Marcin Bielecki, Advanced Macroeconomics QF, Fall 2020
Endogenous Growth Models[1]

1 Expanding product variety

Based on **Romer (1990)** *Endogenous Technological Change*.

1.1 Households

Assume for simplicity constant population L. Utility maximization problem:

$$\max \quad U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$\text{subject to} \quad a_{t+1} = w_t + (1+r_t)\, a_t - c_t$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - a_{t+1} \right] \right]$$

Expanded:

$$\mathcal{L} = \ldots + \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \cdots + \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - a_{t+1} \right]$$
$$+ \beta^{t+1} \lambda_{t+1} \left[w_{t+1} + (1+r_{t+1})\, a_{t+1} - c_{t+1} - a_{t+2} \right] + \ldots$$

First order conditions:

$$c_t \; : \; \beta^t c_t^{-\sigma} - \beta^t \lambda_t = 0 \quad \rightarrow \quad \lambda_t = c_t^{-\sigma}$$
$$a_{t+1} \; : \; -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1+r_{t+1}) = 0 \quad \rightarrow \quad \lambda_t = \beta (1+r_{t+1})\, \lambda_{t+1}$$

Euler equation:

$$c_t^{-\sigma} = \beta (1+r_{t+1})\, c_{t+1}^{-\sigma}$$
$$\left(\frac{c_{t+1}}{c_t} \right)^{\sigma} = \beta (1+r_{t+1})$$
$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\sigma}}$$

where we used the relationship between the discount factor and the discount rate: $\beta = 1/(1+\rho)$.

Approximate (this results in expressions that are accurate in continuous time):[2]

$$\frac{c_{t+1}}{c_t} = \left(\frac{(1+r_{t+1})(1-\rho)}{(1+\rho)(1-\rho)} \right)^{\frac{1}{\sigma}} = \left(\frac{1+r_{t+1}-\rho-\rho r_{t+1}}{1-\rho^2} \right)^{\frac{1}{\sigma}} \approx (1+r_{t+1}-\rho)^{\frac{1}{\sigma}} \approx 1 + \frac{r_{t+1}-\rho}{\sigma}$$
$$\frac{\Delta c_{t+1}}{c_t} \approx \frac{r_{t+1}-\rho}{\sigma}$$

Rate of growth of per capita consumption along the Balanced Growth Path:

$$g_c = \frac{r-\rho}{\sigma}$$

[1] This set of lecture notes is based on chapters 3–5 and 7 from **Aghion and Howitt (2009)** *The Economics of Growth.*
[2] At the end of the first line we used first-order Taylor expansion around $x_0 = 0$: $(1+x)^a = 1^a + a \cdot 1^{a-1} \cdot (x-0) + \mathcal{O}(2)$.

1.2 Producers

Two types of goods:

- homogenous final goods Y_t produced by perfectly competitive, representative firm
- M_t varieties of differentiated intermediate goods (machines) x_{it} produced by monopolists

1.2.1 Final goods

Production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Profit maximization problem:

$$\max \quad L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First order conditions:

$$\begin{aligned} L \quad &: \quad (1-\alpha) L^{-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha) \frac{Y_t}{L} \\ x_{it} \quad &: \quad L^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha} \end{aligned}$$

Demand for intermediate good of type i:

$$x_{it} = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} L$$

1.2.2 Intermediate goods

One unit of intermediate good is produced from one unit of final good. Hence the marginal cost of production in the intermediate goods sector is equal to 1.

Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_{it} = (p_{it} - 1) x_{it} = p_{it} x_{it} - x_{it} \\ \text{subject to} \quad & p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha} \end{aligned}$$

Incorporate the demand schedule into the profit function:

$$\max \quad \Pi_{it} = \alpha x_{it}^{\alpha} L^{1-\alpha} - x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1} L^{1-\alpha}}_{p_{it}} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} = 1$$

Optimal price:

$$p_{it} = \frac{1}{\alpha}$$

Optimal level of production:

$$x_{it} = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} L = \left(\alpha^2\right)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{2}{1-\alpha}} L$$

The level of production of all intermediate goods will be the same and constant over time. We can drop subscripts i and t. Maximal profit of the intermediate goods producer is given by:

$$\Pi = (p-1) x = \left(\frac{1}{\alpha} - 1\right) \alpha^{\frac{2}{1-\alpha}} L = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L$$

1.3 Research and Development

Developing a new variety of intermediate good requires sacrificing $1/\eta$ units of final good. Parameter η measures the productivity of the R&D sector. Let R denote the amount of resources devoted to R&D. Then the number of varieties will increase by:

$$\Delta M_{t+1} = \eta R_t$$

Assume that the research sector is perfectly competitive. Then the cost of invention $1/\eta$ will have to be equal to the present discounted value of profit flows of a new intermediate good producing monopolist:[3]

$$\frac{1}{\eta} = V = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \Pi = \Pi \cdot \frac{\frac{1}{1+r}}{1 - \frac{1}{1+r}} = \Pi \cdot \frac{\frac{1}{1+r}}{\frac{r}{1+r}} = \frac{\Pi}{r}$$

And the real interest rate will have to satisfy in equilibrium:

$$r = \eta \Pi$$

1.4 General Equilibrium

We can now plug the above interest rate into the Euler equation:

$$g_c = \frac{\eta \Pi - \rho}{\sigma} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

By assuming that along the Balanced Growth Path rates of growth of consumption and productivity are equal, as is indeed the case, we get the result that:

$$g = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

Growth rate increases with:

- the productivity of the R&D sector as measured by the parameter η
- the size of the economy as measured by labor supply L

and decreases with:

- the rate of time preference ρ
- the degree of risk aversion σ.

The prediction that g should increase with L was first seen as a virtue of the model, suggesting that larger countries or larger free-trade zones should grow faster. However, **Jones (1995)** pointed out that this prediction is counterfactual. On the other hand, **Kremer (1993)** argued that the above equation approximates well the growth experience of the world economy treated as a whole.

[3]Here we assume that the new variety invented in period t is produced starting from period $t+1$ onwards.

2 Increasing product quality (Schumpeterian growth)

Based on **Aghion and Howitt (1992)** *A Model of Growth Through Creative Destruction.*

2.1 Producers

Two types of goods:

- homogenous final goods Y_t produced by perfectly competitive, representative firm
- constant M varieties of differentiated intermediate goods (machines) x_{it} produced by monopolists, with each characterized by a certain level of quality / productivity A_{it}

2.1.1 Final goods

Production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

Profit maximization problem:

$$\max \quad L^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M} p_{it} x_{it}$$

First order condition (with respect to x_{it}):

$$x_{it} \quad : \quad p_{it} = \alpha L^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1}$$

2.1.2 Intermediate goods

Same as before, one unit of intermediate good is produced from one unit of final good.

Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_{it} = p_{it} x_{it} - x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it} L\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Incorporate the demand schedule:

$$\max \quad \Pi_{it} = \alpha \left(A_{it} L\right)^{1-\alpha} x_{it}^{\alpha} - x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha \left(A_{it} L\right)^{1-\alpha} x_{it}^{\alpha-1}}_{p_{it}} - 1 = 0$$

Optimal price:

$$p_t = \frac{1}{\alpha}$$

Optimal level of production:

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} A_{it} L$$

Profit:

$$\Pi_{it} = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A_{it} L \equiv \Pi \cdot A_{it}$$

Value of the firm with quality level A_{it}:

$$V\left(A_{it}\right) = \frac{\Pi A_{it}}{r+z}$$

where z is the probability of being replaced by a successful innovator. Note here that I assume that both the real interest rate r and innovation probability z are constant in equilibrium, which is indeed the case.

Final goods output:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_{it} L\right)^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}} L \left(\frac{1}{M} \sum_{i=1}^{M} A_{it}\right) = \alpha^{\frac{2\alpha}{1-\alpha}} A_t L$$

where the aggregate productivity A_t is the simple average of all the individual productivity levels:

$$A_t \equiv \frac{1}{M} \sum_{i=1}^{M} A_{it}$$

2.2 Research and Development

A successful innovator replaces the monopolist in an industry i and increases the productivity / quality of the intermediate good by $1+q$, where $q > 0$:

$$A_{i,t+1} = (1+q) A_{it}$$

Success probability z_{it} depends on the amount of devoted R&D resources R_{it}, adjusted by the target quality level $A_{it}^* \equiv (1+q) A_{it}$, since as technology advances it becomes harder to improve upon:

$$z_{it} = \eta \frac{R_{it}}{A_{it}^*}$$

with parameter η reflecting the productivity of the R&D sector.

If successful, the innovator will gain ownership of a firm with quality level A_{it}^*:

$$V_{it}^* = V\left(A_{it}^*\right) = \frac{\Pi A_{it}^*}{r+z}$$

The expected net benefit of R&D activity is:

$$z_{it} V_{it}^* - R_{it} = \eta \frac{R_{it}}{A_{it}^*} \cdot \frac{\Pi A_{it}^*}{r+z} - R_{it} = R_{it} \cdot \frac{\eta \Pi}{r+z} - R_{it}$$

Innovators choose the amount of R&D resources R_{it} to maximize the expected net benefits of R&D:

$$R_{it} \quad : \quad \frac{\eta \Pi}{r+z} - 1 = 0 \quad \rightarrow \quad z = \eta \Pi - r$$

Note that in equlibrium the probability of a successful innovation will be the same for all intermediates.

2.3 General Equilibrium

Solving the standard utility maximization problem of the consumer results in the Euler equation:

$$g_c = \frac{r - \rho}{\sigma}$$

As the number of intermediate good types is large, the growth rate of the economy will be "smooth":

$$A_{t+1} = \frac{1}{M} \sum_{i=1}^{M} \left[z(1+q) A_{it} + (1-z) A_{it}\right] = (1+zq) \frac{1}{M} \sum_{i=1}^{M} A_{it} = (1+zq) A_t$$

$$g_A = \frac{A_{t+1} - A_t}{A_t} = \frac{(1+zq) A_t - A_t}{A_t} = \frac{zq A_t}{A_t} = zq$$

By assuming that along the Balanced Growh Path rates of growth of consumption and productivity are equal, as is indeed the case, we get the following system of three equations linking real interest rate r, innovative success probability z and the rate of growth of the economy g:

$$\begin{cases} \sigma g = r - \rho & \text{Euler equation} \\ z = \eta \Pi - r & \text{Optimal R\&D intensity} \\ g = zq & \text{Growth rate} \end{cases}$$

Solving the system:

$$r = \eta \Pi - z$$
$$z = g/q$$

$$\sigma g = \eta \Pi - z - \rho$$
$$\sigma g = \eta \Pi - g/q - \rho$$

$$g = \frac{\eta \Pi - \rho}{\sigma + 1/q} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma + 1/q}$$

Growth rate increases with:

- the productivity of the R&D sector η
- the size of the innovative step q
- the size of the economy as measured by labor supply L

and decreases with:

- the rate of time preference ρ
- the degree of risk aversion σ

3 Innovation and capital accumulation

Suppose now that intermediate goods are produced using capital, which can accumulate over time. We will now also allow for the population size to change over time, but also assume a production function that eliminates the scale effects (i.e. growth will not depend on L).

Final goods

Production function:

$$Y_t = \left(\frac{L_t}{M_t}\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} = \ell_t^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

where $\ell_t \equiv L_t/M_t$ denotes workers per product line. Here we assume that what matters is not the absolute input L of labor but the input per product L/M.

First we have to specify the process by which product variety increases. One simple scheme is to assume that suppose that each person has a probability ψ of inventing a new intermediate product, with no expenditure at all on research. Suppose also that the exogenous fraction ε of products disappears each period. The number of intermediate products will stabilize at a level proportional to population:

$$\begin{aligned} M_{t+1} &= (1-\varepsilon) M_t + \psi L_t \quad | \quad : L_t \\ (1+n) \frac{M_{t+1}}{L_{t+1}} &= (1-\varepsilon) \frac{M_t}{L_t} + \psi \end{aligned}$$

Steady state:

$$\frac{M}{L} = \frac{\psi}{\varepsilon + n} \quad \rightarrow \quad \frac{L}{M} = \frac{\varepsilon + n}{\psi} \equiv \ell$$

where ℓ denotes workers per product line. We will analyze the economy that has already reached its Balanced Growth Path.

Profit maximization problem:

$$\max \quad \ell^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L_t - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First order condition (with respect to x_{it}):

$$x_{it} \quad : \quad p_{it} = \alpha \ell_t^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1}$$

Intermediate goods

Now in order to produce one unit of an intermediate good, its producer needs to rent one unit of capital at capital rental rate r_t^k. Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_{it} = p_{it} x_{it} - r_t^k x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it} \ell\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Incorporate the demand schedule:

$$\max \quad \Pi_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha} - r_t^k x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1}}_{p_{it}} - r_t^k = 0 \quad \rightarrow \quad \alpha p_{it} = r_t^k \quad \rightarrow \quad p_{it} = \frac{r_t^k}{\alpha}$$

Optimal level of production:

$$x_{it} = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} A_{it} \ell$$

The capital rental rate is determined in the market for capital, where the supply is the (predetermined) capital stock K_t and the demand is the sum of all intermediate goods demands:

$$K_t = \sum_{i=1}^{M_t} x_{it} = \sum_{i=1}^{M_t} \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} L_t \cdot \frac{1}{M_t}\sum_{i=1}^{M_t} A_{it} = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_t L_t$$

The level of capital rental rate depends negatively on the level of capital per effective labor:

$$K_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_t L_t \quad \rightarrow \quad \hat{k}_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} \quad \rightarrow \quad r_t^k = \alpha^2 \hat{k}_t^{\alpha-1}$$

The optimal level of intermediate goods production can be expressed also as:

$$x_{it} = \left(\frac{\alpha^2}{\alpha^2 \hat{k}_t^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} A_{it}\ell = A_{it}\ell\hat{k}_t$$

Profit:

$$\Pi_{it} = \alpha\left(A_{it}\ell\right)^{1-\alpha}\left(A_{it}\ell\hat{k}_t\right)^{\alpha} - \alpha^2\hat{k}_t^{\alpha-1} \cdot A_{it}\ell\hat{k}_t = \alpha\left(1-\alpha\right)\hat{k}_t^{\alpha}\ell \cdot A_{it} \equiv \Pi(\hat{k}_t)\cdot A_{it}$$

Profits increase with capital per effective labor, because an increase in $\hat{k}_t$ reduces the monopolist's per-unit cost of production equal to the rental rate of capital r_t^k.

Final goods output is then given by a familiar Cobb-Douglas production function:

$$Y_t = \ell^{1-\alpha}\sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} = \left(\frac{L_t}{M_t}\right)^{1-\alpha}\sum_{i=1}^{M_t} A_{it}^{1-\alpha}\left(A_{it}\ell\hat{k}_t\right)^{\alpha} = \hat{k}_t^{\alpha} L_t \cdot \frac{1}{M_t}\sum_{i=1}^{M_t} A_{it} = \hat{k}_t^{\alpha} L_t A_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$$

Innovation and Growth

As before, a successful innovation will increase the productivity of an intermediate good by $1+q$ and the success probability depends on target productivity adjusted R&D resources:

$$A_{i,t+1} \equiv A_{it}^* = (1+q)\,A_{it}$$
$$z_{it} = \eta\frac{R_{it}}{A_{it}^*}$$

Technically speaking, the following expression for the value of the firm is incorrect when capital per effective labor changes over time, but we are focusing on the BGP anyway:

$$V\left(A_{it}\right) = \frac{\Pi(\hat{k}^*)A_{it}}{r(\hat{k}^*)+z}$$

The interest rate and capital rental rate are related by:

$$r(\hat{k}^*) = r^k(\hat{k}^*) - \delta = \alpha^2(\hat{k}^*)^{\alpha-1} - \delta$$

The expected net benefit of R&D activity is:

$$\eta\frac{R_{it}}{A_{it}^*}\cdot V\left(A_{it}^*\right) - R_{it} = \eta\frac{R_{it}}{A_{it}^*}\cdot\frac{\Pi(\hat{k}^*)A_{it}^*}{r(\hat{k}^*)+z} - R_{it} = R_{it}\cdot\frac{\eta\Pi(\hat{k}^*)}{r(\hat{k}^*)+z} - R_{it}$$

First order condition:

$$R_{it} \quad : \quad \frac{\eta\Pi(\hat{k}^*)}{r(\hat{k}^*)+z} - 1 = 0 \quad \rightarrow \quad z = \eta\Pi(\hat{k}^*) - r(\hat{k}^*)$$

The growth rate along the BGP now depends positively on capital per effective labor, since higher capital per effective labor means higher profits and lower interest rates:

$$g = zq = q\left[\eta\Pi(\hat{k}^*) - r(\hat{k}^*)\right]$$

General Equilibrium

To keep things simple, we'll assume that the savings rate is a constant fraction of income, as in the Solow-Swan model. The capital accumulation equation is:

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)\,K_t \quad \Big| \quad : A_t L_t \\ (1+n)\,(1+g)\,\hat{k}_{t+1} &= s\hat{k}_t^{\alpha} + (1-\delta)\,\hat{k}_t \end{aligned}$$

and the BGP level of capital per effective labor is given by:

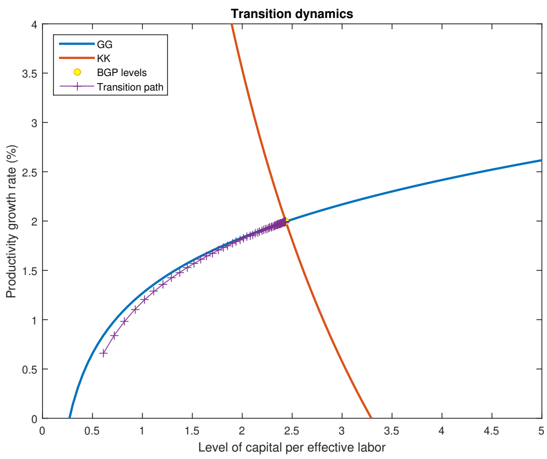
$$\hat{k}^* = \left(\frac{s}{\delta + n + g + ng} \right)^{\frac{1}{1-\alpha}}$$

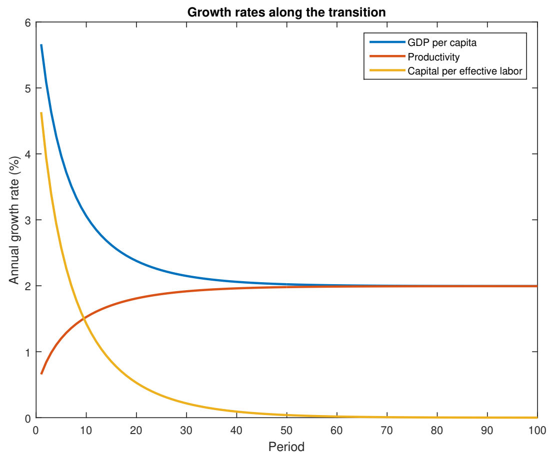
Note that the BGP level of capital per effective labor depends negatively on the growth rate.

The BGP level of capital per effective labor and the BGP productivity growth rate are here jointly determined and influence each other:

$$\begin{cases} g = q\left[\eta\Pi(\hat{k}^*) - r(\hat{k}^*)\right] & \text{BGP productivity growth rate (GG curve)} \\ \hat{k}^* = \left(\dfrac{s}{\delta + n + g + ng} \right)^{\frac{1}{1-\alpha}} & \text{BGP level of capital per effective labor (KK curve)} \end{cases}$$

We can obtain the solution of the transition of the model to the BGP numerically and produce the following graphical illustration:[4]





When the initial level of capital per effective labor is below its BGP level, the growth rate of productivity is also lower than along the BGP as intermediate goods producers face higher production costs and the gains from engaging in R&D are smaller. Also initially the rate of growth of capital per effective labor contributes more to the growth of GDP per capita than productivity growth, but eventually productivity growth becomes the sole driver of growth in GDP per capita in the long run.

[4]The following parameter values were assumed: $\alpha = 0.33$, $\delta = 0.08$, $n = 0.01$, $s = 0.2$, $\ell = 15$ (average employment per firm), $q = 1.05$. To get $\ell = 15$, the rate of firm destruction was assumed at $\varepsilon = 0.12$ annually and $\psi = 0.00865$. Finally, $\eta = 0.085$ was chosen to match the 2% growth rate observed on average in the United States.

4 Technology Transfer and Cross-Country Convergence

We can also analyze the issue of international transfer of technology. Suppose that there exist two groups of countries: technology leaders and technology adopters.[5] The behavior of technology leaders is the same as described in the previous sections. Technology adopters enjoy an "advantage of backwardness" and can increase their productivity by adopting technologies developed in other countries. However, if a country does not innovate at all, then it will stagnate while the rest of the world continues to advance.

Productivity and Distance to Frontier

We will assume that the number of intermediate good types M is constant over time and the same in all countries, but each country might have access to different productivity / quality levels of these intermediate goods. A successful innovator / imitator in any sector gets to implement a technology with a productivity parameter equal to a level $\bar{A}_{it}$, which represents the world technology frontier in this sector and which grows at a rate $\bar{g}$ determined outside the country. Each sector's productivity parameter A_i will evolve according to:

$$A_{i,t+1} = \begin{cases} \bar{A}_{it} & \text{with probability } z \\ A_{it} & \text{with probability } 1-z \end{cases}$$

That is, in the fraction of sectors z productivity increases from A_{it} to $\bar{A}_{it}$, whereas in the remaining fraction productivity remains unchanged. The country's aggregate productivity evolves according to:

$$A_{t+1} = \frac{1}{M}\sum_{i=1}^{M} A_{i,t+1} = \frac{1}{M}\sum_{i=1}^{M}\left[z\bar{A}_{it} + (1-z)\,A_{it}\right] = z\sum_{i=1}^{M}\bar{A}_{it} + (1-z)\,\frac{1}{M}\sum_{i=1}^{M}A_{it} = z\bar{A}_t + (1-z)\,A_t$$

That is, in the fraction of sectors z that manage to innovate / imitate productivity jumps to level $\bar{A}_t$, whereas in the remaining fraction productivity remains the same as in period t.

The country's "proximity" to the world technology frontier is the ratio of its aggregate productivity to the global productivity frontier:

$$a_t = A_t/\bar{A}_t$$

and evolves according to:

$$\begin{aligned} A_{t+1} &= z\bar{A}_t + (1-z)\,A_t \quad | \quad : \bar{A}_t \\ \frac{A_{t+1}}{\bar{A}_t} = \frac{A_{t+1}}{\bar{A}_{t+1}}\frac{\bar{A}_{t+1}}{\bar{A}_t} &= z + (1-z)\,\frac{A_t}{\bar{A}_t} = z + (1-z)\,a_t \\ a_{t+1}\,(1+\bar{g}) &= z + (1-z)\,a_t \\ a_{t+1} &= \frac{z + (1-z)\,a_t}{1+\bar{g}} \end{aligned}$$

There is a unique steady-state proximity a^*, which can be found by setting $a_{t+1} = a_t = a^*$:

$$\begin{aligned} a^*\,(1+\bar{g}) &= z + (1-z)\,a^* \\ a^*\,(\bar{g}+z) &= z \\ a^* &= \frac{z}{\bar{g}+z} < 1 \end{aligned}$$

Once the steady-state proximity is reached, the country's productivity growth rate is given by:

$$g = \frac{A_{t+1}-A_t}{A_t} - 1 = \frac{z\bar{A}_t + (1-z)\,A_t}{A_t} - 1 = \frac{z}{a^*} + (1-z) - 1 = \bar{g} + z - z = \bar{g}$$

Therefore, all technology adopters that innovate ($z > 0$) will converge to the same growth rate, although their steady-state proximity to the technology frontier may differ due to different z.

[5] This is a simplifying assumption. In reality this distinction is not strict, as even highly developed countries are technology adopters in some industries.

Convergence and Divergence

Recall the formula for the probability of innovating / imitating z (whenever the formula would yield negative values, a country does not innovate at all and sets $z = 0$):

$$z = \eta \Pi(\hat{k}) - r(\hat{k})$$

Let us focus first on those technology adopters that innovate / imitate ($z > 0$). Another "advantage of backwardness" is that the growth rate of productivity is faster the further behind the technology frontier a country is. The average innovation size is given by:

$$q_{t+1} \equiv \frac{\bar{A}_t - A_t}{A_t} = \frac{1}{a_t} - 1$$

And the country's productivity growth rate is:

$$g_t = z q_t = z \left(\frac{1}{a_{t-1}} - 1 \right)$$

Therefore, the further behind the frontier the country is, the higher its productivity growth rate will be, conditional on $z > 0$. This fact limits how far behind the frontier a country can fall, because eventually it will get so far behind that its growth rate will be just as large as the growth rate of the frontier, at which point the gap will stop increasing.

However, if countries do not innovate at all ($z = 0$), maybe due to poor macroeconomic conditions, legal environment, education system, or credit markets, or simply due to a low level of capital per effective labor, they will not partake in technology transfers, but will instead stagnate. If this situation persists, their productivity level will remain constant and they will diverge from the club of innovating countries.

Together these two results help to explain the empirical fact that there is a group of countries that are converging to parallel growth paths (i.e., with identical long-run growth rates) and another group of countries that are falling further and further behind. Notice that even countries that are converging to parallel growth paths are not necessarily converging in levels. That is, one country's steady-state proximity to the frontier a^* can differ from another's if they have different values of the critical parameters governing the intensity of R&D.

This result helps us to account for the fact that there are systematic and persistent differences across countries in the level of productivity. That is, convergence in levels is not absolute but conditional. In our model, two countries will end up with the same productivity levels in the long run if they share identical parameter values, but not otherwise.