University of Warsaw
Faculty of Economic Sciences

Business Cycles Facts
Real Business Cycles Model

Advanced Macroeconomics QF: Lecture 9

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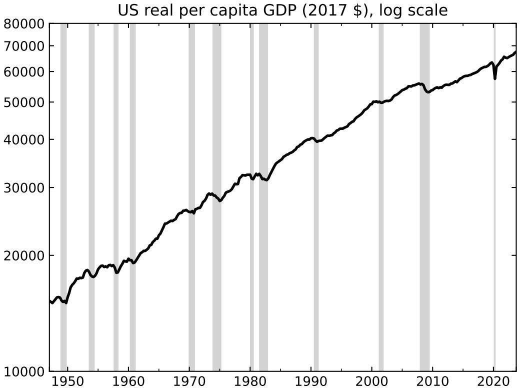
Business Cycles Facts

US real GDP per person



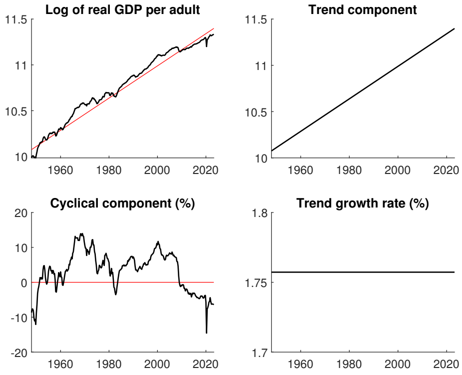
Real GDP per person grows thanks to technological progress (trend)

It fluctuates in the short term (recessions marked with grey bars)

How to separate cyclical component from the trend?

Log-linear trend

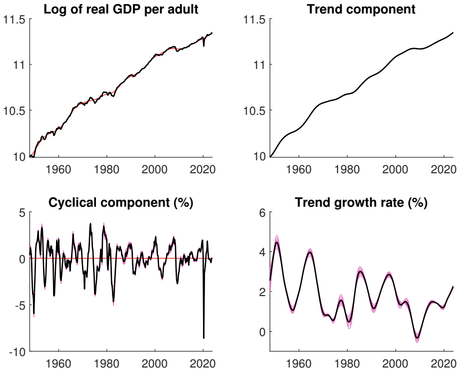
If the rate of technological progress is constant, real GDP **per adult** has log-linear trend



We know now however that the rate of technological progress can change over time

Additionally, the cyclical component obtained that way can be positive during recessions and negative during booms – we need a more flexible approach

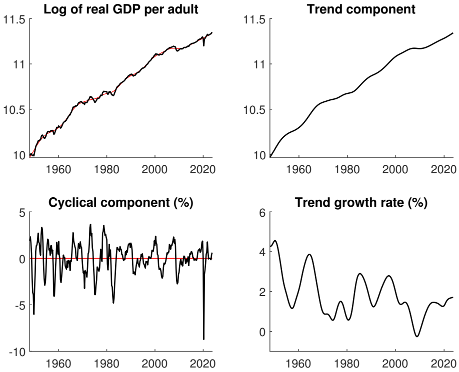
Estimated trend-cycle decomposition



The trend growth rate of real GDP per adult fluctuates around 2% annually

Cyclical component has a sensible property: it becomes negative during recessions

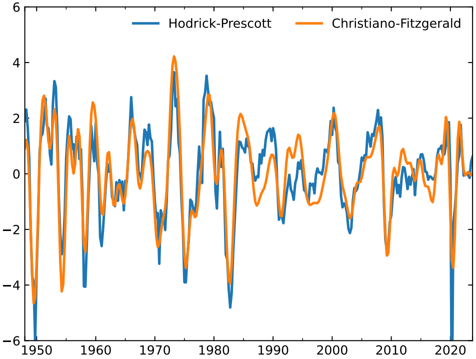
Hodrick-Prescott filter



Similar results can be obtained using a purely statistical Hodrick and Prescott (1980, 1997) filter, with a „smoothing" parameter $\lambda = 1600$ (typical value for quarterly data)

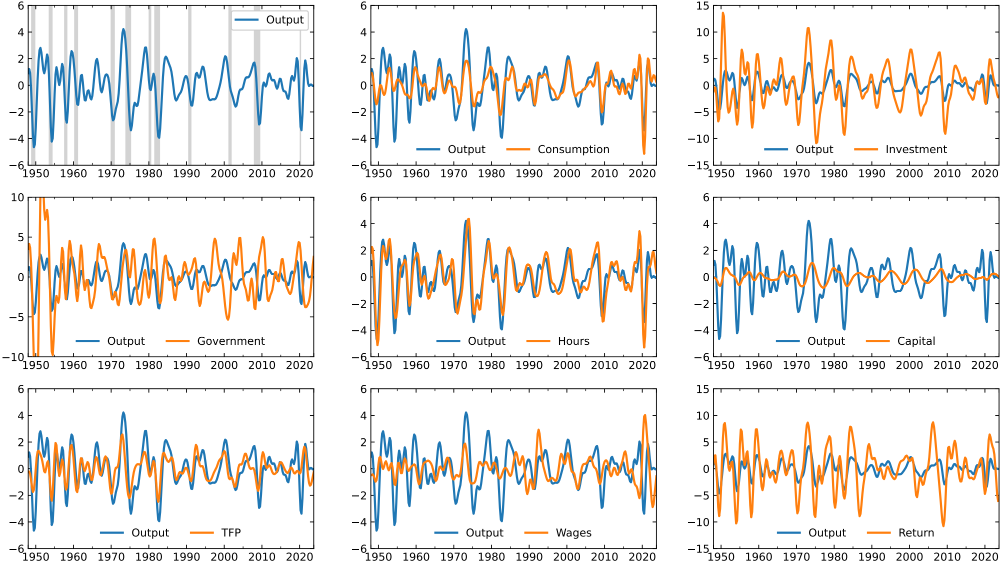
Comparing Hodrick-Prescott and Christiano-Fitzgerald filter results

Christiano-Fitzgerald filter identifies similar cyclical components, but “smoother” (it discards volatility at short-term frequencies)

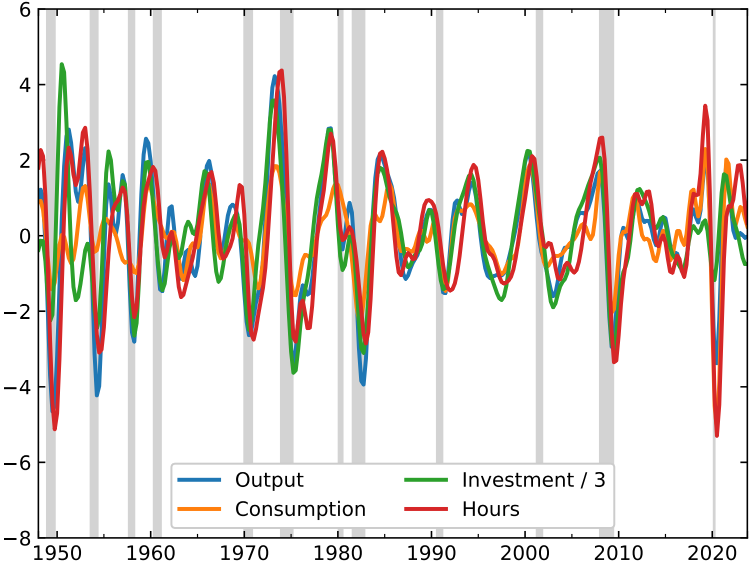


Due to visual niceness we will use Christiano-Fitzgerald filter for graphs, while quantitative results will be generated using Hodrick-Prescott filter

Cyclical components of US macroeconomic variables



Cyclical components of US macroeconomic variables



Cyclical components of US macroeconomic variables

Selected statistical moments of cyclical components (1960Q1-2015Q4)

	Standard deviation	Correlation with Y	Autocorrelation
Output (GDP)	1.47	1.00	0.87
Consumption	0.86	0.80	0.86
Investment	4.40	0.90	0.91
G + NX	2.74	0.11	0.52
Hours worked	1.52	0.78	0.91
Capital	0.45	0.34	0.97
Real hourly wages	0.90	0.27	0.78
Real return on capital	4.38	0.64	0.82
TFP	0.95	0.66	0.73

Real Business Cycles

Real Business Cycles Model (RBC)

Ramsey model with a neoclassical labor market and random "technology" shocks

To simplify notation assume constant population and stationary technology

Closed economy, no government (for now)

Perfect competition in goods and factors of production markets

Homogeneous final good with price normalized to 1 in every period, produced according to a neoclassical production function

All variables and prices expressed in real terms

Two groups of representative agents

- Households
- Firms

Households own factors of production directly

Households

Maximize their **expected** utility, which depends negatively on hours worked L

$$\max_{\{C_t, L_t, A_{t+1}\}_{t=0}^{\infty}} \quad U_0 = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{L_t^{1+\varphi}}{1+\varphi} \right) \right]$$

$$\text{subject to} \quad C_t + A_{t+1} = w_t L_t + (1+r_t) A_t$$

Parameter $\varphi > 0$ is the inverse of Frisch elasticity of labor supply
Parameter $\psi > 0$ is the labor disutility (chosen such that in the steady state $L^* = 1$)

Lagrangian

$$\mathcal{L} = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{L_t^{1+\varphi}}{1+\varphi} \right) + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t L_t + (1+r_t) A_t - C_t - A_{t+1} \right] \right]$$

Households

Expanded Lagrangian (adjusted notation for t denoting "today" and $t+1$ "tomorrow")

$$\mathcal{L} = \frac{\boldsymbol{C_t}^{1-\sigma}}{1-\sigma} - \psi \frac{\boldsymbol{L_t}^{1+\varphi}}{1+\varphi} + \ldots + \lambda_t \left[w_t \boldsymbol{L_t} + (1+r_t) A_t - \boldsymbol{C_t} - \boldsymbol{A_{t+1}} \right]$$
$$+ \beta \mathrm{E}_t \left[\lambda_{t+1} \left[w_{t+1} L_{t+1} + (1+r_{t+1}) \boldsymbol{A_{t+1}} - C_{t+1} - A_{t+2} \right] \right] + \ldots$$

First Order Conditions

$$\boldsymbol{C_t}: \quad C_t^{-\sigma} - \lambda_t = 0 \quad \rightarrow \quad \lambda_t = C_t^{-\sigma}$$
$$\boldsymbol{L_t}: \quad -\psi L_t^{\varphi} + \lambda_t w_t = 0 \quad \rightarrow \quad \lambda_t = \psi L_t^{\varphi} / w_t$$
$$\boldsymbol{A_{t+1}}: \quad -\lambda_t + \beta \mathrm{E}_t \left[\lambda_{t+1} (1+r_{t+1}) \right] = 0 \quad \rightarrow \quad \lambda_t = \beta \mathrm{E}_t \left[\lambda_{t+1} (1+r_{t+1}) \right]$$

Euler equation and consumption-hours choice (labor supply)

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left[C_{t+1}^{-\sigma} (1+r_{t+1}) \right]$$
$$C_t^{-\sigma} = \psi L_t^{\varphi} / w_t \quad \rightarrow \quad L_t = \left(w_t C_t^{-\sigma} / \psi \right)^{1/\varphi}$$

Firms

Firms maximize their profits / dividends in every period,
where Z is a stochastic Total Factor Productivity (TFP)

$$\max_{Y_t,\, K_t,\, L_t} \quad D_t = Y_t - w_t L_t - (r_t + \delta)\, K_t$$
$$\text{subject to} \quad Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

Profit maximization problem

$$\max_{K_t,\, L_t} \quad D_t = Z_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - (r_t + \delta)\, K_t$$

First Order Conditions

$$K_t: \quad \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - (r_t + \delta) = 0 \quad \rightarrow \quad r_t = \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta = \alpha \frac{Y_t}{K_t} - \delta$$
$$L_t: \quad (1-\alpha)\, Z_t K_t^{\alpha} L_t^{-\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha)\, Z_t K_t^{\alpha} L_t^{-\alpha} = (1-\alpha)\, \frac{Y_t}{L_t}$$

We can easily show that economic profits are 0

General equilibrium

The capital market is in equilibrium: $A_t = K_t$ for all t

Plug in the expression for prices into the household budget constraint to arrive at the economy's resource constraint

$$C_t + A_{t+1} = w_t L_t + (1 + r_t)\, A_t$$

$$C_t + K_{t+1} = w_t L_t + (1 + r_t)\, K_t$$

$$C_t + K_{t+1} = (1 - \alpha)\, Y_t / L_t \cdot L_t + (1 + \alpha Y_t / K_t - \delta)\, K_t$$

$$C_t + K_{t+1} = (1 - \alpha)\, Y_t + \alpha Y_t + (1 - \delta)\, K_t$$

$$C_t + K_{t+1} = Y_t + (1 - \delta)\, K_t$$

It is useful to explicitly define investment and track it over business cycle

$$I_t = K_{t+1} - (1 - \delta)\, K_t \quad \rightarrow \quad Y_t = C_t + I_t$$

Stochastic Total Factor Productivity (TFP)

TFP evolves according to an AR(1) process in logs

$$\ln Z_t = \rho_Z \cdot \ln Z_{t-1} + \epsilon_{Z,t}$$

where $\rho_Z \in (0,1)$ and $\epsilon_{Z,t} \sim \mathcal{N}(0, \sigma_Z^2)$

In the absence of shocks (steady state) $\ln Z^* = 0$ and $Z^* = 1$

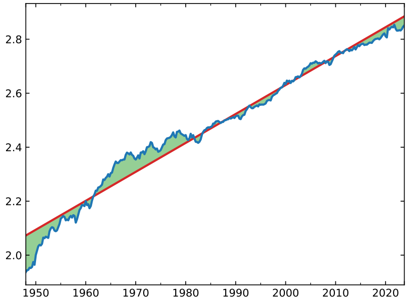
Stochastic Total Factor Productivity (TFP)

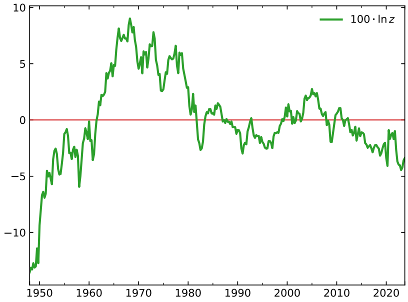
How can we estimate Z from the data? Using the Cobb-Douglas production function

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} \equiv Z_t \left(A_0 \cdot e^{gt}\right)^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}$$

$$\ln TFP_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln L_t$$

$$\ln Z_t = \ln TFP_t - (1-\alpha)(\ln A_0 + gt)$$





$$\ln Z_t = \rho_Z \cdot \ln Z_{t-1} + \epsilon_{Z,t}$$

$$\rho_Z \approx 0.966, \; \sigma_Z \approx 0.82$$

Where can negative TFP shocks come from?

Extend the production function with energy input E, where $\chi \in (0,1)$

$$\tilde{Y}_t = \left[K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}\right]^{\chi} E_t^{1-\chi} \equiv \tilde{A}_t \left[K_t^{\alpha} L_t^{1-\alpha}\right]^{\chi} E_t^{1-\chi} \equiv \tilde{A}_t X_t^{\chi} E_t^{1-\chi}$$

Assume that E comes only from imports and has relative price p^E. GDP then is

$$Y_t = \max_{E_t} \left\{ \tilde{A}_t X_t^{\chi} E_t^{1-\chi} - p_t^E E_t \right\}$$

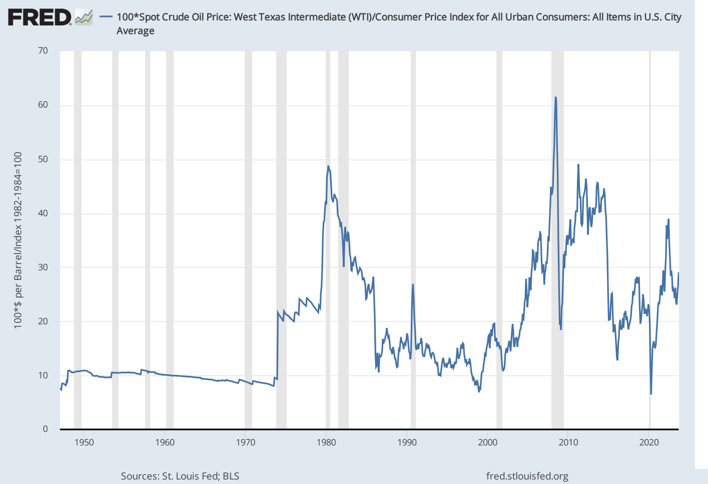
$$E_t: \qquad (1-\chi)\tilde{A}_t X_t^{\chi} E_t^{-\chi} - p_t^E = 0 \qquad \rightarrow \qquad E_t = \left[(1-\chi)\tilde{A}_t / p_t^E\right]^{1/\chi} X_t$$

$$Y_t = \tilde{A}_t X_t^{\chi} \left[\frac{(1-\chi)\tilde{A}_t}{p_t^E}\right]^{(1-\chi)/\chi} X_t^{1-\chi} - p_t^E \left[\frac{(1-\chi)\tilde{A}_t}{p_t^E}\right]^{1/\chi} X_t$$

$$Y_t = \chi \tilde{A}_t^{1/\chi} \left(\frac{1-\chi}{p_t^E}\right)^{(1-\chi)/\chi} X_t \equiv TFP_t \cdot X_t = TFP_t \cdot K_t^{\alpha} L_t^{1-\alpha}$$

Our TFP estimate depends negatively on the relative energy price, $\partial TFP_t / \partial p_t^E < 0$

Relative price of oil in the US



FRED

Full set of general equilibrium conditions

System of 8 equations and 8 unknowns: $Y, \boldsymbol{C}, I, \boldsymbol{L}, \boldsymbol{K}, w, r, \boldsymbol{Z}$

$$\begin{aligned}
\text{Euler equation} &: \quad C_t^{-\sigma} = \beta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(1 + r_{t+1}\right) \right] && (1) \\
\text{Labor supply} &: \quad L_t = \left(w_t C_t^{-\sigma} / \psi \right)^{1/\varphi} && (2) \\
\text{Production function} &: \quad Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} && (3) \\
\text{Real return on capital} &: \quad r_t = \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta && (4) \\
\text{Real hourly wage} &: \quad w_t = (1-\alpha) Y_t / L_t && (5) \\
\text{Investment} &: \quad I_t = K_{t+1} - (1-\delta) K_t && (6) \\
\text{Output accounting} &: \quad Y_t = C_t + I_t && (7) \\
\text{TFP AR(1) process} &: \quad \ln Z_t = \rho_Z \cdot \ln Z_{t-1} + \epsilon_{Z,t} && (8)
\end{aligned}$$

This time we cannot “move” C_{t+1} to the left hand side of the Euler equation

As in the Ramsey model, this equilibrium is fully efficient
and government interventions cannot improve households’ welfare

Steady state: closed form solution

Recall that the RBC model is just an extension of the Ramsey model

We already know that $Z^* = 1$. Next start with the Euler equation (1)

$$C^{-\sigma} = \beta C^{-\sigma}(1+r) \quad \rightarrow \quad r = 1/\beta - 1 = \rho$$

Since $L^* = 1$, we can easily find the level of capital per worker from equation (4)

$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta \quad \rightarrow \quad K^* = \left(\frac{\alpha}{r+\delta}\right)^{1/(1-\alpha)} = \left(\frac{\alpha}{\rho+\delta}\right)^{1/(1-\alpha)}$$

Once K^* is found, we can easily get the steady state values of remaining variables

$$Y = K^\alpha, \quad w = (1-\alpha)Y, \quad I = \delta K, \quad C = Y - I = Y - \delta K$$

Parametr ψ satisfies equation (2) under $L^* = 1$

$$\psi \equiv w \cdot C^{-\sigma}$$

We consider now more variables, but the steady state is identical to Ramsey

Transition dynamics: special case

Assume logarithmic utility $\sigma = 1$ and full capital depreciation $\delta = 1$

Again "guess-and-verify" that the model behaves like a Solow model

$$C_t = (1-s)\,Y_t \quad \text{and} \quad I_t = sY_t$$

Euler equation

$$C_t^{-1} = \beta \mathrm{E}_t \left[C_{t+1}^{-1} (1 + r_{t+1}) \right]$$

$$\frac{1}{(1-s)\,Y_t} = \beta \mathrm{E}_t \left[\frac{1 + \alpha Z_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} - \delta}{(1-s)\,Z_{t+1} K_{t+1}^{\alpha} L_{t+1}^{1-\alpha}} \right]$$

$$1/Y_t = \beta \mathrm{E}_t \left[\alpha / K_{t+1} \right]$$

$$K_{t+1} = \alpha\beta Y_t = \alpha\beta \cdot Z_t K_t^{\alpha} L_t^{1-\alpha} = I_t$$

Consumption and wages

$$C_t = (1-\alpha\beta)\,Y_t = (1-\alpha\beta)\,Z_t K_t^{\alpha} L_t^{1-\alpha} \quad \text{and} \quad w_t = (1-\alpha)\,Z_t K_t^{\alpha} L_t^{-\alpha}$$

Can show that hours worked are constant (and equal to 1 due to normalization)

Transition dynamics: general case

As in the case of Ramsey model, the RBC typically cannot be solved analytically

We can easily solve the model using numerical methods:

- Perturbation around the steady state (with approximation error, but “cheap”)
- Dynamic programming (global solution but computationally (too) heavy)

Usually we employ perturbation methods provided by Dynare software

How does it work for (log-)linear approximation?

We transform the system of forward-looking equations
into a backward-looking vector autoregression (VAR) system

What magic allows this? We assume rational expectations (model-consistent)

Agents make forecast errors (e.g. ϵ_Z cannot be foreseen),
but the forecasts are unbiased (they are correct on average)

We can also solve models under several non-rational expectation formation mechanisms

Parameter values

Parameter values can be chosen in at least two ways:

- Calibration: pick values according to data external to the model (e. g. $\alpha \approx 1/3$) or such that the steady state levels correspond to long sample averages
- Estimation: econometric procedure picks “most probable” values given the properties of supplied time series

First I extend the basic model a bit by adding trend population and productivity growth

I also add public expenditures (summed with net exports) with two components: quasi-permanent and transitory

Parameter values

Some parameters are set in line with long-term averages in the US

- Annual growth rate of adult (age 16 and over) population: 1.26%
- Annual growth rate of GDP per adult in the long run: 1.77%
- Investment to GDP ratio: 25.2%
- Public expenditures and net exports ratio to GDP: 19.7%
- Estimated capital to annual GDP ratio: 2.58 (10.3 times quarterly GDP)

Parameter	Value	Justification
α	0.355	Capital share of income
σ	1	Logarithmic utility function (consumption)
φ	0.5	Logarithmic utility function (leisure)
δ	0.0169	From capital accumulation equation (quarterly)
β	0.9922	From Euler equation (quarterly)
ψ	0.8913	Steady state hours worked = 1

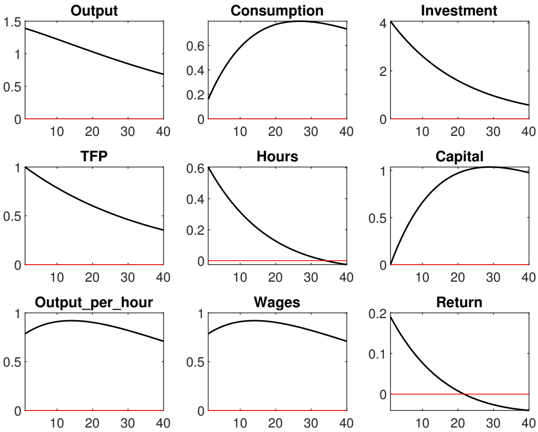
Parameter values

Parameters governing stochastic processes are estimated (results)

Parameter	Value	Justification
ρ_Z	0.9766	TFP shock persistence
ρ_S	0.6796	Quasi-permanent G+NX shock persistence
ρ_F	0.7437	Transitory G+NX shock persistence
σ_Z	0.8226	TFP shock standard deviation
σ_S	0.1453	Quasi-permanent G+NX shock standard deviation
σ_F	0.5159	Transitory G+NX shock standard deviation

Impulse response functions: TFP shock

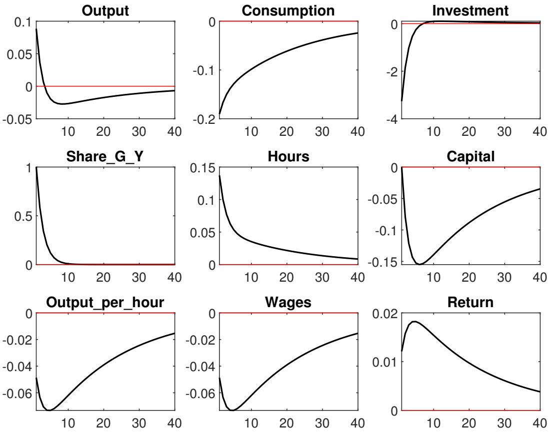
The economy receives a positive shock ϵ_Z whose influence dies out over time



Percent deviations from steady state values (for r percentage points)

Impulse response functions: transitory G shock

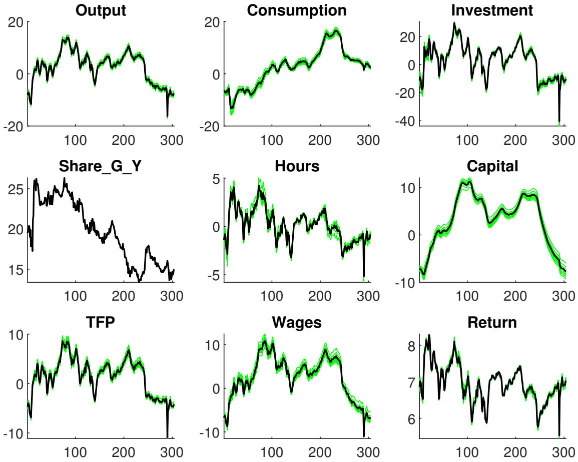
Transitory increase in public expenditures, financed via lump-sum tax



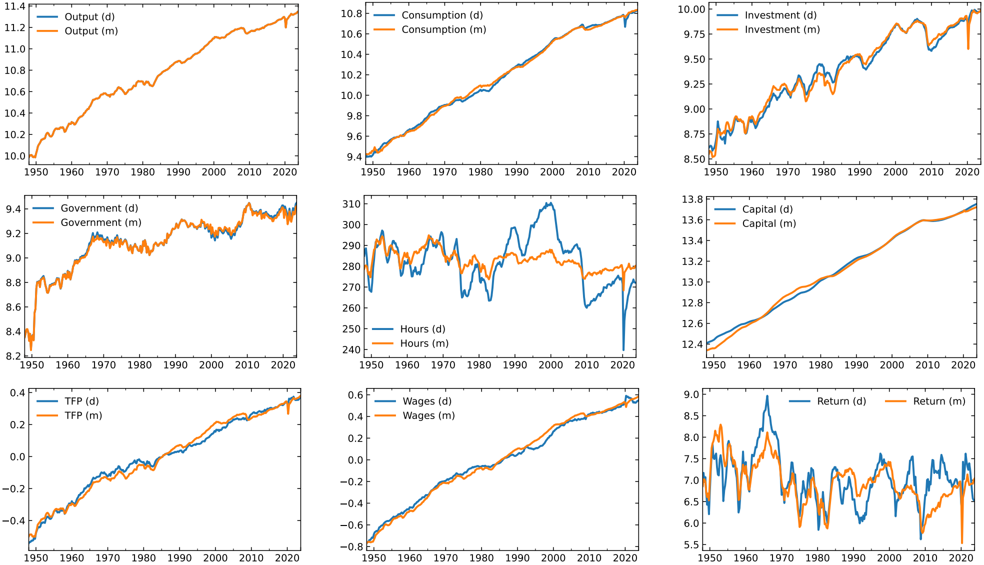
Percent deviations from steady state values (for r percentage points)

Estimated deviations from steady state

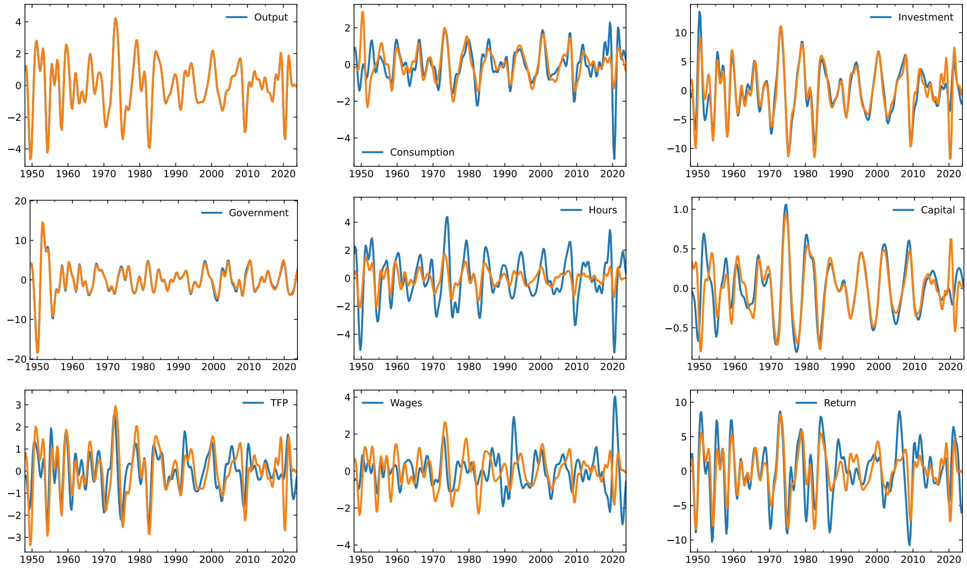
The model is only informed on the time series of quarterly real GDP per adult growth rates and share of public expenditure (+NX) to GDP. Deviations from the steady state:



Model vs data comparison: levels



Model vs data comparison: cyclical components



Model vs data comparison: statistical moments (1960Q1-2015Q4)

	Std. dev.		Corr. with Y		Autocorr.		Correlation
	Data	Model	Data	Model	Data	Model	Data-Model
Output	1.47	1.47	1.00	1.00	0.87	0.87	1.00
Consumption	0.86	0.83	0.80	0.73	0.86	0.94	0.79
Investment	4.40	4.41	0.90	0.96	0.91	0.89	0.95
Hours	**1.52**	**0.60**	0.79	0.97	0.91	0.82	0.66
Capital	0.45	0.43	0.34	0.28	0.97	0.96	0.96
Real wage	0.90	0.92	**0.27**	**0.99**	0.78	0.89	**0.23**
Return on K	4.38	3.29	0.65	0.95	0.82	0.86	0.77
TFP	0.95	1.04	**0.66**	**1.00**	0.82	0.87	0.67

Model performance is surprisingly good! There are some issues though:

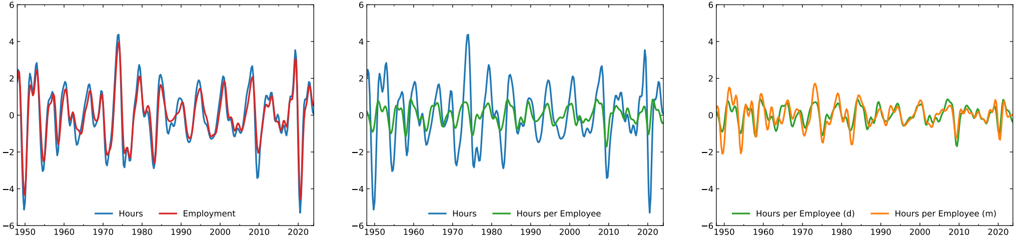
- Hours worked are much more volatile in the data
- Real wages can be pro- or counter-cyclical, in the model they are always pro-cyclical
- TFP and output are correlated in the data, but not 100% correlated as in the model

Why the RBC model does not model labor market well?

The vast majority of hours worked volatility over the business cycle is due to employment-unemployment flows rather than changes in hours worked per employee

Volatility of hours per employee is relatively small (correlation with RBC is 0.81)

RBC model mistakenly attributes remaining volatility of total hours to TFP volatility



Another shortcoming of RBC is assumption that real wages are bargained over, where in reality we typically bargain over nominal wages, while the price level is an “independent” variable over which we form expectations

What lies ahead

We can model both long-run growth and business cycles phenomena starting from the same model (Ramsey) – a huge surprise in the 1980s!

We will replace the neoclassical labor market with a better mechanism

We will also later add the nominal side of the economy