Chapter 1

Consumption

The main cornerstone of “modern” macroeconomics is its focus on dynamic optimization problems, often involving some source of uncertainty. In this chapter we learn how to solve such problems while providing a first-pass answer to the question of what influences the behavior of households’ consumption. The key tradeoff that the consumers face is whether to consume more right now or to defer some consumption toward future time periods. As a consequence, current consumption not only depends on current income, as in the old Keynesian macroeconomic models, but also on the expected path of future income, as well as on the incentives towards saving or borrowing, which are influenced by the (expected) rate of return on financial assets.

Once we acquire sufficient familiarity in the two-period context under certainty, the model gets extended in multiple interesting directions, allowing us to consider the impact of uncertainty, imperfect access to financial markets, or study the effects of government transfers. Finally, the case of any number of periods (even infinitely many) is solved.

To tackle these issues, we first need to review a universal mathematical tool that allows handling any type of multi-variable optimization problems: the method of Lagrange multipliers. Applying this method in our context is quite straightforward, requiring of us only the knowledge on how to set up the Lagrange function (or the Lagrangian for short) and how to obtain the first order conditions, which requires calculating a few partial derivatives. The next part of the procedure involves summarizing this information and solving a system of equations, yielding the optimal consumption choices of a household.

1.1 Two-period optimization

An agent solves the following problem:

$$\max_{c_1,\, c_2,\, a} \quad U = \ln c_1 + \beta \ln c_2$$
$$\text{subject to} \quad c_1 + a = y_1$$
$$c_2 = y_2 + (1+r)\, a$$

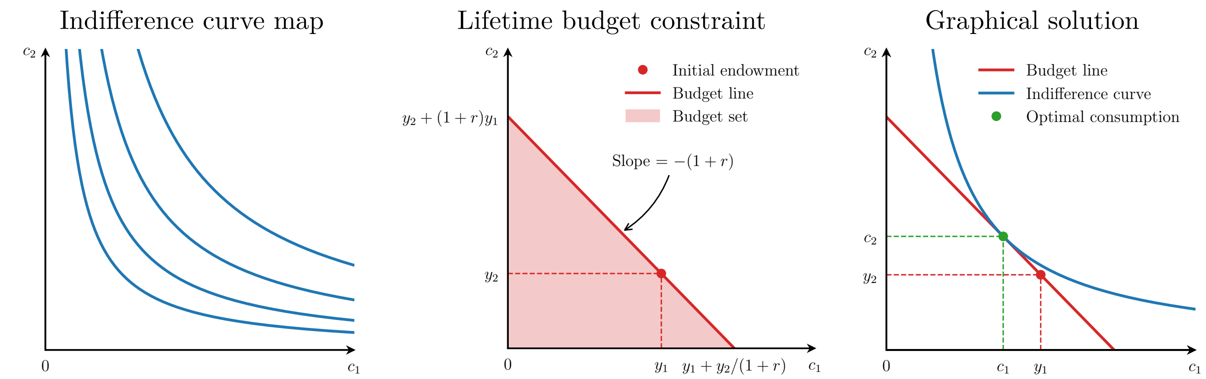
where c_1 and c_2 denote agent's consumption in the first and second period, y_1 and y_2 denote agent's income in the first and second period, $\beta \in [0, 1]$ is the **discount factor**, and a denotes wealth / net assets (possibly negative) held by the agent at the end of period 1, which then yield real interest rate r.

First construct the lifetime budget constraint:

$$a = \frac{c_2 - y_2}{1+r}$$
$$c_1 + \frac{c_2 - y_2}{1+r} = y_1$$
$$\boldsymbol{c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}}$$

The lifetime budget constraint says that the **present discounted value** (PDV) of income (plus initial wealth which is here for simplicity assumed to be 0) has to equal the PDV of consumption.

The problem has an easy graphical interpretation. We are looking for a specific **indifference curve** that is just tangent to the **budget line**. The point of tangency is the **optimal consumption** choice:



To confront theory with the data we need quantitative answers. To that end we will be using the **method of Lagrange multipliers**.

First, rewrite the budget constraint so that one side of the equation equals 0:

$$y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0$$

And then set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

where λ is the Lagrange multiplier with the following economic interpretation: by how much would the agent's utility increase if the budget constraint was marginally relaxed, i.e. the agent had a marginally higher PDV of income ($\partial U / \partial y_1$).

The Lagrangian involves two choice variables: c_1 and c_2. Therefore, we will need to calculate two first order conditions (FOCs), which are simply the partial **derivatives** of the Lagrangian with respect to the choice variables, c_1 and c_2:

$$
\begin{array}{llll}
c_1 & : \quad \dfrac{\partial \mathcal{L}}{\partial c_1} = \dfrac{1}{c_1} + \lambda\left[-1\right] = 0 & \rightarrow & \lambda = \dfrac{1}{c_1} \\
c_2 & : \quad \dfrac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \dfrac{1}{c_2} + \lambda\left[-\dfrac{1}{1+r}\right] = 0 & \rightarrow & \lambda = \beta\left(1+r\right)\dfrac{1}{c_2}
\end{array}
$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta\left(1+r\right)\frac{1}{c_2} \quad \rightarrow \quad \boldsymbol{c_2 = \beta\left(1+r\right)c_1}$$

An equation linking in optimum consumption in different time periods is called the intertemporal (from Latin: "between time (periods)") condition or the **Euler equation**.

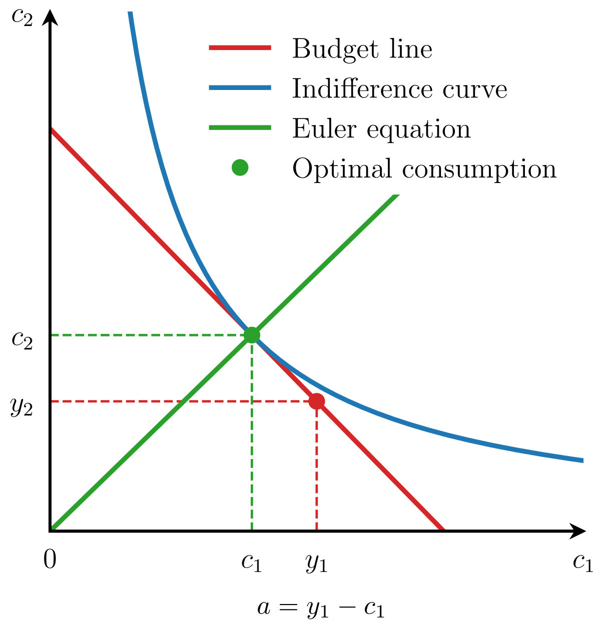
Now we just need to find the point where the optimality condition intersects with the budget line. Plug the optimality condition into the lifetime budget constraint:

$$
\begin{aligned}
c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r} \\
c_1 + \frac{\beta\left(1+r\right)c_1}{1+r} &= y_1 + \frac{y_2}{1+r} \\
c_1 + \beta c_1 &= y_1 + \frac{y_2}{1+r}
\end{aligned}
$$

Finally, we can obtain the optimal levels of consumption and assets at the end of the first period:

$$
\begin{aligned}
c_1 &= \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \\
c_2 &= \frac{\beta}{1+\beta}\left[\left(1+r\right)y_1 + y_2\right] \\
a &= y_1 - c_1 = y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] = \frac{1}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right]
\end{aligned}
$$

Note that assets a are more likely to be positive the higher is y_1 relative to y_2.



1.2 Comparative statics

Let us take a look at how changes in the parameters of our problem affect the agent's choices.

Changes in β

The higher is β, the more patient is our agent. Intuitively, since the agent with higher β cares more about the future, she would save more in the first period (and thus consume less in the first period) in order to consume more in the second period:

$$\begin{aligned}
\frac{\partial c_1}{\partial \beta} &= -\frac{1}{(1+\beta)^2}\left[y_1 + \frac{y_2}{1+r}\right] < 0 \\
\frac{\partial c_2}{\partial \beta} &= \frac{1}{(1+\beta)^2}\left[(1+r)\,y_1 + y_2\right] > 0 \\
\frac{\partial a}{\partial \beta} &= \frac{1}{(1+\beta)^2}\left[y_1 + \frac{y_2}{1+r}\right] > 0
\end{aligned}$$

Changes in y_1

An increase in first period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional first period income to second period consumption, assets will also increase:

$$\begin{aligned}
\frac{\partial c_1}{\partial y_1} &= \frac{1}{1+\beta} > 0 \\
\frac{\partial c_2}{\partial y_1} &= \frac{\beta}{1+\beta}\,(1+r) > 0 \\
\frac{\partial a}{\partial y_1} &= \frac{\beta}{1+\beta} > 0
\end{aligned}$$

Changes in y_2

An increase in second period (expected) income increases consumption in both time periods. Since the agent will want to transfer a part of the additional second period income to first period consumption, assets will decrease:

$$\begin{aligned}
\frac{\partial c_1}{\partial y_2} &= \frac{1}{1+\beta}\frac{1}{1+r} > 0 \\
\frac{\partial c_2}{\partial y_2} &= \frac{\beta}{1+\beta} > 0 \\
\frac{\partial a}{\partial y_2} &= -\frac{1}{1+\beta}\frac{1}{1+r} < 0
\end{aligned}$$

Changes in r

An increase in the interest rate generates two effects: substitution effect and income effect.

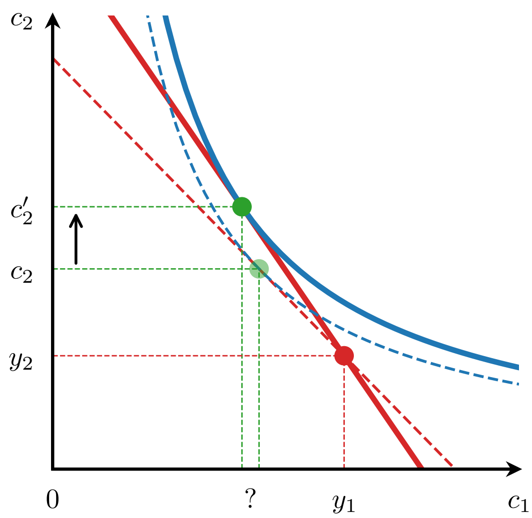
Substitution effect is generated by the change in relative prices of consumption in the first period versus consumption in the second period. It induces the agent to consume more in the second period and consume less in the first period.

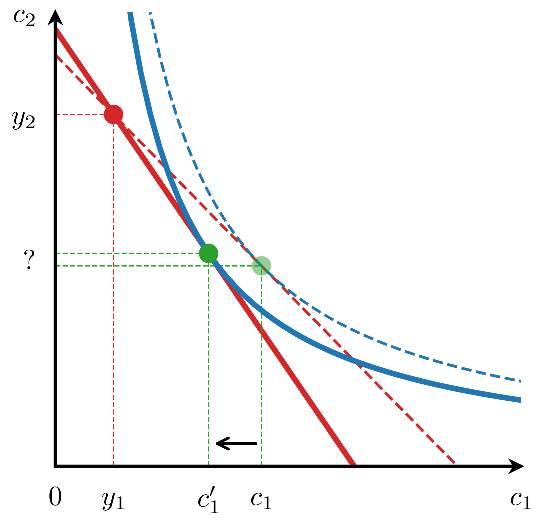
Income effect is generated by the fact that the PDV of income changes due to the changes in the interest rate. The sign of the effect depends on whether an agent is a saver or a borrower. An increase in interest rates creates a positive income effect for the saver, increasing her consumption in both periods. Conversely, an increase in interest rates creates a negative income effect for the borrower, decreasing her consumption in both periods.

These effects can be summarized by the following table, where pluses and minuses describe the direction of change in a variable for which we can be certain of the direction, regardless of the assumed utility function:

Effects of an increase in r	Saver c_1	c_2	a	Borrower c_1	c_2	a
Income	+	+	−	−	−	+
Substitution	−	+	+	−	+	+
Net	?	+	?	−	?	+

There are however also changes in variables marked with question marks, where the direction of change in a variable depends on the relative strength of substitution and income effects. In the case of a saver, if the income effect is stronger than the substitution effect, first period consumption increases and assets decrease. In the case of a borrower, if the income effect is stronger than the substitution effect, second period consumption decreases.





The relative strength of the two effects depends on the shape of the indifference curves. The logarithmic utility is special in the sense that the direction of change is the same for both savers and borrowers, but note well that this is not a general property:

$$\frac{\partial c_1}{\partial r} = \frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] < 0$$

$$\frac{\partial c_2}{\partial r} = \frac{\beta}{1+\beta} y_1 > 0$$

$$\frac{\partial a}{\partial r} = -\frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] > 0$$

In response to the increase in the interest rate, first period consumption decreases (if only $y_2 > 0$), second period consumption increases (if only $y_1 > 0$) and savings increase (if only $y_2 > 0$). The negative relationship between the level of interest rates and aggregate consumption is a usual assumption in macroeconomics, and is present in the undergraduate macroeconomic models such as IS-LM and AS-AD.

1.3 Choice under uncertainty

Consider now the problem where the agent is uncertain about their future level of income y_2 and return on their assets r_2.

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = \ln c_1 + \mathrm{E}\left[\beta \ln c_2\right] \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1 + r_2)\, a \end{aligned}$$

Since the lifetime budget constraint itself is subject to uncertainty, we will use two separate budget constraints, with two Lagrange multipliers λ_1 and λ_2:

$$\mathcal{L} = \ln \boldsymbol{c_1} + \mathrm{E}\left[\beta \ln \boldsymbol{c_2}\right] + \lambda_1 \left[y_1 - \boldsymbol{c_1} - \boldsymbol{a}\right] + \mathrm{E}\left[\lambda_2 \left[y_2 + (1 + r_2)\, \boldsymbol{a} - \boldsymbol{c_2}\right]\right]$$

First order conditions (FOCs):

$$\begin{array}{cccccc} c_1 & : & \dfrac{\partial \mathcal{L}}{\partial c_1} = \dfrac{1}{c_1} + \lambda_1 \left[-1\right] = 0 & \rightarrow & \lambda_1 = \dfrac{1}{c_1} \\ c_2 & : & \dfrac{\partial \mathcal{L}}{\partial c_2} = \mathrm{E}\left[\beta \dfrac{1}{c_2}\right] + \mathrm{E}\left[\lambda_2 (-1)\right] = 0 & \rightarrow & \lambda_2 = \beta \dfrac{1}{c_2} \\ a & : & \dfrac{\partial \mathcal{L}}{\partial a} = \lambda_1 \left[-1\right] + \mathrm{E}\left[\lambda_2 (1 + r_2)\right] = 0 & \rightarrow & \lambda_1 = \mathrm{E}\left[\lambda_2 (1 + r_2)\right] \end{array}$$

Resulting optimality condition:

$$\frac{1}{c_1} = \mathrm{E}\left[\beta \frac{1}{c_2} (1 + r_2)\right]$$

Note well that now we need to be extra careful not to break any expectation operators. Let us rewrite the Euler equation in the following way:

$$1 = \mathrm{E}\left[\beta \frac{c_1}{c_2} (1 + r_2)\right]$$

The above equation can be thought also as an asset pricing equation. In this particular example, the price of a unit of savings is one unit of first period consumption. The payoff from having an asset in the second period will be $(1 + r_2)$ per unit of savings. The term $\beta \cdot c_1 / c_2$ is called the **stochastic discount factor** and measures the relative marginal utility of consumption across time.

A general consumption-based asset pricing equation can be expressed as:

$$p_t = \mathrm{E}_t \left[m_{t+1} \cdot x_{t+1}\right]$$

where p_t denotes current asset price, E_t is the expectation operator conditional on all information available until (and including) period t, m_{t+1} is the stochastic discount factor and x_{t+1} is the future asset payoff. This equation is the basis of John Cochrane's book **Asset Pricing**, which covers a variety of cases:[1]

	Price p_t	Payoff x_{t+1}
Stock	p_t	$p_{t+1} + d_{t+1}$
Zero-coupon bond	p_t	1
Risk-free rate	1	R^f

[1]For more asset pricing relationships consult Mathematical Appendix.

Why the average return on bonds is lower than on stocks: a simplified example

Suppose an endowment economy, where assets are in zero net supply so then $c_1 = y_1$ and $c_2 = y_2$. Also set $\beta = 0.95$ and $y_1 = 1$. The second period endowment can take two values: high $y_2^h = 1.1$ and low $y_2^l = 0.9$, with $q = 0.5$ being the probability of the low state. We are going to apply the consumption-based asset pricing equation $p_1 = \mathrm{E}\left[m_2 \cdot x_2\right]$ to price bonds and stocks in this economy.

First let us find the stochastic discount factor:

$$\mathrm{E}\left[m_2\right] = \mathrm{E}\left[\beta\frac{c_1}{c_2}\right] = \mathrm{E}\left[\beta\frac{y_1}{y_2}\right] = \beta y_1\left[q \cdot \frac{1}{y_2^l} + (1-q)\frac{1}{y_2^h}\right] \approx 0.9596$$

For a bond that pays 1 unit in the second period with certainty its price p_1^b is simply equal to the stochastic discount factor computed above. The return on the bond is given by the reciprocal of price:

$$1 + r_2^b = \frac{1}{p_1^b} = \frac{1}{0.9596} \approx 1.0421 \quad \rightarrow \quad r_2^b \approx 4.2\%$$

Consider now a stock that will pay a dividend $d_2 = 1$ with certainty and cease to exist, so that $p_2^s = 0$. Then the price of the stock will be equal to the price of the bond:

$$p_1^s = \mathrm{E}\left[\beta\frac{y_1}{y_2} \cdot d_2\right] = \mathrm{E}\left[\beta\frac{y_1}{y_2}\right] \cdot d_2 = \beta y_1\left[q \cdot \frac{1}{y_2^l} + (1-q)\frac{1}{y_2^h}\right] \cdot d_2 \approx 0.9596$$

The return on the stock is the ratio of the sum of future dividend and price, and today's price:

$$1 + r_2^s = \frac{d_2 + p_2^s}{p_1^s} = \frac{1+0}{0.9596} \approx 1.0421 \quad \rightarrow \quad r_2^s \approx 4.2\%$$

which is the same as the return on the bond. This should come as no surprise as both assets have the same risk characteristics.

Consider now a case of a stock that pays dividend $d_2^h = 1.2$ when endowment is high and $d_2^l = 0.8$ when endowment is low. It is key to notice that (due to covariance) in general the expected value of a product of two random variables is not equal to the product of expected values of the variables:

$$\mathrm{E}\left[\beta\frac{y_1}{y_2} \cdot d_2\right] \neq \mathrm{E}\left[\beta\frac{y_1}{y_2}\right] \cdot \mathrm{E}\left[d_2\right]$$

The calculation of stock price goes now as follows:

$$p_1^s = \beta y_1\left[q \cdot \frac{d_2^l}{y_2^l} + (1-q)\frac{d_2^h}{y_2^h}\right] \approx 0.9404$$

Note that now the stock is cheaper than in the certain dividend case, although the expected value of dividend has not changed. This is due to the fact that in our example the dividends and income levels exhibit positive covariance and in the low income state the dividend is low as well. The expected return on the stock needs then to be higher to make investors willing to hold the risky asset:

$$\mathrm{E}\left[1 + r_2^s\right] = \frac{\mathrm{E}\left[d_2 + p_2^s\right]}{p_1^s} = \frac{1+0}{0.9404} \approx 1.0634 \quad \rightarrow \quad \mathrm{E}\left[r_2^s\right] \approx 6.3\%$$

The equity risk premium is the difference between the expected rate of return on stock and on bond:

$$\mathrm{E}\left[r_2^s - r_2^b\right] \approx 2.1\%$$

1.4 Non-borrowing constraint

An agent solves the same problem as before, but now the agent cannot borrow and so $a \geq 0$:

$$\begin{aligned}
\max_{c_1,\, c_2,\, a} \quad & U = \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 + a = y_1 \\
& c_2 = y_2 + (1+r)\, a \\
& \boldsymbol{a \geq 0}
\end{aligned}$$

Let us first approach this problem in an "intuitive" manner. Suppose for a while that the inequality constraint does not exist at all. In that case, we simply go back to the problem we have already solved in section 1.1, with the following solution:

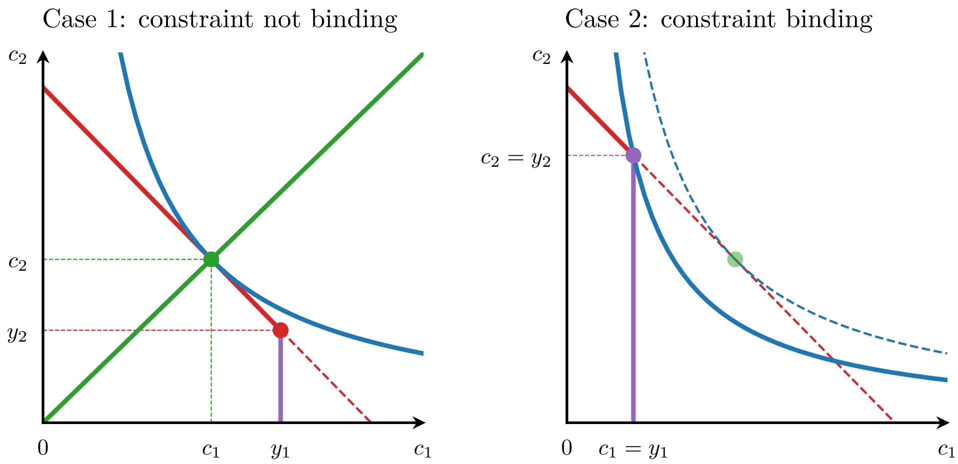
$$\begin{aligned}
c_1 &= \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \\
c_2 &= \frac{\beta}{1+\beta}\left[(1+r)\, y_1 + y_2\right] \\
a &= \frac{1}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right]
\end{aligned}$$

We should now check if the assets are indeed positive, $a > 0$. If that's the case, the non-borrowing constraint does not bind at all and we have our solution, which we can call **Case 1**.

However, if the assets are negative, we are actually in the case where the borrowing constraint is binding. While the agent would like to borrow, she cannot, and so she settles with zero assets, $a = 0$. This is our **Case 2**. The levels of consumption can be then easily obtained from the budget constraints:

$$\begin{aligned}
a &= 0 \\
c_1 &= y_1 - a = y_1 \\
c_2 &= y_1 + (1+r)\, a = y_2
\end{aligned}$$

The graphical depiction of these two cases can be seen below:



We can clearly see that when the agent cannot borrow, they are not able to reach their optimal consumption allocation and find themselves on a lower indifference curve. These "stuck" agents do not react to the small changes in the interest rates. However, they increase their consumption euro-for-euro following any transfers that they receive.

A more technical approach

It is convenient for a to remain explicit in the optimization problem and be an additional choice variable. Again we will use two separate budget constraints, with two Lagrange multipliers λ_1 and λ_2. There is also an additional multiplier μ associated with the inequality constraint $a \geq 0$:

$$\mathcal{L} = \ln \boldsymbol{c_1} + \beta \ln \boldsymbol{c_2} + \lambda_1 [y_1 - \boldsymbol{c_1} - \boldsymbol{a}] + \lambda_2 [y_2 + (1+r)\, \boldsymbol{a} - \boldsymbol{c_2}] + \mu \boldsymbol{a}$$

The Lagrangian involves three choice variables: c_1, c_2 and a. Therefore, we will need to calculate three first order conditions. Additionally, since one of the constraints is an inequality constraint, we will add a complementary slackness (CS) constraint that says that either $a = 0$ and the constraint is binding ($\mu \geq 0$), or $a \geq 0$ and the constraint is not binding ($\mu = 0$).

First order conditions (FOCs):

$$\begin{array}{rcllcl}
c_1 & : & \dfrac{\partial \mathcal{L}}{\partial c_1} = \dfrac{1}{c_1} + \lambda_1 [-1] = 0 & & \rightarrow & \lambda_1 = \dfrac{1}{c_1} \\
c_2 & : & \dfrac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \dfrac{1}{c_2} + \lambda_2 [-1] = 0 & & \rightarrow & \lambda_2 = \beta \dfrac{1}{c_2} \\
a & : & \dfrac{\partial \mathcal{L}}{\partial a} = \lambda_1 [-1] + \lambda_2 [1+r] + \mu = 0 & & \rightarrow & \lambda_1 = \lambda_2 (1+r) + \mu \\
\text{CS} & : & \mu a = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad a \geq 0 & & &
\end{array}$$

Because of the potentially not binding constraint, we have again the following two cases:

Case 1: constraint not binding, $\mu = 0$, $a \geq 0$:

The FOC for assets simplifies to:

$$\lambda_1 = \lambda_2 (1+r)$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta (1+r) \frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta (1+r)\, c_1$$

Construct the lifetime budget constraint:

$$\begin{aligned}
a &= \frac{c_2 - y_2}{1+r} \\
c_1 + \frac{c_2 - y_2}{1+r} &= y_1 \\
c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r}
\end{aligned}$$

Plug the optimality condition into the budget constraint:

$$\begin{aligned}
c_1 + \frac{\beta (1+r)\, c_1}{1+r} &= y_1 + \frac{y_2}{1+r} \\
c_1 (1+\beta) &= y_1 + \frac{y_2}{1+r}
\end{aligned}$$

Solution:

$$\begin{aligned}
c_1 &= \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right] \\
c_2 &= \frac{\beta}{1+\beta} \left[(1+r)\, y_1 + y_2 \right] \\
a &= \frac{1}{1+\beta} \left[\beta y_1 - \frac{y_2}{1+r} \right]
\end{aligned}$$

We should now check whether $a \geq 0$. If yes, then this is the optimal solution. Otherwise the optimal solution lies in **Case 2**.

Case 2: constraint binding, $\mu \geq 0$, $a = 0$:

The levels of consumption can be obtained from the budget constraints for separate time periods:

$$\begin{aligned} a &= 0 \\ c_1 &= y_1 - a = y_1 \\ c_2 &= y_2 + (1+r)\, a = y_2 \end{aligned}$$

Now we can find the value of μ, which should be positive if the agent is indeed constrained:

$$\begin{aligned} \mu &= \lambda_1 - \lambda_2 (1+r) \\ \mu &= \frac{1}{c_1} - \frac{\beta}{c_2} (1+r) \\ \mu &= \frac{1}{y_1} - \frac{\beta}{y_2} (1+r) \end{aligned}$$

The multiplier μ has the following interpretation: by how much would the utility of an agent improve if we allowed this agent to borrow a marginal amount. In other words, the higher is μ the more utility is "lost" because of the non-borrowing constraint. It is evident that μ increases with y_2 and decreases with y_1. Intuitively, the higher is y_2 relative to y_1, the more is the agent induced to decrease assets, which eventually would become negative. At this point, however, the non-borrowing constraint kicks in and the agents "suffers".

1.5 Differing interest rates on saving and borrowing

Examine now the problem of an agent that can borrow, but at an interest rate higher than that on saving, $r^b > r$. Let us introduce a separate variable $b \geq 0$ for the amount that the consumer borrows:

$$\begin{aligned}
\max_{c_1,\, c_2,\, a,\, b} \quad & U = \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 + a = y_1 + b \\
& c_2 + \left(1 + r^b\right) b = y_2 + (1 + r)\, a \\
& a \geq 0 \\
& b \geq 0
\end{aligned}$$

Since $r^b > r$, the agent would never choose to hold positive assets and borrow at the same time. The solution then has to lie in one of three cases:

1. The agent holds positive assets and does not borrow and $(a > 0,\, b = 0)$
2. The agent holds zero assets and does borrow $(a = 0,\, b > 0)$
3. The agent holds zero assets and does not borrow $(a = 0,\, b = 0)$

We already know how to approach **Case 1**. We set $b = 0$, assume that the agent holds positive assets, and ignore all inequality constraints. We arrive at the familiar solution:

$$\begin{aligned}
c_1 &= \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \\
c_2 &= \frac{\beta}{1+\beta}\left[(1+r)\, y_1 + y_2\right] \\
a &= \frac{1}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right]
\end{aligned}$$

and check if $a > 0$. If that is the case, we are done, otherwise we continue with other cases.

Let us now analyze **Case 2**. Analogously, we set $a = 0$, assume that the agent borrows, and ignore all inequality constraints. The problem becomes then very similar to the one in section 1.1:

$$\begin{aligned}
\max_{c_1,\, c_2,\, b} \quad & U = \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 = y_1 + b \\
& c_2 + \left(1 + r^b\right) b = y_2
\end{aligned}$$

The lifetime budget constraint can be constructed as:

$$\begin{aligned}
b &= \frac{y_2 - c_2}{1 + r^b} \\
c_1 &= y_1 + \frac{y_2 - c_2}{1 + r^b} \\
\boldsymbol{c_1 + \frac{c_2}{1 + r^b}} &\boldsymbol{= y_1 + \frac{y_2}{1 + r^b}}
\end{aligned}$$

Set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1 + r^b} - c_1 - \frac{c_2}{1 + r^b}\right]$$

First order conditions (FOCs):

$$\begin{aligned}
c_1 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda\left[-1\right] = 0 \quad &\rightarrow \quad \lambda = \frac{1}{c_1} \\
c_2 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda\left[-\frac{1}{1 + r^b}\right] = 0 \quad &\rightarrow \quad \lambda = \beta\left(1 + r^b\right)\frac{1}{c_2}
\end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta\left(1+r^b\right)\frac{1}{c_2} \quad \rightarrow \quad \boldsymbol{c_2 = \beta\left(1+r^b\right)c_1}$$

Let us now plug the Euler equation into the budget constraint:

$$\begin{aligned} c_1 + \frac{\beta\left(1+r^b\right)c_1}{1+r^b} &= y_1 + \frac{y_2}{1+r^b} \\ c_1 + \beta c_1 &= y_1 + \frac{y_2}{1+r^b} \end{aligned}$$

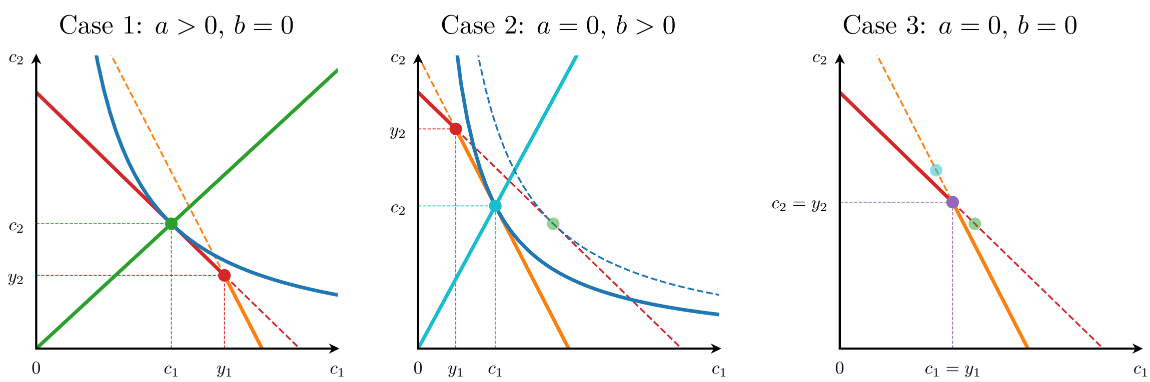
Solution:

$$\begin{aligned} c_1 &= \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r^b}\right] \\ c_2 &= \frac{\beta}{1+\beta}\left[\left(1+r^b\right)y_1 + y_2\right] \\ b &= c_1 - y_1 = \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r^b}\right] - y_1 = \frac{1}{1+\beta}\left[\frac{y_2}{1+r^b} - \beta y_1\right] \end{aligned}$$

Note well that b is more likely to be positive the higher is y_2 relative to y_1. If $b > 0$, we are indeed in **Case 2** and this is our solution.

However, if we find that we are neither in **Case 1** nor **Case 2**, then necessarily we are in **Case 3**. The interest rate on savings is too low for the agent to hold positive assets, but the interest rate on borrowing is too high for the agent to want to borrow. The agent then chooses:

$$\begin{aligned} a &= b = 0 \\ c_1 &= y_1 + b - a = y_1 \\ c_2 &= y_1 + (1+r)\,a - \left(1+r^b\right)b = y_2 \end{aligned}$$



Note that even if the agent is allowed to borrow, whenever they find themselves in **Case 3** they will behave (locally) as if they were borrowing constrained, whereas if they find themselves in either **Case 1** or **Case 2** they will behave (locally) as a "standard" optimizing agent.

1.6 Ricardian equivalence

One of many "irrelevance results" in economics, that can be demonstrated using the two-period framework, is the **Ricardian equivalence**. It posits that a decrease in taxation will not result in an increase in private consumption because households know that the resulting debt will have to be repaid by the taxpayers (themselves!) in the next period through higher taxes. Therefore, households instead of consuming more increase their savings to be able to cover expected higher taxes in the future.

Government sector

The government's budget constraints are:

$$
\begin{aligned}
g_1 &= \tau_1 + b \\
g_2 + (1+r)\, b &= \tau_2
\end{aligned}
$$

where g_1 and g_2 denote government spending in the first and second period, b are government bonds issued in the first period that have to be bought back in the second period, and τ_1 and τ_2 denote **lump-sum taxes** levied on the households in the first and second period.

Here we impose that the government cannot be indebted in the last period, which seems restrictive, but when generalized to the case of longer time horizons it simply posits that the government debt does not explode to infinity and the government does not go bankrupt.

Households

Households solve the following utility maximization problem:

$$
\begin{aligned}
\max_{c_1,\, c_2,\, a} \quad & U = \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 + a = y_1 - \tau_1 \\
& c_2 = y_2 - \tau_2 + (1+r)\, a
\end{aligned}
$$

Private savings a are comprised of government bonds b and other assets a' that pay the same interest rate as bonds. The expression $y_1 - \tau_1$, or income after taxation, is often called the **disposable income**.

Lifetime budget constraint:

$$
c_1 + \frac{c_2}{1+r} = y_1 - \tau_1 + \frac{y_2 - \tau_2}{1+r}
$$

Lagrangian:

$$
\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 - \tau_1 + \frac{y_2 - \tau_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]
$$

First order conditions (FOCs):

$$
\begin{aligned}
c_1 &: \quad \frac{1}{c_1} - \lambda = 0 && \rightarrow \quad \lambda = \frac{1}{c_1} \\
c_2 &: \quad \beta \frac{1}{c_2} - \frac{\lambda}{1+r} = 0 && \rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2}
\end{aligned}
$$

Euler equation:

$$
c_2 = \beta\,(1+r)\, c_1
$$

In this setup, government bonds and other assets are perfect substitutes, so from the perspective of the households their relative mix is indeterminate. We can pin down the level of savings in other assets by using the households' and government budget constraints in the first time period:

$$
\begin{aligned}
c_1 + b + a' &= y_1 - \tau_1 \\
b &= g_1 - \tau_1 \\
c_1 + g_1 - \tau_1 + a' &= y_1 - \tau_1 \\
a' &= y_1 - g_1 - c_1
\end{aligned}
$$

We can also rewrite the households' budget constraint in the second time period:

$$\begin{aligned}
c_2 &= y_2 - \tau_2 + (1+r)\, b + (1+r)\, a' \\
b &= \frac{\tau_2 - g_2}{1+r} \\
c_2 &= y_2 - \tau_2 + \tau_2 - g_2 + (1+r)\, a' \\
c_2 &= y_2 - g_2 + (1+r)\, a'
\end{aligned}$$

Now let us join the budget constraints:

$$\begin{aligned}
c_2 &= y_2 - g_2 + (1+r)\,(y_1 - g_1 - c_1) \\
c_1 + \frac{c_2}{1+r} &= y_1 - g_1 + \frac{y_2 - g_2}{1+r}
\end{aligned}$$

And plug in the Euler equation:

$$\begin{aligned}
c_1 + \frac{\beta\,(1+r)\,c_1}{1+r} &= y_1 - g_1 + \frac{y_2 - g_2}{1+r} \\
(1+\beta)\,c_1 &= y_1 - g_1 + \frac{y_2 - g_2}{1+r}
\end{aligned}$$

Finally we can obtain the optimal levels of consumption:

$$\begin{aligned}
c_1 &= \frac{1}{1+\beta}\left[y_1 - g_1 + \frac{y_2 - g_2}{1+r}\right] \\
c_2 &= \frac{\beta}{1+\beta}\left[(1+r)\,(y_1 - g_1) + y_2 - g_2\right]
\end{aligned}$$

The levels of asset holdings in the economy are pinned down as follows: government bonds are determined by $b = g_1 - \tau_1$ and other assets by $a' = y_1 - g_1 - c_1$. Total assets are given by:

$$a = b + a' = y_1 - \tau_1 - c_1$$

Maybe surprisingly, neither the level of government bonds issuance nor the level of taxation influence the optimal consumption levels and savings in assets other than government bonds. What does influence them, though, is the present discounted value of government expenditures. This would imply that the fiscal policy can be only effective via changes in the level of government spending, and not via changes in levels of taxation.

Breaking the Ricardian equivalence

Distortionary taxation

All taxes, except lump-sum taxes considered above, create distortions by changing the economic incentives. For example, if income was not exogenous, but determined by the labor supply, and taxes were levied on labor income, then a drop in income taxation in the first time period would result in more labor supplied in the first time period and less labor supplied in the second period, when taxes would be higher. It would affect the PDV of income and also the consumption-labor incentives. We should then expect changes in consumption levels in response to changes in the bond-tax mix.

Differing rates of return on assets

The assumption that we made was that all assets held by the households, including government bonds, yield identical rates of return. In reality government bond yields in advanced countries are significantly lower than returns on other assets. This means that it is “cheaper” for the government to be doing the borrowing in the economy than for the private agents and as such running positive government debt and deferring taxes towards the future might increase the PDV of private after-tax incomes.

Treating government debt as net wealth

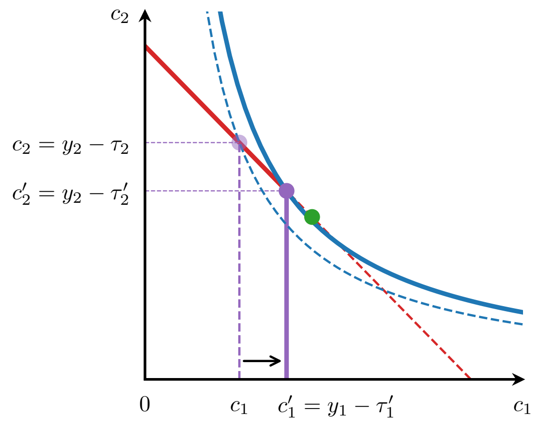
If people treat government bonds as assets, they may think that they are "wealthy" and decide to spend more. However, the government bonds also represent future liabilities, that is, in the future they will have to be repaid with higher taxes. In our above example, the same agent is taxed to repay the government bonds she owns. Thus treating government debt as net wealth can be interpreted as a form of myopic behavior.

Myopy (shortsightedness), time-inconsistent and temptation-driven preferences

The above model assumed that people care not only about their current, but also future consumption, and realize the overall structure of the problem they are facing. However, if people do not fully realize how the economy and the intertemporal government budget constraints work, or have problems with "sticking" to their consumption plans, then changes in current disposable income will influence consumption. It can be difficult to distinguish these effects in practice, as they are often observationally identical.

Non-borrowing constraints and different interest rates on saving and borrowing

If some of the households cannot borrow (or are at the "kink" of the budget constraint in the two interest rates case), they might not be able to reach their optimal consumption allocation. A drop in taxes in the first period boosts their disposable income and makes the nonborrowing constraint less painful, or even nonbinding. In the example below, a reduction in taxes in the first period, from τ_1 to τ_1', generates an increase in first period consumption, from c_1 to c_1', with $\Delta c_1 = -\Delta\tau_1$. Note however, that if the tax reduction would push the first period disposable income beyond the green point, the household would no longer be borrowing constrained, and further reductions in taxes would have no impact on consumption.



Finite lifetime

In our considerations we have assumed that consumers and the government "live" for the same number of periods. Since people have finite lifetimes and the governments can "outlive" them, individuals may expect that the higher taxes will affect the economy after they die and thus a current tax break will boost the PDV of their individual income.

Spending the 2008 Rebate, by Age

Age group	Percent mostly spending
29 or less	11.7
30–39	14.2
40–49	16.9
50–64	19.9
65 or over	28.4

Source: **Shapiro and Slemrod (2009)**

1.7 Finite time horizon optimization

Consider now a general case of an agent planning for an arbitrary, but finite number of periods $T+1$. For notational convenience, let us denote the "first" time period with time subscript 0, the "second" time period with time subscript 1, and so on (and so the "final" period has the time subscript T). We allow the agent to start with some initial assets a_0 (not necessarily positive), and add the condition that the agent's assets at the end of the planning horizon cannot be negative.

The agent solves the following utility maximization problem:

$$\begin{aligned}
\max_{\{c_t\}_{t=0}^T,\,\{a_{t+1}\}_{t=0}^T} \quad & U = \ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2 + \ldots + \beta^T \ln c_T = \sum_{t=0}^{T} \beta^t \ln c_t \\
\text{subject to} \quad & c_t + a_{t+1} = y_t + (1+r)\,a_t \quad \text{for all } t = 0,1,2,\ldots,T \\
& a_{T+1} \geq 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

In these multi-period cases it is much more convenient to construct the Lagrangian using a series of per-period budget constraints. Also, we will pre-multiply the Lagrange multipliers with the appropriate discount factors (the multiplier λ_t has then the interpretation of the current value marginal utility from consumption in period t):

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{T} \beta^t \ln c_t + \sum_{t=0}^{T} \beta^t \lambda_t \left[y_t + (1+r)\,a_t - c_t - a_{t+1}\right] + \mu a_{T+1} \\
&= \sum_{t=0}^{T} \beta^t \left[\ln c_t + \lambda_t \left[y_t + (1+r)\,a_t - c_t - a_{t+1}\right]\right] + \mu a_{T+1}
\end{aligned}$$

We can expand the Lagrangian for an easier derivation of the first order conditions with respect to consumption and asset decisions made in any time period t (c_t and a_{t+1}), as well as terminal assets a_{T+1}:

$$\begin{aligned}
\mathcal{L} = \ldots &+ \beta^t \left[\ln \boldsymbol{c_t} + \lambda_t \left[y_t + (1+r)\,a_t - \boldsymbol{c_t} - \boldsymbol{a_{t+1}}\right]\right] \\
&+ \beta^{t+1} \left[\ln c_{t+1} + \lambda_{t+1} \left[y_{t+1} + (1+r)\,\boldsymbol{a_{t+1}} - c_{t+1} - a_{t+2}\right]\right] + \ldots \\
&+ \beta^T \left[\ln c_T + \lambda_T \left[y_T + (1+r)\,a_T - c_T - \boldsymbol{a_{T+1}}\right]\right] + \mu \boldsymbol{a_{T+1}}
\end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned}
c_t :&\quad \beta^t \left[\frac{1}{c_t} - \lambda_t\right] = 0 && \rightarrow \quad \lambda_t = \frac{1}{c_t} && \text{for all } t = 0,1,2,\ldots,T \\
a_{t+1} :&\quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1+r) = 0 && \rightarrow \quad \lambda_t = \beta (1+r)\,\lambda_{t+1} && \text{for all } t = 0,1,2,\ldots,T-1 \\
a_{T+1} :&\quad -\beta^T \lambda_T + \mu = 0 \\
\text{CS} :&\quad \mu a_{T+1} = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad a_{T+1} \geq 0
\end{aligned}$$

Let us first focus on the complementary slackness condition and the FOC with respect to a_{T+1}:

$$\mu = \beta^T \lambda_T = \beta^T \frac{1}{c_T} > 0$$

The multiplier μ has to be positive, unless consumption in period T is infinite and the marginal utility of consumption is 0. Therefore, the complementary slackness condition implies that in optimum the agent holds no assets at the end of the planning horizon, which we have been implicitly assuming before:

$$\mu > 0 \quad \text{and} \quad \mu a_{T+1} = 0 \quad \rightarrow \quad a_{T+1} = 0$$

Note that this allows us to restate the original constraint of the no end-of-life debt in a form that is often called a **transversality condition**, and is necessary for solving infinite time horizon problems:

$$\beta^T \lambda_T a_{T+1} = 0$$

By combining the FOCs for consumption and assets, we arrive at the familiar Euler equation:

$$\lambda_t = \beta (1+r) \lambda_{t+1} \quad \text{and} \quad \lambda_t = \frac{1}{c_t}$$

$$\frac{1}{c_t} = \beta (1+r) \frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta (1+r) c_t$$

Since the above Euler equation is true for any two neighboring time periods, we can also rewrite it as:

$$\begin{aligned} c_{t+1} &= \beta (1+r) c_t \\ c_{t+2} &= \beta (1+r) c_{t+1} \end{aligned}$$

By substitution we can get:

$$c_{t+2} = [\beta (1+r)]^2 c_t$$

And we can relate $t+i$ period consumption (for any i) to t period consumption as follows:

$$c_{t+i} = [\beta (1+r)]^i c_t \quad \rightarrow \quad c_t = [\beta (1+r)]^t c_0$$

Let us now construct the lifetime budget constraint. Start from the first time period:

$$\begin{aligned} c_0 + a_1 &= y_0 + (1+r) a_0 \\ (1+r) a_0 &= c_0 - y_0 + a_1 \\ a_0 &= \frac{c_0 - y_0}{1+r} + \frac{a_1}{1+r} \end{aligned}$$

Now consider the second time period:

$$\begin{aligned} c_1 + a_2 &= y_1 + (1+r) a_1 \\ (1+r) a_1 &= c_1 - y_1 + a_2 \\ a_1 &= \frac{c_1 - y_1}{1+r} + \frac{a_2}{1+r} \end{aligned}$$

We can clearly see a repeating pattern. Plug the expression for a_1 into the first period budget constraint:

$$(1+r) a_0 = c_0 - y_0 + a_1 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{a_2}{1+r}$$

We could also obtain the expression for a_2 from the third period budget constraint:

$$a_2 = \frac{c_2 - y_2}{1+r} + \frac{a_3}{1+r}$$

and plug the expression for a_2 into the first period budget constraint:

$$(1+r) a_0 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{c_2 - y_2}{(1+r)^2} + \frac{a_3}{(1+r)^2}$$

If we continue this substitution process, we shall finally arrive at the following lifetime budget constraint:

$$(1+r) a_0 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{c_2 - y_2}{(1+r)^2} + \ldots + \frac{c_T - y_T}{(1+r)^T} + \frac{a_{T+1}}{(1+r)^T}$$

which we can express compactly by using the summation notation:

$$(1+r) a_0 = \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T}$$

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T} = (1+r) a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

By deriving and reworking the FOCs, we have obtained the following results:

$$a_{T+1} = 0 \quad \text{and} \quad c_t = [\beta(1+r)]^t c_0$$

Plug these two pieces of information into the lifetime budget constraint:

$$\sum_{t=0}^{T} \frac{[\beta(1+r)]^t c_0}{(1+r)^t} = (1+r)a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

$$c_0 \cdot \sum_{t=0}^{T} \beta^t = (1+r)a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

We have arrived at the expression for the optimal consumption expenditure in the first time period, which is a generalized version of the formula we have developed for the two period case:

$$c_0 = \frac{1}{\sum_{t=0}^{T} \beta^t} \left[(1+r)a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \right]$$

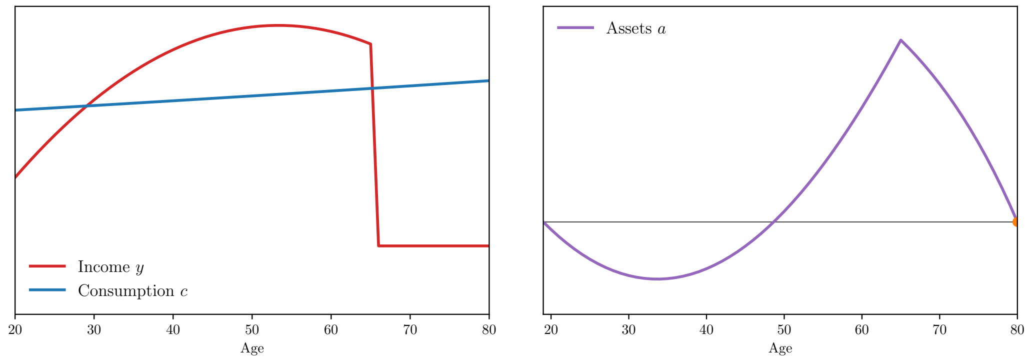
$$c_t = [\beta(1+r)]^t c_0$$

$$a_{t+1} = y_t - c_t + (1+r)a_t$$

We can also apply the formula for a **sum of the finite geometric series** to get:

$$c_0 = \frac{1-\beta}{1-\beta^{T+1}} \left[(1+r)a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \right]$$

An illustration of an example of a life-cycle of a consumer is shown below:



1.8 Infinite time horizon optimization

The agent solves the following infinite-horizon utility maximization problem:

$$\begin{aligned}
\max_{\{c_t\}_{t=0}^{\infty},\,\{a_{t+1}\}_{t=0}^{\infty}} \quad & U = \sum_{t=0}^{\infty} \beta^t \ln c_t \\
\text{subject to} \quad & c_t + a_{t+1} = y_t + (1+r)\,a_t \quad \text{for all } t = 0,1,2,\ldots,\infty \\
& \lim_{T\to\infty} \beta^T \lambda_T a_{T+1} = 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

Again we include the condition that the agents' assets at the "end" cannot be negative. In this problem the planning horizon is infinite and so we need to make use of the transversality condition. The transversality condition eliminates such paths where consumer becomes more and more indebted over time.

Set up the Lagrangian and expand it for an easier derivation of the first order conditions:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \lambda_t \left[y_t + (1+r)\,a_t - c_t - a_{t+1}\right]\right] \\
&= \ldots + \beta^t \left[\ln \boldsymbol{c_t} + \lambda_t \left[y_t + (1+r)\,a_t - \boldsymbol{c_t} - \boldsymbol{a_{t+1}}\right]\right] \\
&\qquad + \beta^{t+1} \left[\ln c_{t+1} + \lambda_{t+1} \left[y_{t+1} + (1+r)\,\boldsymbol{a_{t+1}} - c_{t+1} - a_{t+2}\right]\right] + \ldots
\end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned}
c_t \ : \quad & \beta^t \left[\frac{1}{c_t} - \lambda_t\right] = 0 & \rightarrow \quad & \lambda_t = \frac{1}{c_t} & \text{for all } t = 0,1,2,\ldots,\infty \\
a_{t+1} \ : \quad & -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1+r) = 0 & \rightarrow \quad & \lambda_t = \beta\,(1+r)\,\lambda_{t+1} & \text{for all } t = 0,1,2,\ldots,\infty
\end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_t} = \beta\,(1+r)\,\frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta\,(1+r)\,c_t$$

And as before, we can generalize the Euler equation to link consumption in any two time periods:

$$c_{t+i} = \left[\beta\,(1+r)\right]^i c_t \quad \rightarrow \quad c_t = \left[\beta\,(1+r)\right]^t c_0$$

The process for constructing the lifetime budget constraint is analogous to the finite horizon case. Imagine we have used our substitution procedure for the first T time periods and so we have that:

$$(1+r)\,a_0 = \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T}$$

Now consider what happens as $T \to \infty$. We know from the assets FOC that:

$$\lambda_t = \beta\,(1+r)\,\lambda_{t+1} \quad \rightarrow \quad \lambda_{t+1} = \frac{\lambda_t}{\beta\,(1+r)} \quad \rightarrow \quad \lambda_{t+i} = \frac{\lambda_t}{\left[\beta\,(1+r)\right]^i} \quad \rightarrow \quad \lambda_T = \frac{\lambda_0}{\left[\beta\,(1+r)\right]^T}$$

The transversality condition guarantees that in the limit (since $\lambda_0 = \frac{1}{c_0} > 0$):

$$\lim_{T\to\infty} \beta^T \lambda_T a_{T+1} = \lim_{T\to\infty} \beta^T \frac{\lambda_0}{\beta^T (1+r)^T} a_{T+1} = \lim_{T\to\infty} \lambda_0 \frac{a_{T+1}}{(1+r)^T} = 0 \quad \rightarrow \quad \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} = 0$$

Going back to the lifetime budget constraint we then have:

$$(1+r)\,a_0 = \sum_{t=0}^{\infty} \frac{c_t - y_t}{(1+r)^t} \quad \rightarrow \quad \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = (1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}$$

If we plug in the Euler equations for c_t, we get:

$$\sum_{t=0}^{\infty} \frac{[\beta(1+r)]^t c_0}{(1+r)^t} = (1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}$$

$$c_0 \cdot \sum_{t=0}^{\infty} \beta^t = (1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}$$

$$c_0 = \frac{1}{\sum_{t=0}^{\infty} \beta^t} \left[(1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right]$$

From the properties of the infinite geometric series we get:

$$c_0 = (1-\beta)\left[(1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right]$$

Since the planning horizon is infinite, the above expression can be applied to any time period t:

$$c_t = (1-\beta)\left[(1+r)\,a_t + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+r)^i}\right]$$

Note also that this is the limit of the expression found for the finite horizon case, as T goes to infinity:

$$\lim_{T\to\infty} \frac{1-\beta}{1-\beta^{T+1}} \left[(1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}\right] = (1-\beta)\left[(1+r)\,a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right]$$

Transitory vs permanent income changes

Consider now how consumption in time period t would react to an increase in time period t income only:

$$\Delta c_t = (1-\beta)\left[\frac{\Delta y_t}{(1+r)^0}\right] = (1-\beta)\,\Delta y_t$$

If we consider yearly time intervals, then $\beta \approx 0.96$ and the reaction of consumption is very small.

On the other hand, if income is expected to increase in all future time periods, then we get:

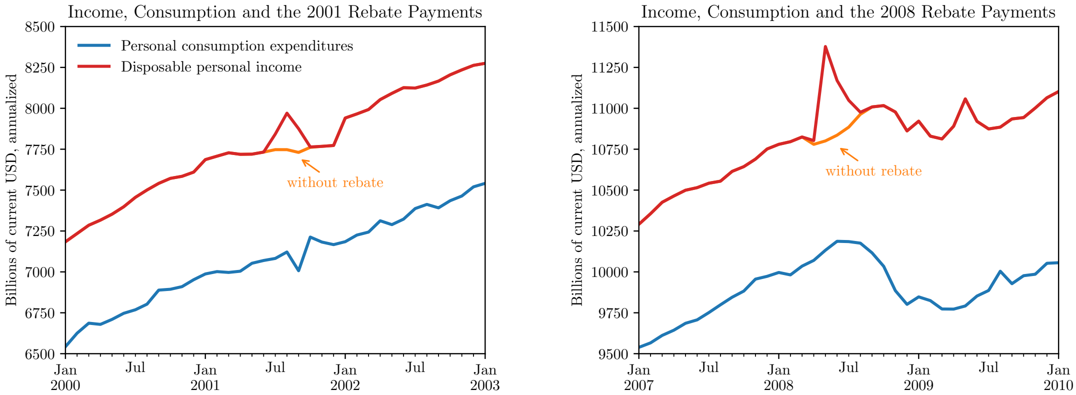
$$\Delta c_t = (1-\beta)\left[\sum_{i=0}^{\infty} \frac{\Delta \bar{y}_t}{(1+r)^i}\right] = (1-\beta)\left[\Delta \bar{y}_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i}\right] = (1-\beta)\,\frac{1}{1-\frac{1}{1+r}}\Delta \bar{y}_t = (1-\beta)\,\frac{1+r}{r}\Delta \bar{y}_t$$

Now if we assume that $\beta(1+r) = 1 \quad \rightarrow \quad \beta = 1/(1+r)$, then $\Delta c_t = \Delta \bar{y}_t$.

According to the **permanent income hypothesis**, forward-looking consumers do not change their consumption much when experiencing a temporary change in income, but react strongly when "permanent" income changes.

1.9 Consumption behavior: empirics

Transitory changes in taxes



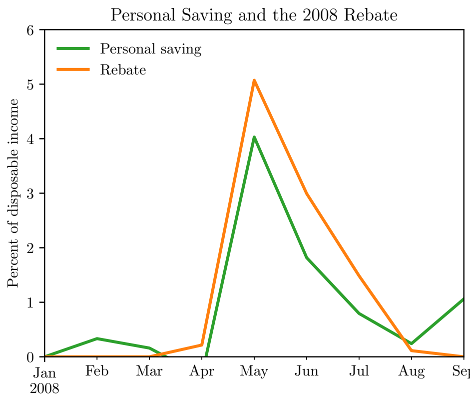
Source: **Taylor (2009)**, US Bureau of Economic Analysis.

PCE Regressions with Rebate Payments

Lagged PCE	0.794	0.832
	(0.057)	(0.056)
Rebate payments	0.048	0.081
	(0.055)	(0.054)
Disposable personal income (w/o rebate)	0.206	0.188
	(0.056)	(0.055)
Oil price ($/bbl lagged 3 months)		-1.007
		(0.325)
R^2	0.999	0.999

Notes: The dependent variable is personal consumption expenditures. Standard errors are reported in parentheses. The oil price is for West Texas Intermediate. The sample period is January 2000 to October 2008.

Source: **Taylor (2009)**



Source: **Shapiro and Slemrod (2009)**, US Bureau of Economic Analysis.

Responses to 2001 and 2008 Rebate Surveys

	2001		2008	
	Number	Percent	Number	Percent
Mostly spend	256	21.8	447	19.9
Mostly save	376	32.0	715	31.8
Mostly pay off debt	544	46.2	1083	48.2
Will not get rebate	223		212	
Don't know / refused	45		61	
Total	1444	100	2518	100

Source: **Shapiro and Slemrod (2003)** and **Shapiro and Slemrod (2009)**.

Transitory and permanent changes in income

The **life-cycle** / **permanent income** hypothesis of consumer behavior that we have developed makes a number of predictions that we can test in the data. Some of them are:

1. Consumption is forward-looking, depending not just on current income but future income as well.
2. Consumption reacts more to permanent changes in income than transitory changes in income.
3. Current consumption does not react much to predictable, anticipated changes in income.

The results from the literature are mixed, implying that at least a significant fraction of households does not behave as the theory predicts, even when augmented with liquidity constraints.

In one of the first empirical tests of aggregate consumption behavior, **Hall (1978)** postulates that under some additional assumptions the consumption should behave as a random walk process:

$$u'\left(c_{t-1}\right) = \mathrm{E}_{t-1}\left[\beta u'\left(c_t\right)\left(1+r_t\right)\right] \quad + \quad \text{additional assumptions} \quad \rightarrow \quad c_t = c_{t-1} + \varepsilon_t$$

and indeed, this null hypothesis is not rejected. Obviously, what is most interesting is whether some macroeconomic variables have predictive power for the error term ε_t. If the theory is correct, no variable x known in time periods $t-1$ and $t-2$ could be able to predict consumption growth between periods $t-1$ and t, and in the following equation the coefficient α should be indistinguishable from 0:

$$\Delta c_t = \alpha \Delta x_{t-1} + e_t$$

The null hypothesis cannot be rejected for several considered variables, including changes in disposable income. However in some cases, most notably in case of stock prices growth, the null was rejected.

Further research has shown that there are two apparent deviations from our theory: on one hand, consumption exhibits *excess sensitivity* to changes in current income, and on the other, *excess smoothness* to changes in permanent income. This means that consumption reacts stronger to changes in current income than theory predicts, and also weaker to changes in permanent income, which actually is quite volatile. **Flavin (1981)** assumes that changes in income are to some extent autocorrelated, where ε_t is the unpredictable part, and consumption is allowed to react to this innovation:

$$\begin{aligned} \Delta y_t &= \mu + \lambda \Delta y_{t-1} + \varepsilon_t \\ \Delta c_t &= \beta \Delta y_t + \theta \varepsilon_t + \nu_t \end{aligned}$$

If $\beta > 0$, then consumption overreacts to changes in current income (excess sensitivity), and that is what she finds ($\beta \approx 0.36$). On the other hand, **Campbell and Deaton (1989)** find on the basis of a VAR model that consumption exhibits excess smoothness as it depends very strongly on its past values and does not respond much to the estimated permanent income process.

It appears that within an economy there are two groups of households: those behaving according to the LC/PIH theory, and others being liquidity constrained / myopic, which could potentially account for both excess sensitivity and smoothness in aggregate data. Campbell and Mankiw (1989) test this idea against aggregate data and estimate the following equation:

$$\Delta c_t = \lambda \Delta y_t + (1 - \lambda)\, e_t$$

where λ is a fraction of borrowing-constrained households. They find that, depending on the specification, $\lambda \approx 0.42$ or $\lambda \approx 0.52$.

A group of papers also finds departures from our theory in response to predictable government transfers and wage changes. Parker (1999) finds that households increase their consumption after they reach the annual cap in social security contributions and their disposable income increases. Similar behavior in response to income tax refunds is found by Souleles (1999). Shapiro and Slemrod (2003) point out that apart from transitory tax rebates, the 2001 tax reforms also implied positive changes in permanent income, to which households did not react. On the other hand and using higher quality data, Johnson et al. (2006) find that households have spent their 2001 rebate transfers over the course of few months, and they were especially stimulative for low income and low wealth households. More recently, Evans and Moore (2012) find that mortality rates decline before the first day of the month and spike after the first, which suggests that the mortality cycle is linked to short-term variation in levels of economic activity as a result of households being liquidity constrained by the end of the month. Stephens (2003) makes similar observations regarding the behavior of social security recipients.

In some cases though our theory is corroborated. Paxson (1993) shows that Thai households who experience large seasonal variations in their income are able to smooth their consumption over the year. Browning and Collado (2001) document the fact that some Spanish workers receive additional bonuses in June and December, but their consumption behavior is indistinguishable from behavior of workers who do not receive bonuses. Hsieh (2003) investigates the effects of predictable dividend payments from the Alaska Permanent Fund and finds that households behave consistently with PIH, including those households that overreact to income tax refunds.

Lusardi et al. (2011) document that around 1/3 of U.S. households would “certainly be unable to cope with a financial emergency that required them to come up with \$2,000 in the next month.”, and among those giving that answer, a high proportion of individuals are at middle class levels of income. In Broda and Parker (2012), 40% of households report that they do not have “at least two months of income available in cash, bank accounts, or easily accessible funds.”

Kaplan and Violante (2014) use micro-level data and show that while around 10% of US households are poor hand-to-mouth (very low or negative net worth), there are around 33% are wealthy hand-to-mouth: households who have significant positive net worth in form of illiquid assets (housing, pension accounts, etc.) but very low level of liquid assets and in the short run behave as borrowing-constrained agents, responding strongly to (relatively small) changes to their disposable income. They able to replicate the fraction of 2001 tax rebate payments that were immediately spent (around 25%) in a model with forward-looking, optimizing consumers who can save in two types of assets (low-return liquid and high-return illiquid) with transaction costs between them. The model is able to account for size-asymmetry in the consumption responses to small and large payments: large rebates trigger many households to pay the transaction cost and deposit the extra income into the illiquid account, but when they adjust, these households are unconstrained and therefore save the bulk of their rebate.

1.10 Mathematical Appendix

Deriving the Euler equation the "Euler way"

The Euler equation can be also obtained in the "Euler way": using calculus of variation. Suppose that the agent has found the optimal solution to the problem. She now considers consuming x units less in the first period so that she can consume $(1+r)\,x$ units more in the second period. Her utility level is now given by:

$$U = \ln(c_1 - x) + \beta \ln(c_2 + (1+r)\,x)$$

If the solution was truly optimal, the derivative of the utility function with respect to x, evaluated at $x = 0$, should be equal to 0:

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{1}{c_1 - x} \cdot (-1) + \beta \frac{1}{c_2 + (1+r)\,x} \cdot (1+r) \\ \frac{\partial U}{\partial x}|_{x=0} &= -\frac{1}{c_1} + \beta\,(1+r)\,\frac{1}{c_2} = 0 \quad \rightarrow \quad \boldsymbol{c_2 = \beta\,(1+r)\,c_1} \end{aligned}$$

Problem with uncertainty with generic time-separable utility function

Consider now the problem where the agent is uncertain about their future level of income y_2 and return on their assets r_2. Additionally, instead of assuming a specific functional form for the utility function, we consider a generic function u, satisfying the standard assumptions that $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c \to 0} u(c) = -\infty$.

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = u(c_1) + \mathrm{E}\left[\beta \cdot u(c_2)\right] \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r_2)\,a \end{aligned}$$

Set up the Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & u(c_1) + \mathrm{E}\left[\beta \cdot u(c_2)\right] + \lambda_1 \left[y_1 - c_1 - a\right] \\ & + \mathrm{E}\left[\lambda_2 \left[y_2 + (1+r_2)\,a - c_2\right]\right] \end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned} c_1 &: \quad u'(c_1) - \lambda_1 = 0 && \rightarrow \quad \lambda_1 = u'(c_1) \\ c_2 &: \quad \mathrm{E}\left[\beta \cdot u'(c_2)\right] - \mathrm{E}\left[\lambda_2\right] = 0 && \rightarrow \quad \lambda_2 = \beta \cdot u'(c_2) \\ a &: \quad -\lambda_1 + \mathrm{E}\left[\lambda_2 (1+r_2)\right] = 0 && \rightarrow \quad \lambda_1 = \mathrm{E}\left[\lambda_2 (1+r_2)\right] \end{aligned}$$

Resulting optimality condition:

$$u'(c_1) = \mathrm{E}\left[\beta \cdot u'(c_2)\,(1+r_2)\right]$$

Note well that now we need to be extra careful not to break any expectation operators. Let us rewrite the Euler equation in the following way:

$$1 = \mathrm{E}\left[\beta \frac{u'(c_2)}{u'(c_1)} \cdot (1+r_2)\right]$$

The above equation can be thought also as an asset pricing equation. In this particular example, the price of a unit of savings is one unit of first period consumption. The payoff from having an asset in the second period will be $(1+r_2)$ per unit of savings. The term $\beta \cdot u'(c_2)/u'(c_1)$ is called the **stochastic discount factor** and measures the relative marginal utility of consumption across periods.

A general form of consumption-based asset equation can be expressed as $p_t = \mathrm{E}_t\left[m_{t+1} \cdot x_{t+1}\right]$, where p_t denotes asset price, E_t is the expectation operator conditional on all information available until (and including) period t, m_{t+1} is the stochastic discount factor and x_{t+1} is the future asset payoff. This equation is the basis of John Cochrane's book **Asset Pricing**, which covers a variety of cases:

	Price p_t	Payoff x_{t+1}
Stock	p_t	$p_{t+1}+d_{t+1}$
Return	1	R_{t+1}
Price-dividend ratio	$\frac{p_t}{d_t}$	$\left(\frac{p_{t+1}}{d_{t+1}}+1\right)\frac{d_{t+1}}{d_t}$
Excess return	0	$R^e_{t+1}=R^a_{t+1}-R^b_{t+1}$
Managed portfolio	z_t	$z_t R_{t+1}$
Moment condition	$\mathrm{E}\left[p_t z_t\right]$	$x_{t+1}z_t$
Zero-coupon bond	p_t	1
Risk-free rate	1	R^f
Option	C	$\max\left\{S_T-K,0\right\}$

Handling multiple inequality constraints: two differing interest rates

Lagrangian:

$$\mathcal{L}=\ln c_1+\beta\ln c_2+\lambda_1\left[y_1+b-c_1-a\right]+\lambda_2\left[y_2+(1+r)\,a-c_2-\left(1+r^b\right)b\right]+\mu_a a+\mu_b b$$

The Lagrangian involves four choice variables: c_1, c_2, a and b. We will need to calculate four first order conditions. We will also have two complementary slackness constraints.

First order conditions (FOCs):

$$\begin{aligned}
c_1 &: \quad \frac{1}{c_1}-\lambda_1=0 && \rightarrow \quad \lambda_1=\frac{1}{c_1}\\
c_2 &: \quad \beta\cdot\frac{1}{c_2}-\lambda_2=0 && \rightarrow \quad \lambda_2=\frac{\beta}{c_2}\\
a &: \quad -\lambda_1+\lambda_2\,(1+r)+\mu_a=0 && \rightarrow \quad \lambda_1=\lambda_2\,(1+r)+\mu_a\\
b &: \quad \lambda_1-\lambda_2\left(1+r^b\right)+\mu_b=0 && \rightarrow \quad \lambda_1=\lambda_2\left(1+r^b\right)-\mu_b\\
\mathrm{CS}_a &: \quad \mu_a a=0 \quad\text{and}\quad \mu_a\geq 0 \quad\text{and}\quad a\geq 0\\
\mathrm{CS}_b &: \quad \mu_b b=0 \quad\text{and}\quad \mu_b\geq 0 \quad\text{and}\quad b\geq 0
\end{aligned}$$

We have potentially four cases:

1. $a>0,\ b=0$
2. $a=0,\ b>0$
3. $a=0,\ b=0$
4. $a>0,\ b>0$

Case 1: positive assets and no loans: $a>0$, $b=0$ and $\mu_a=0$, $\mu_b>0$:

The budget constraints simplify to:

$$\begin{aligned}
c_1+a&=y_1\\
c_2&=y_2+(1+r)\,a
\end{aligned}$$

And the FOC for assets implies:

$$\mu_a=0 \quad\rightarrow\quad \lambda_1=\lambda_2\,(1+r)$$

This is the standard case from section 1.1:

$$\begin{aligned}
c_1&=\frac{1}{1+\beta}\left[y_1+\frac{y_2}{1+r}\right]\\
c_2&=\frac{\beta}{1+\beta}\left[(1+r)\,y_1+y_2\right]\\
a&=\frac{1}{1+\beta}\left[\beta y_1-\frac{y_2}{1+r}\right]
\end{aligned}$$

To check if we are indeed in this case, we need to check if the value of assets and the inequality multiplier for borrowing are indeed positive:

$$\mu_b = \lambda_2 \left(1 + r^b\right) - \lambda_1 = \frac{\beta}{c_2} \left(1 + r^b\right) - \frac{1}{c_1}$$

Case 2: no assets and positive loans: $a = 0$, $b > 0$ and $\mu_a > 0$, $\mu_b = 0$:

The budget constraints simplify to:

$$\begin{aligned} c_1 &= y_1 + b \\ c_2 + \left(1 + r^b\right) b &= y_2 \end{aligned}$$

This is almost the same problem as with the case of positive assets. The lifetime budget constraint can be constructed as:

$$\begin{aligned} b &= \frac{y_2 - c_2}{1 + r^b} \\ c_1 &= y_1 + \frac{y_2 - c_2}{1 + r^b} \\ c_1 + \frac{c_2}{1 + r^b} &= y_1 + \frac{y_2}{1 + r^b} \end{aligned}$$

Note well that we are discounting future flows with the interest rate on borrowed amount. The relevant intertemporal condition results from the FOC for borrowing:

$$\begin{aligned} &\mu_b = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 \left(1 + r^b\right) \\ &\frac{1}{c_1} = \frac{\beta}{c_2} \left(1 + r^b\right) \\ &c_2 = \beta \left(1 + r^b\right) c_1 \end{aligned}$$

Let us now plug the Euler equation into the budget constraint:

$$\begin{aligned} c_1 + \frac{\beta \left(1 + r^b\right) c_1}{1 + r^b} &= y_1 + \frac{y_2}{1 + r^b} \\ c_1 + \beta c_1 &= y_1 + \frac{y_2}{1 + r^b} \end{aligned}$$

Solution:

$$\begin{aligned} c_1 &= \frac{1}{1 + \beta} \left[y_1 + \frac{y_2}{1 + r^b}\right] \\ c_2 &= \frac{\beta}{1 + \beta} \left[\left(1 + r^b\right) y_1 + y_2\right] \\ b = c_1 - y_1 &= \frac{1}{1 + \beta} \left[y_1 + \frac{y_2}{1 + r^b}\right] - y_1 = \frac{1}{1 + \beta} \left[\frac{y_2}{1 + r^b} - \beta y_1\right] \end{aligned}$$

Note well that b is more likely to be positive the higher is y_2 relative to y_1.

To check if we are indeed in this case, we need to check if the value of borrowing and the inequality multiplier for assets are indeed positive:

$$\mu_a = \lambda_1 - \lambda_2 (1 + r) = \frac{1}{c_1} - \frac{\beta}{c_2} (1 + r)$$

Case 3: no assets and no loans: $a = 0$, $b = 0$ and $\mu_a > 0$, $\mu_b > 0$:

This is the easiest case, as the budget constraints simplify to:

$$c_1 = y_1$$
$$c_2 = y_2$$

Implying

$$\lambda_1 = \frac{1}{c_1} = \frac{1}{y_1}$$
$$\lambda_2 = \frac{\beta}{c_2} = \frac{\beta}{y_2}$$

To check if we are indeed in this case, we need to calculate the values of two inequality Lagrange multipliers and check if they are indeed positive:

$$\mu_a = \lambda_1 - \lambda_2 (1 + r) = \frac{1}{y_1} - \frac{\beta}{y_2} (1 + r)$$
$$\mu_b = \lambda_2 \left(1 + r^b\right) - \lambda_1 = \frac{\beta}{y_2} \left(1 + r^b\right) - \frac{1}{y_1}$$

Case 4: positive assets and loans: $a > 0$, $b > 0$ and $\mu_a = 0$, $\mu_b = 0$:

Although in other setups it is possible to make agents want to hold both positive assets and borrowing (usually after introducing some degree of uncertainty), in the current setup this would not make sense. To easily see that, focus on FOCs on assets and borrowing:

$$\mu_a = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 (1 + r)$$
$$\mu_b = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 \left(1 + r^b\right)$$

If we divide the two conditions sideways, we get:

$$1 = \frac{1 + r}{1 + r^b}$$

which could only be true if $r^b = r$.

Below you can find an illustration of the solution for an example with $y_1 = 1$, $y_2 = 1$, $\beta = 1/(1 + 5\%)$ and with a 5% spread between saving and borrowing interest rates: