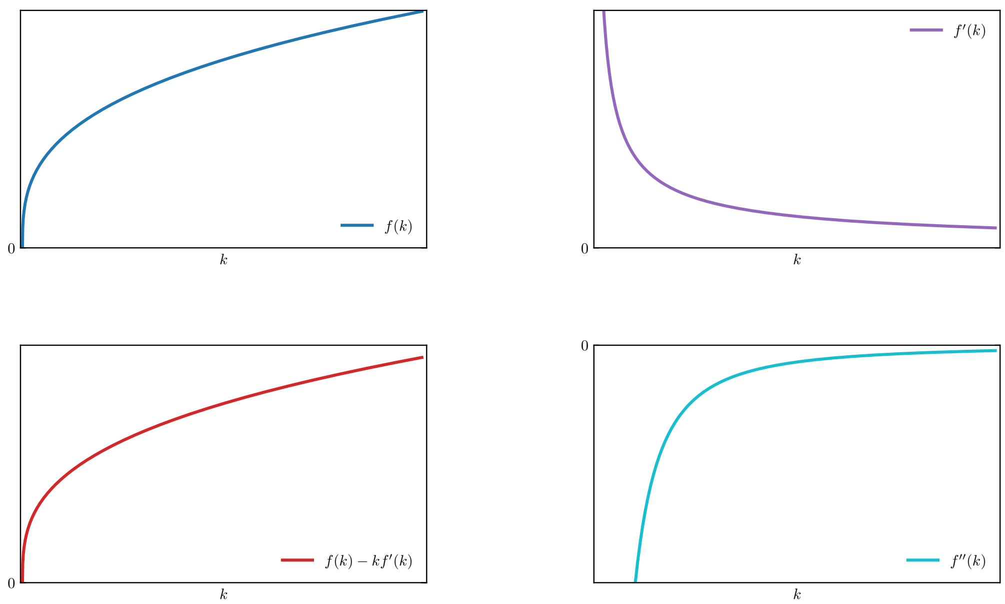
We can invoke the constant returns to scale property of the neoclassical production function (4.1) to express the production function in its intensive (per-worker) form:

$$y_t = \frac{Y_t}{L_t} = \frac{1}{L_t} \cdot F\,(K_t, L_t, A) = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}, A\right) = F\,(k_t, 1, A) \equiv f\,(k_t)$$

The results derived from the Euler theorem (4.4) together with the properties of marginal products of factors of production (4.2) and Inada's conditions (4.3) guarantee that the production function in its intensive form satisfies the following properties:

$$f\,(0) = 0, \quad f'\,(k) > 0, \quad f''\,(k) < 0, \quad \lim_{k \to 0} f'\,(k) = \infty, \quad \lim_{k \to \infty} f'\,(k) = 0, \quad f\,(k) - kf'\,(k) > 0 \quad (4.5)$$

The graphs below depict these properties of f:



Often we will be using the Cobb-Douglas production function:

$$Y = F\,(K, L, A) = AK^{\alpha}L^{1-\alpha}$$

with the following intensive form:

$$y = \frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L^{\alpha} \cdot L^{1-\alpha}} = A\,(K/L)^{\alpha} = Ak^{\alpha}$$

I leave the verification of properties (4.1-4.5) for the Cobb-Douglas production function as an exercise.

Let us now use the remaining ingredients of the model: the capital accumulation equation and the simplifying assumption of a constant ratio of investment to output ($I_t = sY_t$):

$$\begin{aligned}
K_{t+1} &= I_t + (1-\delta) K_t \\
K_{t+1} &= sY_t + (1-\delta) K_t \quad | \quad : L_t \\
\frac{K_{t+1}}{L_t} &= s\frac{Y_t}{L_t} + (1-\delta)\frac{K_t}{L_t} \\
\frac{L_{t+1}}{L_t} \cdot \frac{K_{t+1}}{L_{t+1}} &= sy_t + (1-\delta) k_t \\
(1+n) k_{t+1} &= sf(k_t) + (1-\delta) k_t
\end{aligned}$$

We arrive at the fundamental equation of Solow-Swan model (in discrete time), which describes the dynamics of capital per worker over time:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta) k_t}{1+n}$$

The fundamental equation can be expressed in form of change in k or the growth rate of k:

$$\Delta k_{t+1} \equiv k_{t+1} - k_t = \frac{sf(k_t) + (1-\delta) k_t - (1+n) k_t}{1+n} = \frac{sf(k_t) - (\delta+n) k_t}{1+n} \simeq sf(k_t) - (\delta+n) k_t$$

$$g_k \equiv \frac{\Delta k_{t+1}}{k_t} \simeq s\frac{f(k_t)}{k_t} - (\delta+n)$$

The growth rate of capital per worker depends negatively on k and reaches exactly 0 in the steady state:

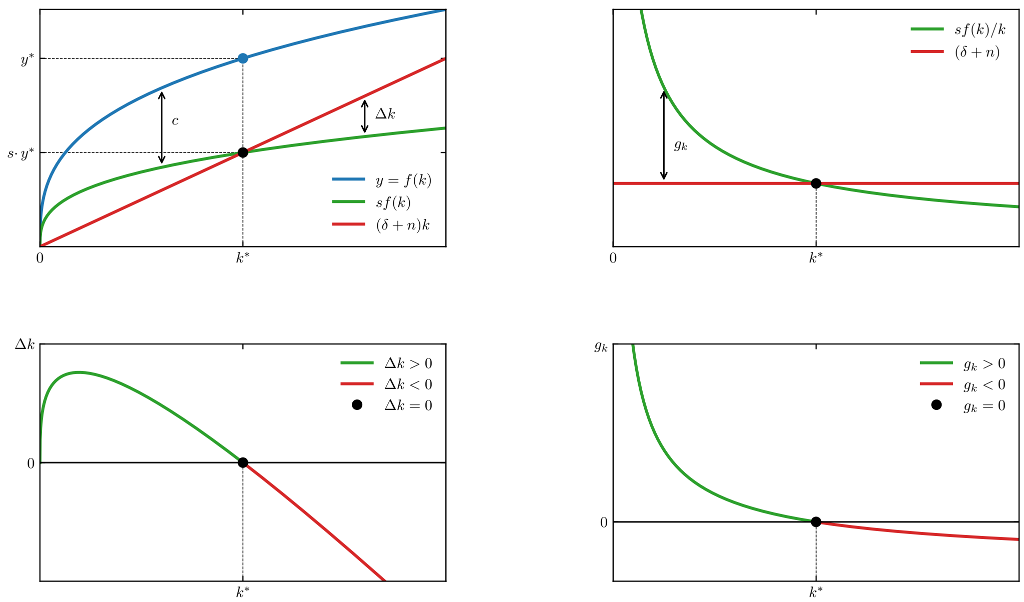
$$s\frac{f(k^*)}{k^*} - (\delta+n) = 0 \quad \rightarrow \quad s\frac{f(k^*)}{k^*} = (\delta+n) \quad \rightarrow \quad \frac{k^*}{f(k^*)} = \frac{s}{\delta+n}$$

In the steady state the capital to output ratio (per worker, but also in aggregate terms) is a positive function of s and a negative function on δ and n. To show that, we need to establish that $f(k)/k$ depends negatively on k (and so $k/f(k)$ depends positively on k and vice versa):

$$\frac{\partial (f(k)/k)}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{f(k) - f'(k)k}{k^2} < 0$$

where the sign is a consequence of the properties of the production function in its intensive form (4.5).

The graphs below illustrate the behavior of the system:



To be more specific in our results we need to assume a particular production function. For the Cobb-Douglas production function the fundamental equation becomes:

$$(1+n)\,k_{t+1} = sAk_t^{\alpha} + (1-\delta)\,k_t$$

The steady state can also be found by equating $k_{t+1} = k_t = k^*$:

$$(1+n)\,k^* = sA(k^*)^{\alpha} + (1-\delta)\,k^* \quad \rightarrow \quad (\delta+n)\,k^* = sA(k^*)^{\alpha} \quad \rightarrow \quad (k^*)^{1-\alpha} = \frac{sA}{\delta+n}$$

$$k^* = \left(\frac{sA}{\delta+n}\right)^{1/(1-\alpha)}$$

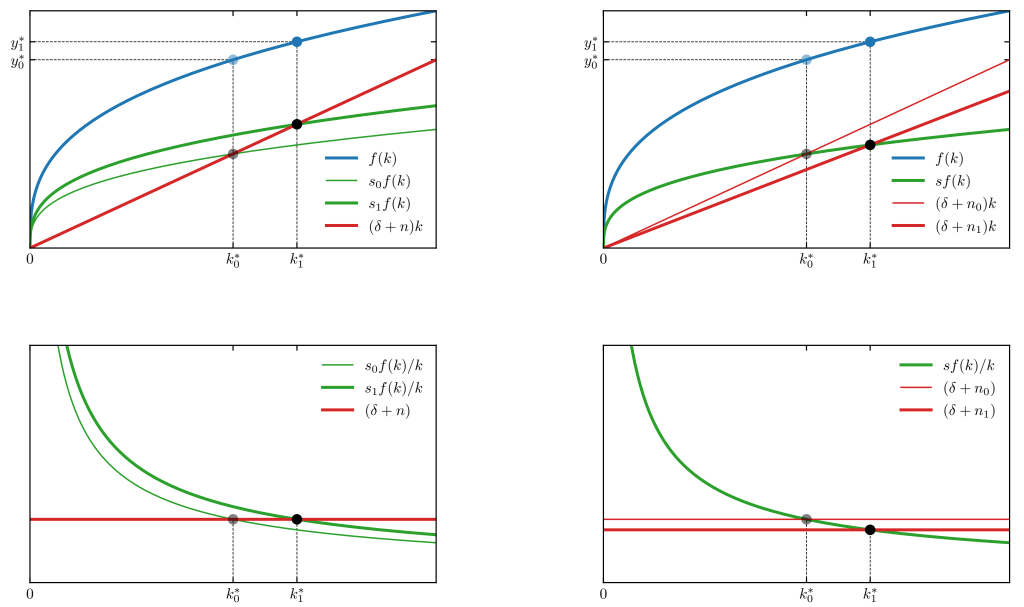
And the remaining variables per worker can be easily calculated:

$$y^* = A(k^*)^{\alpha} = A\left(\frac{sA}{\delta+n}\right)^{\alpha/(1-\alpha)}$$

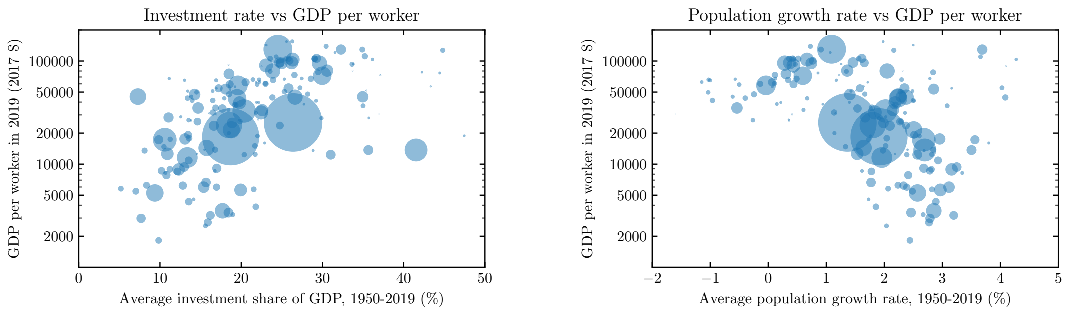
$$c^* = (1-s)\,y^* = (1-s)\,A\left(\frac{sA}{\delta+n}\right)^{\alpha/(1-\alpha)}$$

The model predicts then that steady state output (GDP) per worker is higher in countries with higher A, higher s, lower δ and lower n. We can depict how changes in s and n influence the steady state and the system dynamics:

$s_0 \nearrow s_1$ $\qquad$ $n_0 \searrow n_1$



Indeed in the data countries with higher average investment share of GDP or countries with lower average population growth rate in years 1950-2019 tended to have higher GDP per worker in 2019:



4.3 Golden rule saving rate and dynamic inefficiency

While higher saving / investment rate lead to higher level of output per worker in the steady state, economic growth cannot continue indefinitely via capital accumulation alone. This is due to the diminishing marginal returns to capital (per worker): eventually extra output generated by higher capital stock just enough covers the cost of depreciation δ and the dilution of capital stock between changing number of employees n. Moreover, increasing the saving rate beyond a certain threshold results in lowering steady state consumption per worker.

What would be then the saving rate that would maximize consumption per worker in the long run? It turns out that it is a bit easier to find the answer to a related question: what is the steady state level of capital per worker that is associated with maximal consumption per worker? Start with the national accounting identity and use the fact that in the steady state investment per worker equals effective depreciation of capital:

$$Y_t = C_t + I_t \quad \rightarrow \quad c_t = y_t - sy_t \quad \rightarrow \quad c^* = f(k^*) - (\delta + n)\, k^*$$

We can now calculate the derivative of c^* with respect to k^* and equate it to 0:

$$\partial c^* / \partial k^* = f'(k^*) - (\delta + n) = 0 \quad \rightarrow \quad k^*_{GR} = (f')^{-1}\,(\delta + n)$$

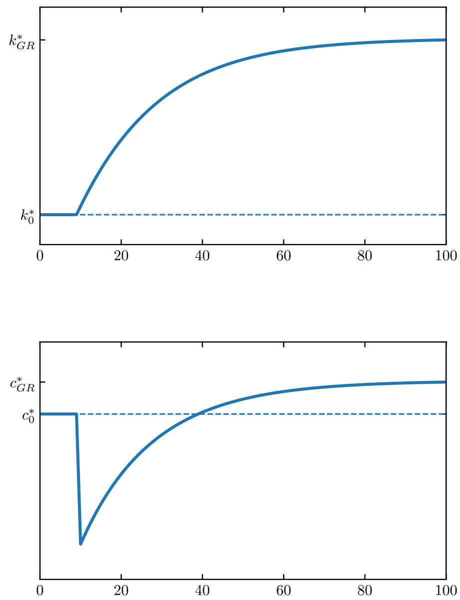
where k^*_{GR}, called the "golden rule" level of capital per worker, maximizes consumption per worker in the long run. As k^* monotonously increases with s, there exists a unique s_{GR}, called the "golden rule" saving rate, that implements this particular long-run equilibrium. Let us find their levels for the Cobb-Douglas production function:

$$\alpha A (k^*)^{\alpha - 1} = \delta + n \quad \rightarrow \quad k^*_{GR} = [(\alpha A)\,/\,(\delta + n)]^{1/(1-\alpha)} \quad \rightarrow \quad s_{GR} = \alpha$$

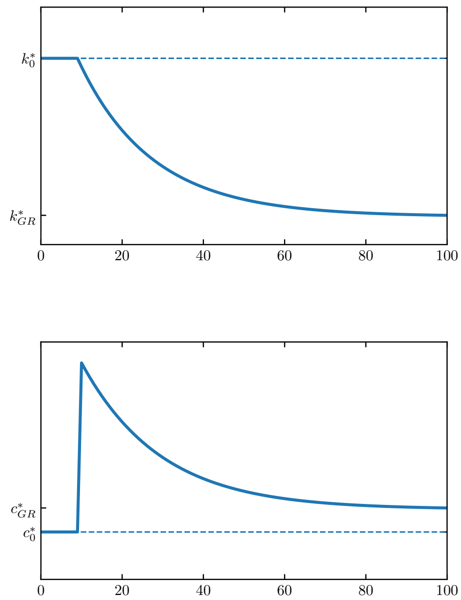
However, since s is exogenous, there is no good reason for it to be exactly equal to s_{GR}. Let us now examine two possible cases:

1. Dynamic efficiency: when $s < s_{GR}$ (and $k^* < k^*_{GR}$) then increasing the saving rate (up to s_{GR}) leads to a decrease in consumption in the short run, and an increase in consumption over the long run above its initial level.
2. Dynamic inefficiency: when $s > s_{GR}$ (and $k^* > k^*_{GR}$) then decreasing the saving rate (down to s_{GR}) leads to an increase in consumption both in the short run and in the long run. That means that the initial equilibrium was (dynamically) inefficient!

Case 1: $s \nearrow s_{GR}$



Case 2: $s \searrow s_{GR}$



4.4 Rates of growth outside the steady state

By definition of the steady state the growth rates of variables per worker are 0. What is the growth rate of e.g. output per worker outside of the steady state? We will often employ the following convenient approximation (which is exact for $x \to 0$):

$$\ln(1+x) \simeq x$$

Then:

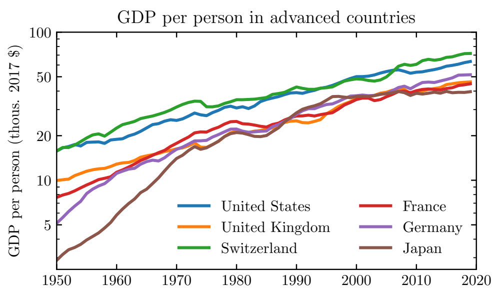
$$g_y \equiv \frac{y_{t+1} - y_t}{y_t} = \frac{y_{t+1}}{y_t} - 1 \quad \to \quad 1 + g_y = \frac{y_{t+1}}{y_t} \quad \to \quad g_y \simeq \ln(1+g_y) = \ln y_{t+1} - \ln y_t$$

And so (for the Cobb-Douglas production function):

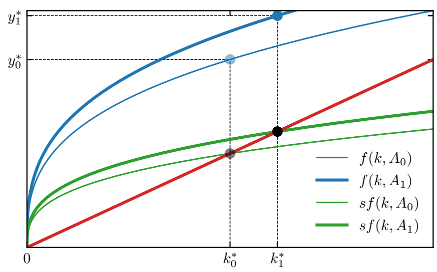
$$\begin{aligned} g_y &\simeq \ln y_{t+1} - \ln y_t = \ln\left(Ak_{t+1}^{\alpha}\right) - \ln\left(Ak_t^{\alpha}\right) = \ln A + \alpha \ln k_{t+1} - \ln A - \alpha \ln k_t = \alpha\left(\ln k_{t+1} - \ln k_t\right) \\ &\simeq \alpha g_k = \alpha\left(sAk_t^{\alpha-1} - (\delta + n)\right) \end{aligned}$$

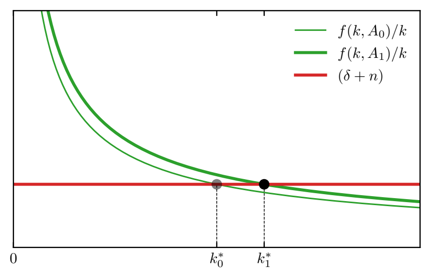
We can again conclude that an increase in s (or a decrease in n) can temporarily increase the growth rate of the economy and it starts transitioning to a higher level of steady state output per worker, but growth disappears once it reaches the new steady state.

Up until this point we assumed that the level of technology is constant. If that was the case however, we would not observe the phenomenon of sustained economic growth in the data, as all variables per worker / per person would stabilize in the long run at some constant value. However, even the most advanced economies still exhibit positive rates of growth of GDP per person:



The "Solow surprise" was that in order for economies to enjoy positive growth in the long run, technological progress is essential and is the only driver of growth in the (very) long run. Before we augment the model with sustained technological progress, let us examine the effects of a one-time increase in A:





Two effects are at play here. First, an increase in A boosts the productivity of the factors of production, enabling the economy to produce more goods and services with the same inputs, which is a level effect. Second, it generates a growth effect where the variables per worker start to converge to a new, higher steady state.

4.5 Balanced growth path

Consider the fundamental equation of a one-sector closed economy:

$$K_{t+1} = I_t + (1-\delta)\, K_t \tag{4.6}$$

And rewrite it in difference form:

$$\Delta K_{t+1} = I_t - \delta K_t = Y_t - C_t - \delta K_t$$

It turns out that this equation generates a lot of interesting results.

Definition: balanced growth path (BGP)

A balanced growth path is a path $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ along which the quantities Y_t, K_t and C_t are positive and grow at constant rates, which we denote g_Y, g_K and g_C, respectively.

Proposition: equivalence of balanced growth and constancy of key ratios

Let $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t and $I_t = Y_t - C_t$ are positive for all $t \geq 0$. Then, given the capital accumulation equation 4.6, the following holds:

1. If there is balanced growth, then $g_Y = g_K = g_C = g_I$ and the ratios K/Y, C/Y and I/Y are constant.
2. If K/Y and C/Y are constant, then Y, K, C and I all grow at the same constant rate, i.e. there is not only balanced growth, but balanced growth where $g_Y = g_K = g_C = g_I$.

Note that 1. means that any economy on a balanced growth path has to generate constant key ratios. 2. ensures that if we observe constant ratios in the data, the economy is on a balanced growth path.

Proof of 1. See Appendix.

Proof of 2.

Suppose that K/Y and C/Y are constant. Then $g_Y = g_K = g_C$ and as a consequence $(I/Y = 1 - C/Y)$ $g_Y = g_I = g_K$. Now we need to show that the growth rates are also constant. To that end we use equation 4.6:

$$\Delta K_{t+1} = I_t - \delta K_t \quad | \quad : K_t$$
$$g_K = \frac{I_t}{K_t} - \delta$$

Since $g_I = g_K$ then I/K is constant and in consequence g_K and all other key growth rates are constant.

4.6 Technological progress

There are at least three possibilities in how technological progress might manifest itself in our production function. Let $\tilde{F}$ denote the "true" production function and $\tilde{A}$ the "true" form of technological progress:

- Hicks-neutral: technology acts as a multiplicative constant: $\tilde{F}(K_t, L_t, \tilde{A}_t) = A_t \cdot F(K_t, L_t)$
- Solow-neutral: technology increases productivity of capital: $\tilde{F}(K_t, L_t, \tilde{A}_t) = F(A_t \cdot K_t, L_t)$
- Harrod-neutral: technology increases productivity of labor: $\tilde{F}(K_t, L_t, \tilde{A}_t) = F(K_t, A_t \cdot L_t)$

Obviously the technological progress might manifest itself as a combination of the above possibilities: $\tilde{F}(K_t, L_t, \tilde{A}_t) = A_{H,t} \cdot F(A_{K,t} \cdot K_t, A_{L,t} \cdot L_t)$

Despite all of the above forms appearing ex ante plausible, balanced growth requires that the "true" production function has a representation of the Harrod-neutral form.

Theorem: Uzawa's balanced growth path theorem

Let $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$, where $0 < C_t < Y_t$ for all $t \geq 0$ be a path satisfying the capital accumulation equation 4.6. Suppose also that $\tilde{F}(K_t, L_t, \tilde{A}_t)$ exhibits constant returns to scale in K and L and worker population grows at a rate n such that $L_t = L_0 \cdot (1+n)^t$.

Then a necessary condition for this path to be a balanced growth path is that:

$$Y_t = \tilde{F}(K_t, L_t, \tilde{A}_t) = F(K_t, A_t \cdot L_t)$$

where $A_t = A_0 \cdot (1+g)^t$ with $g \equiv g_Y - n$.

Proof

Suppose the path $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ is a balanced growth path. Then g_Y and g_K are equal and constant. As a consequence, we can write $Y_t = Y_0 \cdot (1+g_K)^t$ and $K_t = K_0 \cdot (1+g_K)^t$. We then have:

$$Y_t \cdot (1+g_K)^{-t} = Y_0 = \tilde{F}(K_0, L_0, \tilde{A}_0) = \tilde{F}\left(K_t \cdot (1+g_K)^{-t}, L_t \cdot (1+n)^{-t}, \tilde{A}_0\right)$$

Because $g_Y = g_K$ and the function $\tilde{F}$ satifies CRS property, we can rewrite the above equation as:

$$Y_t = \tilde{F}\left(K_t \cdot (1+g_K)^{-t} \cdot (1+g_Y)^t, L_t \cdot (1+n)^{-t} \cdot (1+g_Y)^t, \tilde{A}_0\right) = \tilde{F}\left(K_t, L_t \cdot \left(\frac{1+g_Y}{1+n}\right)^t, \tilde{A}_0\right)$$

Note that in the above formulation $\tilde{A}$ is reduced to a constant $\tilde{A}_0$. Therefore, there exists a function $F(K, L)$ such that:

$$Y_t = F\left(K_t, L_t \cdot \left(\frac{1+g_Y}{1+n}\right)^t\right)$$

which we can rewrite as

$$Y_t = F(K_t, A_t \cdot L_t)$$

where $A_t = [(1+g_Y)/(1+n)]^t$ and $g = (1+g_Y)/(1+n) - 1 \approx g_Y - n$ is the rate of "rewritten" technology growth.

Note that it does not mean that literally all technological progress directly increases labor productivity, just that the "true" production function can be rewritten in the required functional form.

4.7 Solow-Swan model: the case with technological progress

Assume that the number of workers L, all employed, changes at the same rate as the total population N at rate n:

$$L_{t+1}/L_t = N_{t+1}/N_t = 1 + n$$

and that technology A improves at a constant rate $g \geq 0$:

$$A_{t+1} = (1 + g)\, A_t$$

The production function will observe the following form:

$$Y_t = F\left(K_t, A_t L_t\right)$$

The economy this time will not converge to a steady state (unless $g = 0$), but rather to the balanced growth path with a positive rate of growth in output per worker. Since the variables per worker are then also trending, we need to introduce additional notation:

- capital per effective units of labor $\hat{k}_t = K_t/\left(A_t L_t\right)$
- output per effective units of labor $\hat{y}_t = Y_t/\left(A_t L_t\right)$
- consumption per effective units of labor $\hat{c}_t = C_t/\left(A_t L_t\right)$

Again use the constant returns to scale property of the neoclassical production function:

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = \frac{F\left(K_t, A_t L_t\right)}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, \frac{A_t L_t}{A_t L_t}\right) = F(\hat{k}_t, 1) \equiv f(\hat{k}_t)$$

where (with some abuse of notation) f denotes now the production function per effective units of labor. For the Cobb-Douglas
Again start with the capital accumulation equation and use the assumption that $I_t = sY_t$:

$$\begin{aligned}
K_{t+1} &= sY_t + (1-\delta)\, K_t \quad | \quad : A_t L_t \\
\frac{K_{t+1}}{A_t L_t} &= s\frac{Y_t}{A_t L_t} + (1-\delta)\,\frac{K_t}{A_t L_t} \\
\frac{A_{t+1}}{A_t}\frac{L_{t+1}}{L_t}\frac{K_{t+1}}{A_{t+1}L_{t+1}} &= s\hat{y}_t + (1-\delta)\,\hat{k}_t \\
(1+g)\,(1+n)\,\hat{k}_{t+1} &= sf(\hat{k}_t) + (1-\delta)\,\hat{k}_t
\end{aligned}$$

The fundamental equation now describes the dynamics of capital per effective labor over time:

$$\hat{k}_{t+1} = \frac{sf(\hat{k}_t) + (1-\delta)\,\hat{k}_t}{(1+n)\,(1+g)}$$

Again use the Cobb-Douglas production function (this time with technology directly enhancing the labor input) to get more specific results:

$$Y_t = F\left(K_t, A_t L_t\right) = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$$

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}}{\left(A_t L_t\right)^{\alpha}\left(A_t L_t\right)^{1-\alpha}} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = \hat{k}_t^{\alpha}$$

Let us go back to the fundamental equation for the technological progress case:

$$(1+g)(1+n)\,\hat{k}_{t+1} = s\hat{k}_t^{\alpha} + (1-\delta)\,\hat{k}_t$$

And first find the Balanced Growth Path by equating $\hat{k}_{t+1} = \hat{k}_t = \hat{k}^*$:

$$(1+g)(1+n)\,\hat{k}^* = s(\hat{k}^*)^{\alpha} + (1-\delta)\,\hat{k}^* \quad \rightarrow \quad (\delta+n+g+ng)\,\hat{k}^* = s(\hat{k}^*)^{\alpha}$$

$$(\hat{k}^*)^{1-\alpha} = \frac{s}{\delta+n+g+ng} \quad \rightarrow \quad \hat{k}^* = \left(\frac{s}{\delta+n+g+ng}\right)^{1/(1-\alpha)} \simeq \left(\frac{s}{\delta+n+g}\right)^{1/(1-\alpha)}$$

where the ng term can be usually approximated away, and it disappears completely in continuous time version of the model.

The remaining variables per effective labor along the BGP are then given by:

$$\hat{y}^* = (\hat{k}^*)^{\alpha} = \left(\frac{s}{\delta+n+g}\right)^{\alpha/(1-\alpha)}$$
$$\hat{c}^* = (1-s)\,\hat{y}^* = (1-s)\left(\frac{s}{\delta+n+g}\right)^{\alpha/(1-\alpha)}$$

Behavior of factor prices along the balanced growth path

In the Solow-Swan economy, factor prices (that is, wages and the interest rate) are determined by the relative availability of capital and labor. To see this relationship more clearly, let us solve the problem of profit maximization of the representative firm:

$$\max_{Y_t,\,K_t,\,L_t} \quad D_t = Y_t - w_t L_t - (r_t+\delta)\,K_t$$
$$\text{subject to} \quad Y_t = F\left(K_t, A_t L_t\right)$$

Which can be simplified if we plug in the production function directly into the profit function:

$$\max_{K_t,\,L_t} \quad D_t = F\left(K_t, A_t L_t\right) - w_t L_t - (r_t+\delta)\,K_t$$

It is here more convenient to rewrite the problem again in the "per effective labor" form:

$$\max_{\hat{k}_t,\,L_t} \quad A_t L_t\left[f(\hat{k}_t) - \hat{w}_t - (r_t+\delta)\,\hat{k}_t\right]$$

where $\hat{w}_t \equiv w_t/A_t$. First order conditions:

$$\hat{k}_t \quad : \quad A_t L_t\left[f'(\hat{k}_t) - (r_t+\delta)\right] = 0 \qquad \rightarrow \qquad r_t = f'(\hat{k}_t) - \delta$$
$$L_t \quad : \quad A_t\left[f(\hat{k}_t) - \hat{w}_t - (r_t+\delta)\,\hat{k}_t\right] = 0 \quad \rightarrow \quad \hat{w}_t = f(\hat{k}_t) - (r_t+\delta)\,\hat{k}_t = f(\hat{k}_t) - \hat{k}_t \cdot f'(\hat{k}_t)$$

The interest rate is decreasing in capital per effective labor, while the "effective" wage $\hat{w}$ is increasing in capital per effective labor.[1] Along the balanced growth path, interest rate is predicted to be constant, while the real wage is predicted to grow at the rate of technological progress, which is exactly what we see in the data for advanced economies.

[1] To see this, take the derivative of the effective wage with respect to capital per effective labor:

$$\partial\hat{w}/\partial\hat{k} = f'(\hat{k}) - \left[f'(\hat{k}) + \hat{k}\cdot f''(\hat{k})\right] = -\hat{k}\cdot f''(\hat{k}) > 0$$

The above expression is positive due to the property of diminishing marginal products (4.2).

Rates of growth of variables along the balanced growth path

By definition, variables per effective labor do not change along the BGP, i.e. $g^*_{\hat{k}} = g^*_{\hat{y}} = g^*_{\hat{c}} = 0$. Using our definitions of variables per effective labor and per worker we can easily obtain the rates of growth of variables per worker (for example consumption):

$$\begin{aligned} c_t &= A_t \cdot \hat{c}_t \\ \ln c_t &= \ln A_t + \ln \hat{c}_t \\ (\ln c_{t+1} - \ln c_t) &= (\ln A_{t+1} - \ln A_t) + (\ln \hat{c}_{t+1} - \ln \hat{c}_t) \\ g_c &= g_A + g_{\hat{c}} \quad \to \quad g^*_c = g_A + g^*_{\hat{c}} = g + 0 = g \end{aligned}$$

Along the BGP all per worker variables grow at a rate equal to the rate of technological progress. This is a remarkable result as it implies that in the (very) long run the only factor responsible for the improvement of our lives is the rate of improvement in labor productivity. So if we want to make future generations better off we should focus on inventing new and/or better ways of production.

The BGP rate of growth of aggregate variables is also easy to find:

$$\begin{aligned} C_t &= c_t \cdot L_t \\ \ln C_t &= \ln c_t + \ln L_t \\ (\ln C_{t+1} - \ln C_t) &= (\ln c_{t+1} - \ln c_t) + (\ln L_{t+1} - \ln L_t) \\ g_C &= g_c + g_L \quad \to \quad g^*_C = g + n \end{aligned}$$

Transition dynamics

Outside the balanced growth path the economy is in the process of convergence to the BGP. That means that if, say, $\hat{k}_t < \hat{k}^*$ then $g_{\hat{k}} > 0$. As a consequence, both per worker and aggregate variables also grow faster than g and $g + n$, respectively. We describe this as the "catching-up process", as the economies that are further away from their steady states grow faster than the ones that are closer to theirs.

The general formula for the rate of growth of capital per effective worker can be obtained from the fundamental equation:

$$\begin{aligned} \Delta \hat{k}_{t+1} \equiv \hat{k}_{t+1} - \hat{k}_t &= \frac{sf(\hat{k}_t) + (1-\delta)\,\hat{k}_t - (1+n+g+ng)\,\hat{k}_t}{(1+n)\,(1+g)} = \frac{sf(\hat{k}_t) - (\delta+n+g+ng)\,\hat{k}_t}{(1+n)\,(1+g)} \\ &\simeq sf(\hat{k}_t) - (\delta+n+g)\,\hat{k}_t \\ g_{\hat{k}} \equiv \frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} &\simeq s\frac{f(\hat{k}_t)}{\hat{k}_t} - (\delta+n+g) \end{aligned} \tag{4.7}$$

Recall that we have shown that $\hat{k}/f(\hat{k})$ increases with $\hat{k}$, which means that $f(\hat{k})/\hat{k}$ has to decrease with $\hat{k}$. Again, that means that if $\hat{k}_t < \hat{k}^*$ then $g_{\hat{k}} > 0$ (and if $\hat{k}_t > \hat{k}^*$ then $g_{\hat{k}} < 0$). We would also like to assign some numerical values for the rates of growth outside the BGP.

Example (Cobb-Douglas production function)

Let $f(\hat{k}) = \hat{k}_t^{\alpha}$. Then the rate of growth of capital per effective labor is given by:

$$g_{\hat{k}} = s\hat{k}_t^{\alpha-1} - (\delta+n+g)$$

But usually we are much more interested in the rate of growth of output per worker. After some algebra (see Appendix for details) the formula one can obtain the following formula relating the growth rate of output per worker to the distance of the economy to its steady state:

$$g_y \approx g + (1-\alpha)\,(\delta+n+g)\,(\ln y^*_t - \ln y_t)$$

The above equation is very often used in empirical research, which typically finds the rate of convergence to be around 2% per year.

4.8 Appendix

Balanced Growth Path

Proof of **Proposition 1** that if there is balanced growth, then $g_Y = g_K = g_C = g_I$ and the ratios K/Y, C/Y and I/Y are constant.
Consider an economy on a balanced growth path. Then, by definition, g_Y, g_K and g_C are constant. If we use equation 4.6, we get the result that $g_I = g_K$:

$$\Delta K_{t+1} = I_t - \delta K_t \quad | \quad : K_t$$
$$g_K = \frac{\Delta K_{t+1}}{K_t} = \frac{I_t}{K_t} - \delta$$
$$\frac{I_t}{K_t} = g_K + \delta$$

Since the right hand side of the above equation is constant, then the I/K ratio is constant and that means that the rates of growth of I and K have to be identical.

Focus now on the national accounting relationship:

$$Y_t = C_t + I_t \quad \rightarrow \quad \Delta Y_{t+1} = \Delta C_{t+1} + \Delta I_{t+1}$$

$$g_Y = \frac{\Delta Y_{t+1}}{Y_t} = \frac{\Delta C_{t+1}}{Y_t} + \frac{\Delta I_{t+1}}{Y_t} = \frac{\Delta C_{t+1}}{C_t}\frac{C_t}{Y_t} + \frac{\Delta I_{t+1}}{I_t}\frac{I_t}{Y_t} = g_C\frac{C_t}{Y_t} + g_I\frac{Y_t - C_t}{Y_t} = \frac{C_t}{Y_t}(g_C - g_I) + g_I$$
$$g_Y = \frac{C_t}{Y_t}(g_C - g_K) + g_K$$

where in the last line the equality between growth rates of I and K was used.

We have now two possibilities: either $g_C = g_K$ (and in consequence $g_Y = g_K$) or $g_C \neq g_K$. Let us assume that $g_C \neq g_K$. Then we can rewrite the above expression as:

$$\frac{C_t}{Y_t} = \frac{g_Y - g_K}{g_C - g_K}$$

The right hand side contains only growth rates, which are constant by the definition of the BGP. That means that the C/Y ratio is constant as well, and $g_C = g_Y$. But this implies that the right hand side is equal to 1 and $C_t = Y_t$, which contradicts our assumption that $I_t = Y_t - C_t$ is positive.

That means that our assumption $g_C \neq g_K$ was wrong and so $g_C = g_K$. This also implies that $g_Y = g_K = g_I = g_C$.

Speed of convergence

For a general production function we don't have access to closed-form solutions outside the BGP. However, we can obtain the approximation for the growth rate of $\hat{k}$ using a linear approximation around the steady state. This technique relies on a first-order Taylor expansion of equation 4.7 with respect to $\ln \hat{k}$ around the steady state value of $\ln \hat{k}^*$.

$$g_{\hat{k}} \approx \underbrace{\left[s\frac{f(\hat{k}^*)}{\hat{k}^*} - (\delta + n + g)\right]}_{0} + \left\{\frac{\mathrm{d}}{\mathrm{d}\ln \hat{k}_t}\left[s\frac{f(\hat{k}_t)}{\hat{k}_t} - (\delta + n + g)\right]\right\}_{\hat{k}=\hat{k}^*}(\ln \hat{k}_t - \ln \hat{k}^*)$$

The first term in the square brackets evaluates to 0. We now need to find the expression for the derivative of $g_{\hat{k}}$ with respect to $\ln \hat{k}$.

One nice property that we will use is the following chain-rule result:

$$\frac{\mathrm{d}y}{\mathrm{d}\ln x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\ln x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}e^{\ln x}}{\mathrm{d}\ln x} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot e^{\ln x} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot x$$

After making use of the above relationship we get that:

$$\frac{\mathrm{d}}{\mathrm{d}\ln \hat{k}_t}\left[s\frac{f(\hat{k}_t)}{\hat{k}_t} - (\delta+n+g)\right] = s\cdot\frac{\mathrm{d}}{\mathrm{d}\hat{k}_t}\left[\frac{f(\hat{k}_t)}{\hat{k}_t}\right]\cdot\hat{k}_t = s\hat{k}_t\frac{f'(\hat{k}_t)\cdot\hat{k}_t - f(\hat{k}_t)}{\hat{k}_t^2} = s\frac{f(\hat{k}_t)}{\hat{k}_t}\left[\frac{f'(\hat{k}_t)\cdot\hat{k}_t}{f(\hat{k}_t)} - 1\right]$$

The above expression can be simplified if we use the definition of the elasticity (here elasticity of output w.r.t. capital). For the Cobb-Douglas production function this elasticity is always equal to α.

$$\varepsilon_{\hat{k}} \equiv \frac{\mathrm{d}f(\hat{k}_t)}{\mathrm{d}\hat{k}_t}\frac{\hat{k}_t}{f(\hat{k}_t)} = \frac{\mathrm{d}\ln f(\hat{k}_t)}{\mathrm{d}\ln \hat{k}_t} \tag{4.8}$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\ln \hat{k}_t}\left[s\frac{f(\hat{k}_t)}{\hat{k}_t} - (\delta+n+g)\right]\right\}_{\hat{k}=\hat{k}^*} = s\frac{f(\hat{k}_t)}{\hat{k}^*}(\varepsilon^*_{\hat{k}} - 1) = (\delta+n+g)\,(\varepsilon^*_{\hat{k}} - 1)$$

Finally, we get the expression for the approximate rate of growth of capital per effective labor:

$$g_{\hat{k}} \approx -(1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln\hat{k}_t - \ln\hat{k}^*)$$

Now we would like to translate the difference in logs of capital per effective labor to logs of output per effective labor.

$$\ln\hat{y}_t = \ln f(\hat{k}_t) \approx \underbrace{\{\ln f(\hat{k}_t)\}_{\hat{k}=\hat{k}^*}}_{\ln\hat{y}^*} + \left\{\frac{\mathrm{d}\ln f(\hat{k}_t)}{\mathrm{d}\ln\hat{k}_t}\right\}_{\hat{k}=\hat{k}^*}(\ln\hat{k}_t - \ln\hat{k}^*)$$

$$\ln\hat{y}_t - \ln\hat{y}^* \approx \varepsilon^*_{\hat{k}}(\ln\hat{k}_t - \ln\hat{k}^*) \quad\rightarrow\quad \ln\hat{k}_t - \ln\hat{k}^* \approx (\ln\hat{y}_t - \ln\hat{y}^*)\,/\varepsilon^*_{\hat{k}}$$

where we have used the definition of elasticity of output w.r.t. capital (equation 4.8).

One can also substitute the log-difference in output per effective labor with the log-difference in output per worker:

$$\ln y_t - \ln y^*_t = \ln(A_t\hat{y}_t) - \ln(A_t\hat{y}^*) = \ln A_t + \ln\hat{y}_t - \ln A_t - \ln\hat{y}^* = \ln\hat{y}_t - \ln\hat{y}^*$$

We can now use the above formulas to obtain the approximate rate of growth of output (GDP) per worker:[2]

$$\begin{aligned}
y_t &= A_t\hat{y}_t = A_t f(\hat{k}_t) \quad | \quad \ln(\cdot)\\
\ln y_t &= \ln A_t + \ln f(\hat{k}_t) \quad | \quad \frac{\mathrm{d}}{\mathrm{d}t}\\
g_y \equiv \frac{\mathrm{d}\ln y_t}{\mathrm{d}t} &= \frac{\mathrm{d}\ln A_t}{\mathrm{d}t} + \frac{\mathrm{d}\ln f(\hat{k}_t)}{\mathrm{d}\ln\hat{k}_t}\frac{\mathrm{d}\ln\hat{k}_t}{\mathrm{d}t}\\
g_y &= g + \varepsilon_{\hat{k}}g_{\hat{k}}\\
g_y &\approx g + \varepsilon^*_{\hat{k}}[-(1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln\hat{k}_t - \ln\hat{k}^*)]\\
g_y &\approx g + \varepsilon^*_{\hat{k}}[-(1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln\hat{y}_t - \ln\hat{y}^*)\,/\varepsilon^*_{\hat{k}}]\\
g_y &\approx g - (1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln\hat{y}_t - \ln\hat{y}^*)\\
g_y &\approx g - (1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln y_t - \ln y^*_t)\\
g_y &\approx g + (1-\varepsilon^*_{\hat{k}})\,(\delta+n+g)\,(\ln y^*_t - \ln y_t)
\end{aligned}$$

This result constitutes the basis for the econometric "growth regressions".

[2] In continuous time the growth rate of a variable can be obtained by taking the time derivative of the logarithm of this variable. It corresponds to the discrete time concept that $g_x = \Delta x_{t+1}/x_t \approx \ln x_{t+1} - \ln x_t = (\ln x_{t+\Delta t} - \ln x_t)\,/\,(\Delta t = 1)$.