University of Warsaw
Faculty of Economic Sciences

Growth facts. Solow-Swan model

Advanced Macroeconomics QF: Lecture 4

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Brief review on GDP

Gross domestic product (GDP) is the current dollar value of all final goods and services that are produced within a country within a given period of time

Real GDP: Nominal GDP adjusted for inflation
(and differences in relative prices across countries via PPP adjustment)

(Real) GDP measurement:

$$
\begin{aligned}
\text{(1) Expenditure approach:} \quad & Y_t = C_t + I_t + G_t + NX_t \\
\text{(2) Income approach:} \quad & Y_t = w_t L_t + (r_t + \delta) K_t + D_t + T_t \\
\text{(3) Value added approach:} \quad & Y_t = Y_{1t} + Y_{2t} + \ldots + Y_{Mt} = F(K_t, L_t, A_t)
\end{aligned}
$$

Real GDP per person: Y_t / N_t

If you need more of a review: Macroeconomics Principles, Chapters 1-3

GDP per person and welfare

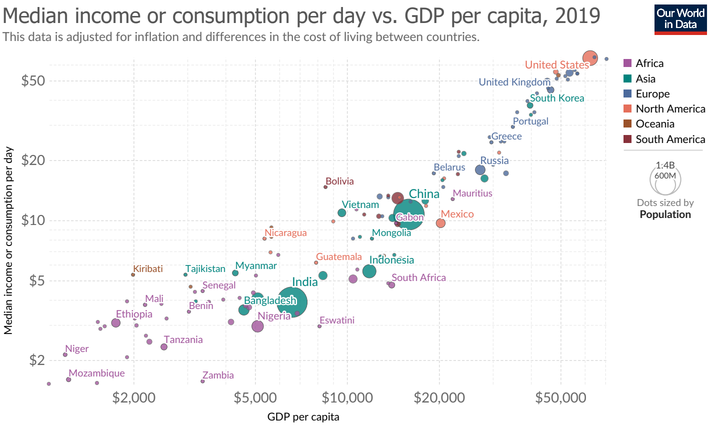
GDP per person is not designed to measure welfare, but it's a useful summary statistic

GDP per person ignores distribution of income within a country

Comparing GDP per person across countries is not trivial in practice:

- You have to convert between currencies
- Countries have different relative prices for goods
- Large uncertainty in comparing real GDP across countries and over time: Johnson et al. (2013) *Is newer better? Penn World Table Revisions and their impact on growth estimates*

GDP per person and welfare: consumption



Source: World Bank Poverty and Inequality Platform (2022), Data compiled from multiple sources by World Bank
Note: This data is expressed in international-$ at 2017 prices. Depending on the country and year, median data relates to income measured after taxes and benefits, or to consumption, per capita.

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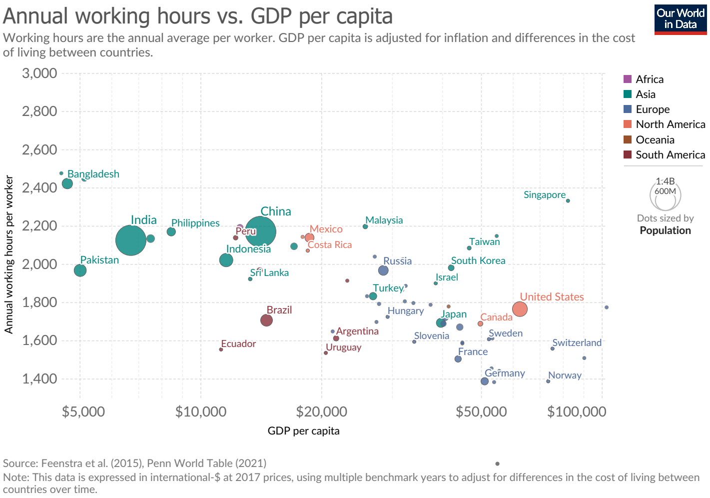
GDP per person and welfare: hours worked

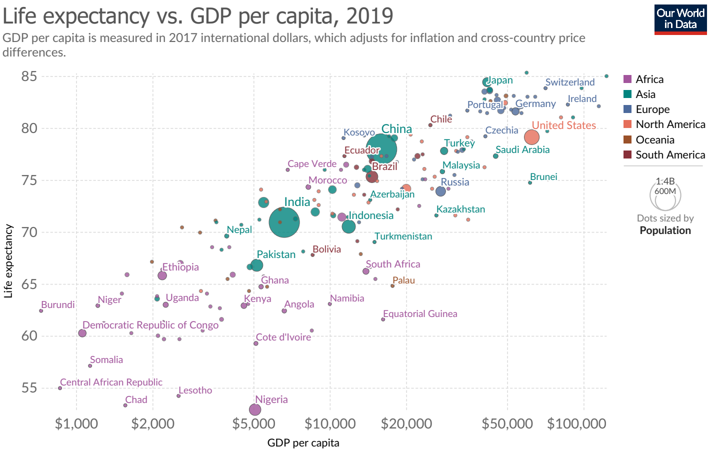


Our World in Data

GDP per person and welfare: life expectancy

Life expectancy vs. GDP per capita, 2019

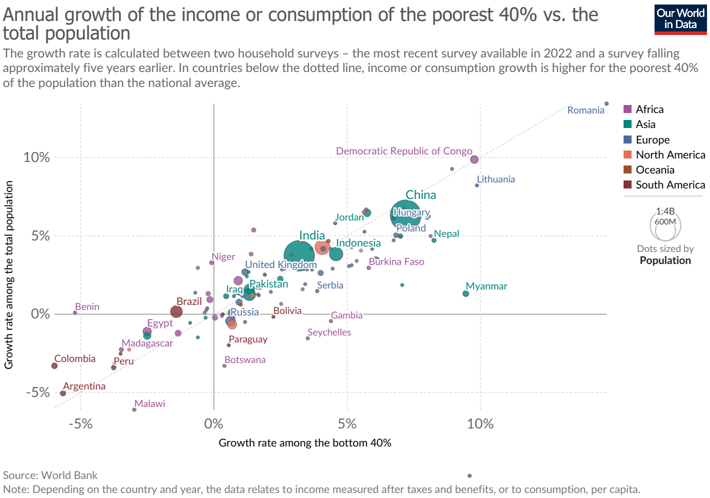
GDP per capita is measured in 2017 international dollars, which adjusts for inflation and cross-country price differences.



Source: UN, World Population Prospects (2022); Data compiled from multiple sources by World Bank

Our World in Data

Growth in average income vs income of bottom 40%

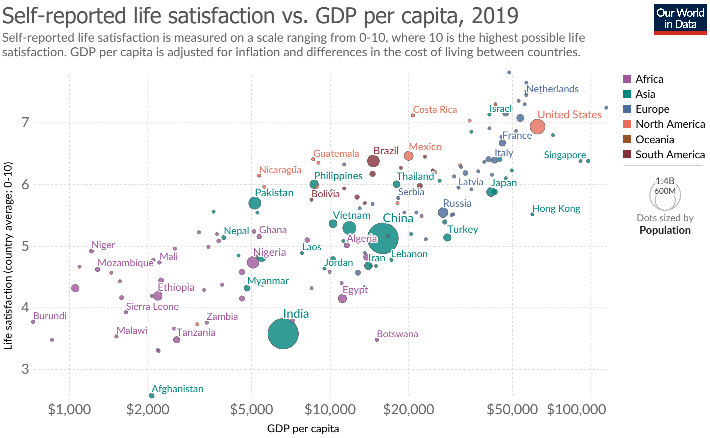


Our World in Data

GDP per person and welfare: life satisfaction

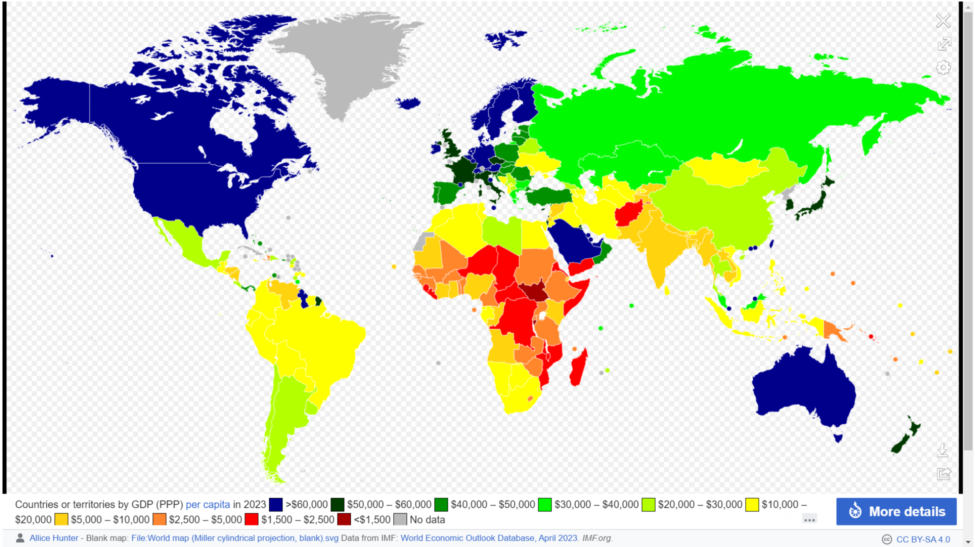
Self-reported life satisfaction vs. GDP per capita, 2019

Self-reported life satisfaction is measured on a scale ranging from 0-10, where 10 is the highest possible life satisfaction. GDP per capita is adjusted for inflation and differences in the cost of living between countries.



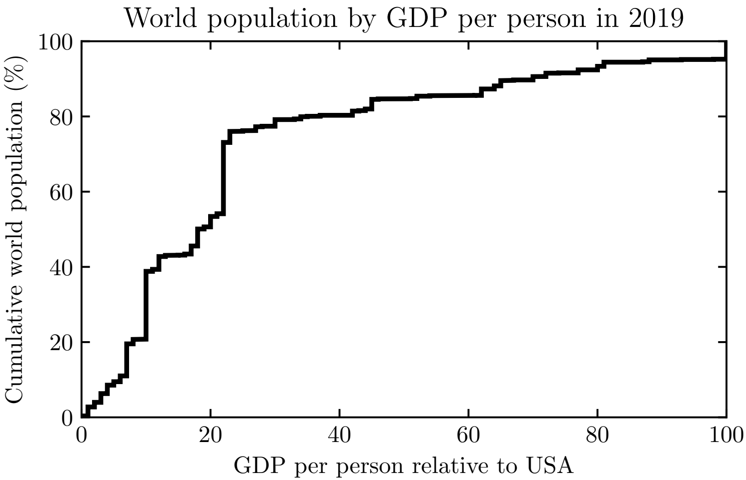
Source: World Happiness Report (2023), Data compiled from multiple sources by World Bank
Note: GDP per capita is expressed in international-$ at 2017 prices.

There is enormous variation in GDP per person across economies

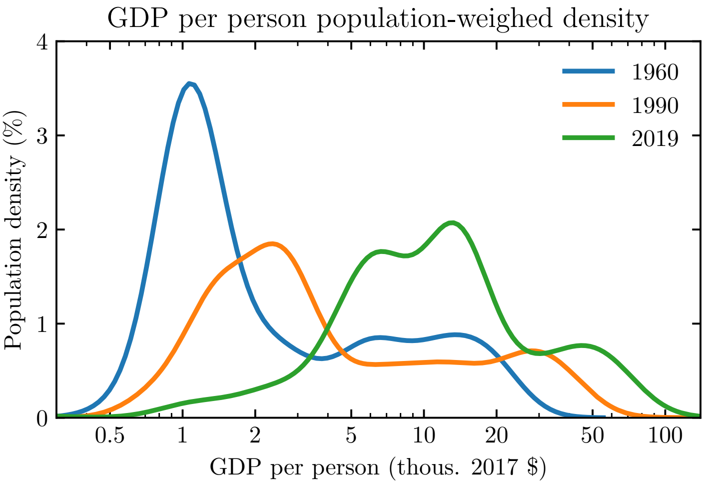


Wikipedia, IMF

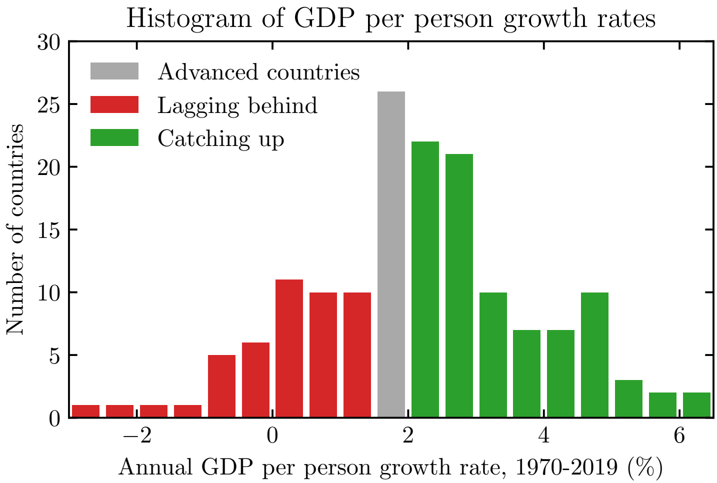
There is enormous variation in GDP per person across economies



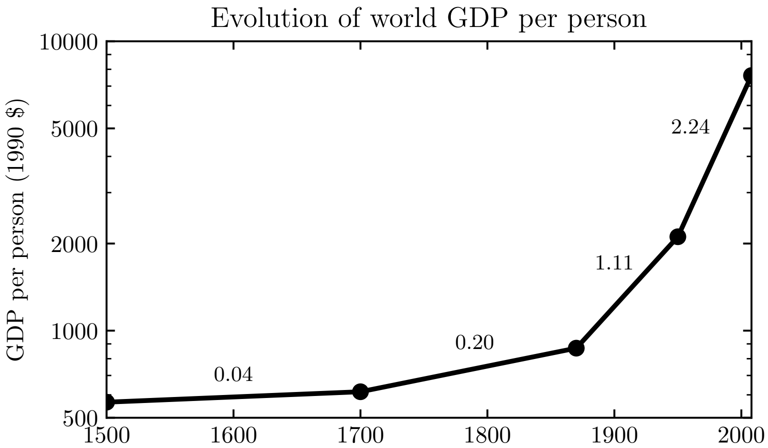
Less variation now than in the previous decades



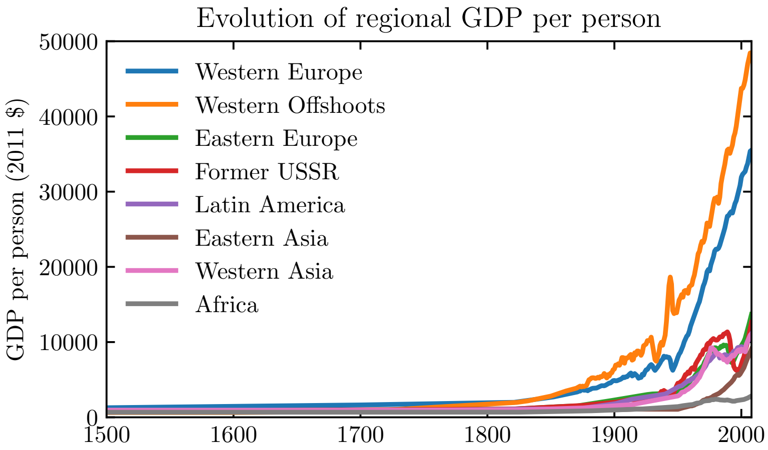
Rates of economic growth vary substantially across countries



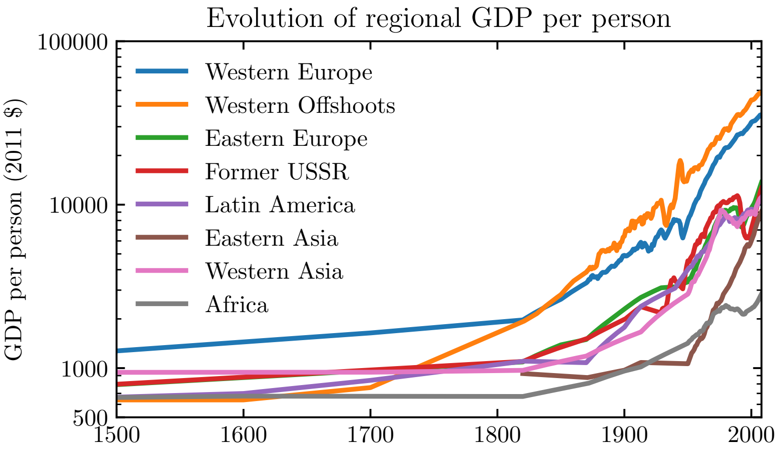
Economic growth is a "recent" phenomenon



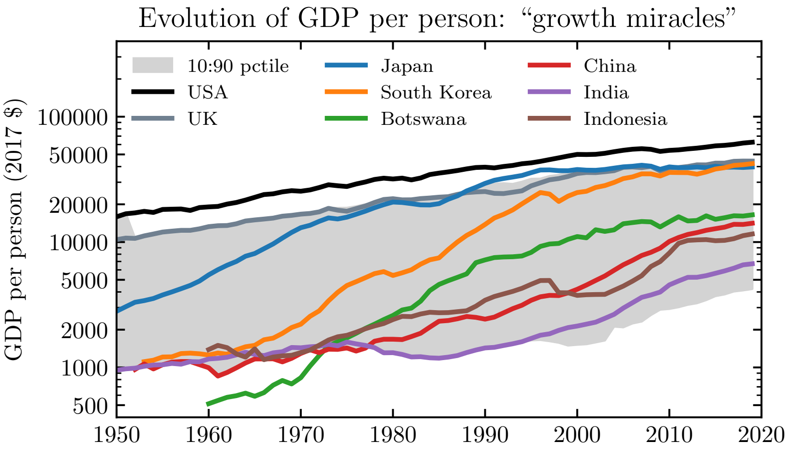
Until the 19th century everyone was similarly poor



Growth took off with different timing across world regions



Countries can go from being “poor” to being “rich”

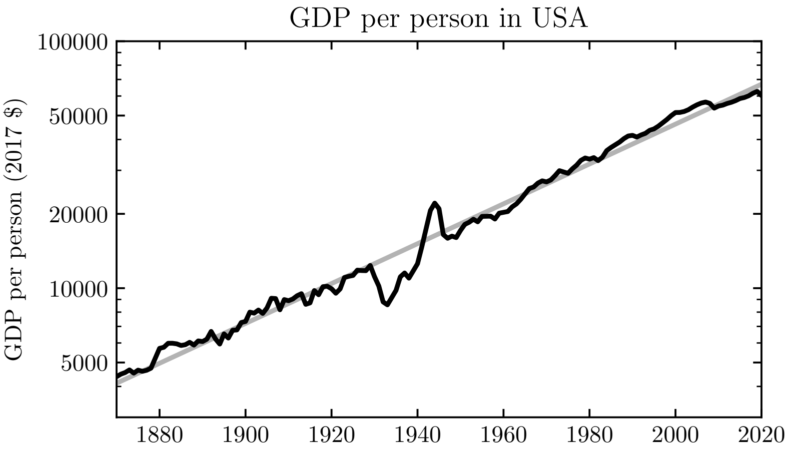


Kaldor's stylized facts

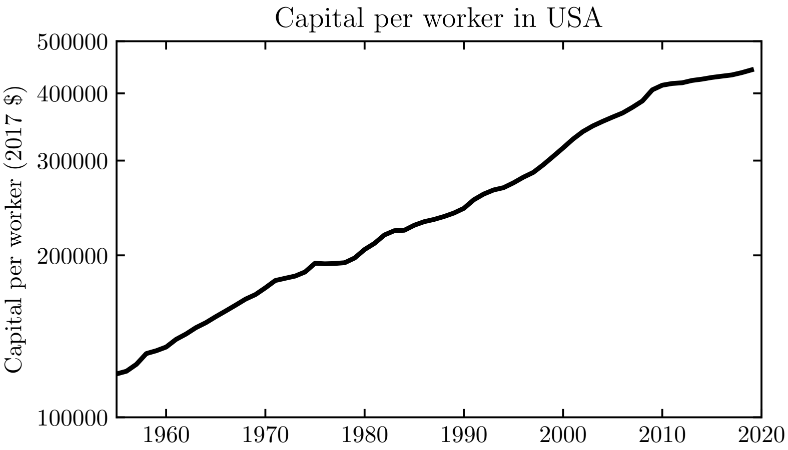
Kaldor (1957, 1961): **In the USA (and other developed countries):**

1. GDP per person sustainably grows at positive rate
2. Physical capital per worker grows over time
3. The rate of return to capital is not trending
4. The ratio of physical capital to output is nearly constant
5. The shares of labor and physical capital in national income are nearly constant
6. Real wages grow over time

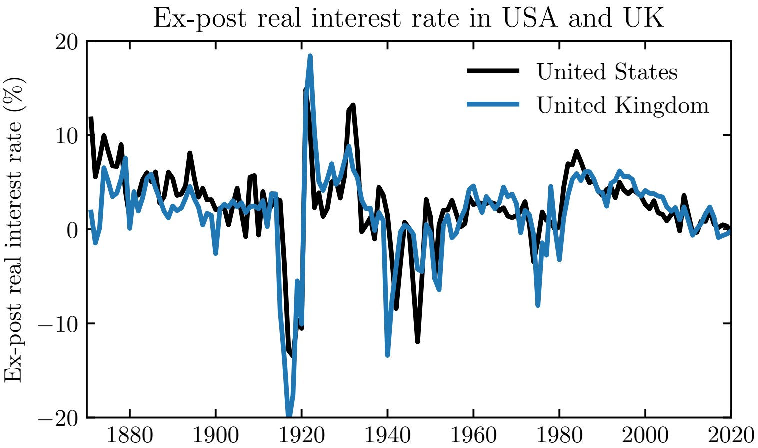
K1: GDP per person sustainably grows at positive rate



K2: Physical capital per worker grows over time



K3: The rate of return to capital is not trending



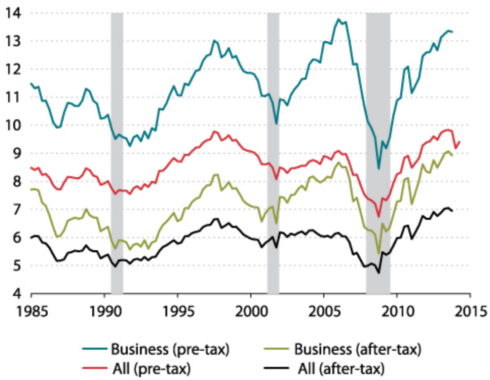
K3: The rate of return to capital is not trending

DeLong (2015)

K3: The rate of return to capital is not trending

Figure 2

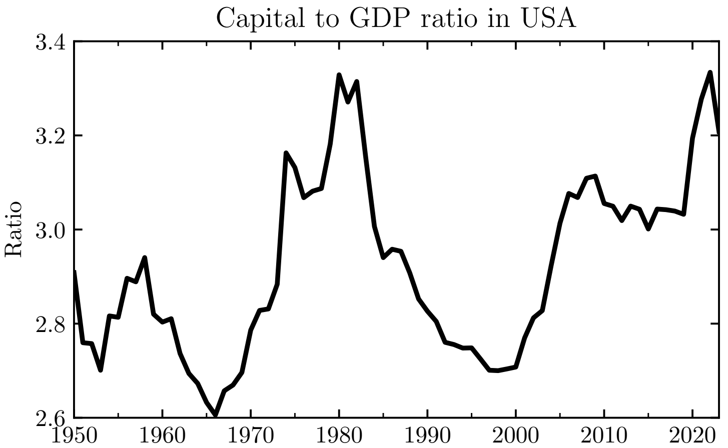
Real Returns on Capital (percent)



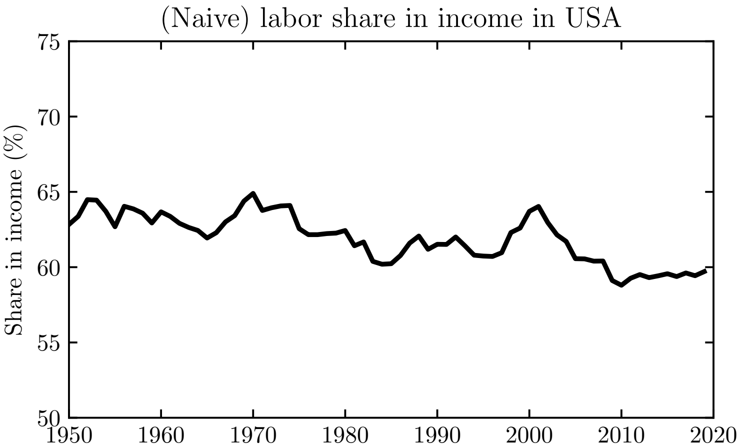
SOURCE: Authors' calculations; for details, see Gomme, Ravikumar, and Rupert (2011).

Gomme, Ravikumar and Rupert (2015) *Secular Stagnation and Returns on Capital*

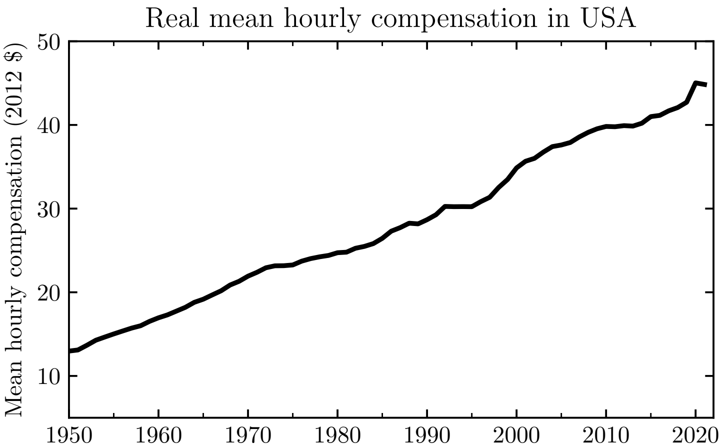
K4: The ratio of physical capital to output is nearly constant



K5: The labor share of national income is nearly constant



K6: Real wages grow over time



Explaining growth

We want to explain:

- Why some countries are “poor” and other “rich”?
- Why some countries that were previously “poor” became “rich”?
- Why not all “poor” countries catch up to “rich” countries?
- Why do “rich” countries still grow?

Solow-Swan model

Developed by Robert Solow (1956) and Trevor Swan (1956)

Growth in income per capita comes from two sources:

- Capital accumulation (endogenous)
- Improvements in technology (exogenous)

But capital accumulation alone cannot sustain growth in the absence of technology improvements

Does not explain "deep" sources of economic growth: Proximate vs fundamental causes

Departure point for growth theory

Simplifications and assumptions

Closed economy ($NX = 0$)

No government ($G = T = 0$)

Single, homogenous final good with its price normalized to 1 in each period

$\hookrightarrow$ all variables and prices are expressed in real terms

Two types of representative agents:

- Firms
- Households

Production

Real GDP is produced according to a neoclassical production function:

$$Y_t = F\left(K_t,\ A_t L_t\right)$$

where Y is real GDP, F is a neoclassical production function, K is capital stock, A is the technology level and L is the number of workers

Technology grows at a rate $g > 0$: $\quad A_{t+1} = (1+g)\, A_t$

Very often we use a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

Like other neoclassical production functions, it exhibits constant returns to scale
$\hookrightarrow$ doubling inputs K and L doubles the amount produced:

$$(zK_t)^{\alpha} \left(A_t \cdot zL_t\right)^{1-\alpha} = z^{\alpha} z^{1-\alpha} K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha} = zY_t$$

Firms

Perfectly competitive firms maximize their profit:

$$\max_{K_t,\, L_t} \quad D_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} - \left(r_t + \delta\right) K_t - w_t L_t$$

First order conditions:

$$\begin{aligned} K_t \quad &: \quad \alpha K_t^{\alpha-1}\left(A_t L_t\right)^{1-\alpha} - \left(r_t + \delta\right) = 0 \quad &\rightarrow \quad r_t = \alpha \frac{Y_t}{K_t} - \delta \\ L_t \quad &: \quad \left(1-\alpha\right) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} - w_t = 0 \quad &\rightarrow \quad w_t = \left(1-\alpha\right)\frac{Y_t}{L_t} \end{aligned}$$

Total factor payments are equal to GDP:

$$\left(r_t + \delta\right) K_t + w_t L_t = \alpha \frac{Y_t}{K_t} K_t + \left(1-\alpha\right)\frac{Y_t}{L_t} L_t = \alpha Y_t + \left(1-\alpha\right) Y_t = Y_t$$

Factor shares

Calculate the fraction of GDP that is paid to each factor:

$$\frac{w_t L_t}{Y_t} = \frac{(1-\alpha)\,\dfrac{Y_t}{L_t} \cdot L_t}{Y_t} = (1-\alpha) \quad \text{and} \quad \frac{(r_t+\delta)\,K_t}{Y_t} = \frac{\alpha\dfrac{Y_t}{K_t} \cdot K_t}{Y_t} = \alpha$$

Cobb-Douglas function implies constant shares of labor and physical capital in income

Confronting with the US data, we can obtain $\alpha \approx \frac{1}{3}$ and $(1-\alpha) \approx \frac{2}{3}$

Households

Own factors of production directly and earn income from renting them to firms

Number of workers is proportional to total population: $L_t \propto N_t$

Both population and number of workers change at rate n:

$$\frac{L_{t+1}}{L_t} = \frac{N_{t+1}}{N_t} = 1 + n$$

Capital accumulates from investment I_t and depreciates at rate δ:

$$K_{t+1} = I_t + (1 - \delta)\, K_t$$

Income of households is consumed or saved / invested:

$$Y_t = w_t L_t + (r_t + \delta)\, K_t = C_t + S_t = C_t + I_t$$

Households **don't optimize**, invest / save a constant fraction s of income:

$$I_t = sY_t \quad \text{and} \quad C_t = (1 - s)\, Y_t$$

GDP per worker and per effective units of labor

Usually we are most interested in GDP per worker (or per person), y:

$$y_t \equiv \frac{Y_t}{L_t} = \frac{K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}}{L_t^{\alpha} L_t^{1-\alpha}} = A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \equiv A_t \hat{k}_t^{\alpha}$$

where $\hat{k}$ is capital K divided per effective unit of labor (AL)

GDP per worker increases due to capital accumulation and improvements in technology

Production function exhibits diminishing marginal returns to capital
$\hookrightarrow$ GDP per worker increases with $\hat{k}$, but the size of the increase falls with $\hat{k}$

It is also useful to define GDP per effective units of labor $\hat{y}$:

$$\hat{y}_t = \frac{y_t}{A_t} = \hat{k}_t^{\alpha}$$

Capital accumulation

Capital accumulates according to:

$$K_{t+1} = sY_t + (1-\delta) K_t$$

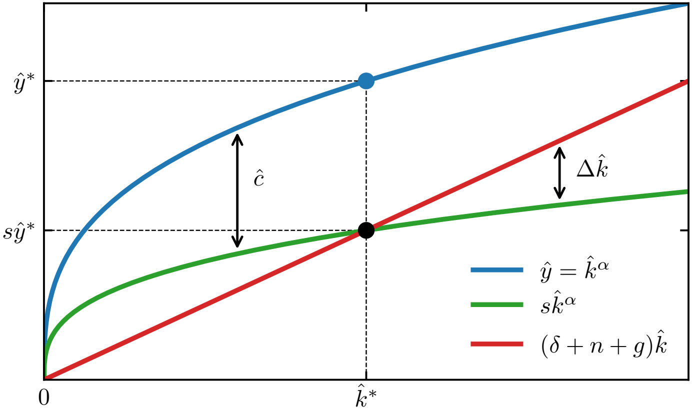
And capital per effective labor according to:

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta) K_t \quad \Big| \quad : A_t L_t \\ \frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}}{A_t} \frac{L_{t+1}}{L_t} &= s\frac{Y_t}{A_t L_t} + (1-\delta) \frac{K_t}{A_t L_t} \\ \hat{k}_{t+1} (1+g)(1+n) &= s\hat{y}_t + (1-\delta) \hat{k}_t \end{aligned}$$

Fundamental equation of the Solow-Swan model:

$$\hat{k}_{t+1} = \frac{s\hat{k}_t^{\alpha} + (1-\delta) \hat{k}_t}{(1+g)(1+n)}$$

Capital accumulation



Capital accumulation

The change in capital per effective labor equals:

$$\Delta\hat{k}_{t+1} \equiv \hat{k}_{t+1} - \hat{k}_t = \frac{s\hat{k}_t^{\alpha} + (1-\delta)\,\hat{k}_t}{(1+g)\,(1+n)} - \frac{(1+n+g+ng)\,\hat{k}_t}{(1+g)\,(1+n)}$$

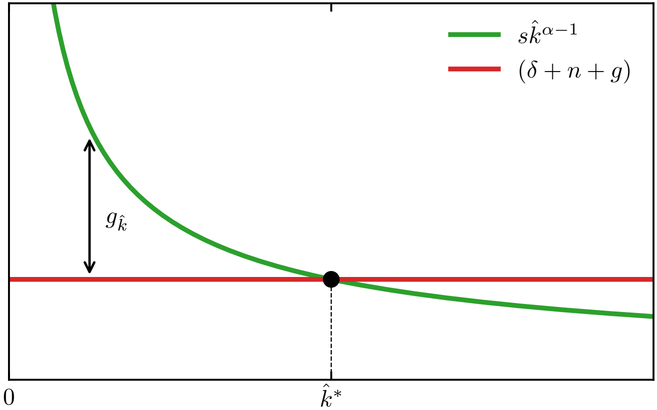
$$= \frac{s\hat{k}_t^{\alpha} - (\delta+n+g+ng)\,\hat{k}_t}{(1+g)\,(1+n)} \simeq s\hat{k}_t^{\alpha} - (\delta+n+g)\,\hat{k}_t$$

where the symbol $\simeq$ means an accurate representation in continuous time

The growth rate of capital per effective labor equals:

$$g_{\hat{k}} \equiv \frac{\Delta\hat{k}_{t+1}}{\hat{k}_t} = \frac{s\hat{k}_t^{\alpha-1} - (\delta+n+g+ng)}{(1+g)\,(1+n)} \simeq s\hat{k}_t^{\alpha-1} - (\delta+n+g)$$

Capital accumulation



Balanced growth path (steady state)

Variables per effective labor converge to their steady state values

If $\hat{k}_{t+1} = \hat{k}_t = \hat{k}^*$ then:

$$\hat{k}^* \left(1 + n + g + ng\right) = s(\hat{k}^*)^\alpha + \left(1 - \delta\right) \hat{k}^*$$
$$\hat{k}^* \left(\delta + n + g + ng\right) = s(\hat{k}^*)^\alpha$$
$$(\hat{k}^*)^{1-\alpha} = \frac{s}{\delta + n + g + ng}$$

$$\hat{k}^* = \left(\frac{s}{\delta + n + g + ng}\right)^{1/(1-\alpha)} \simeq \left(\frac{s}{\delta + n + g}\right)^{1/(1-\alpha)}$$
$$\hat{y}^* = \left(\frac{s}{\delta + n + g + ng}\right)^{\alpha/(1-\alpha)} \simeq \left(\frac{s}{\delta + n + g}\right)^{\alpha/(1-\alpha)}$$
$$\hat{c}^* = \left(1 - s\right) \hat{y}^*$$

Balanced growth path (steady state)

Along the balanced growth path (BGP) variables per worker grow together with increases in technology:

$$y_t^* = A_t \hat{y}^* \quad \rightarrow \quad g_y^* \simeq g_A + g_{\hat{y}}^* = g + 0 = g$$

And aggregate variables like aggregate capital and GDP grow at the sum of rates of increase in population and technology:

$$Y_t^* = A_t L_t \hat{y}^* \quad \rightarrow \quad g_Y^* \simeq g_A + g_L + g_{\hat{y}}^* = g + n$$

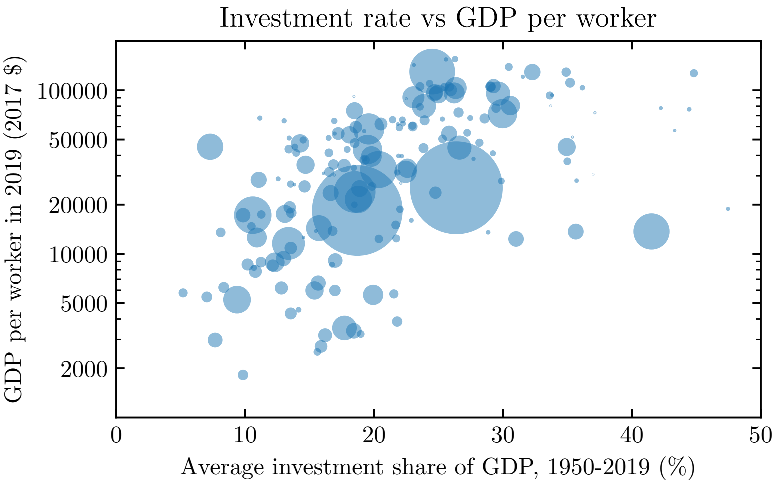
Comparative statics

Solow-Swan model predicts that the BGP level of GDP per worker:

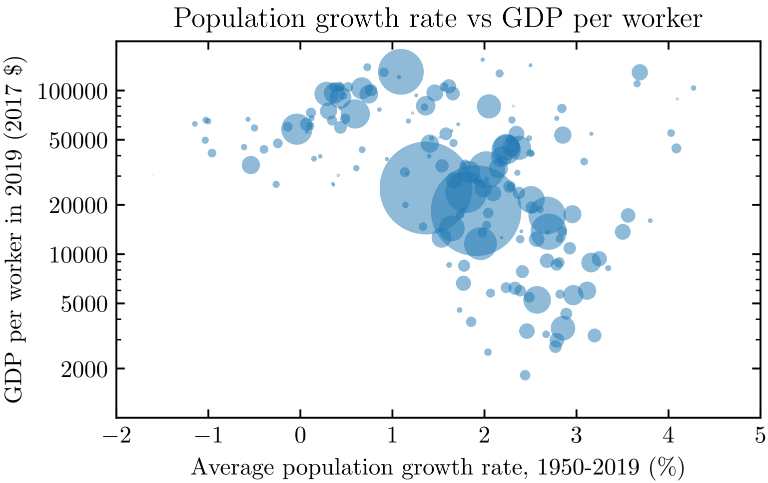
$$y_t^* \simeq A_t \left(\frac{s}{\delta + n + g}\right)^{\alpha/(1-\alpha)}$$

is higher in countries with higher technology level A,
is higher in countries with higher investment share of GDP s
and lower in countries with higher population growth rate n

Investment share of GDP s vs GDP per worker y



Population growth rate n vs GDP per worker y



Transitional dynamics

We are also interested in the growth rates of GDP per worker outside the BGP

Start with growth rates of GDP per effective labor:

$$g_{\hat{y}} \simeq \ln\left(\hat{y}_{t+1}/\hat{y}_t\right) = \ln(\hat{k}_{t+1}^{\alpha}/\hat{k}_t^{\alpha}) = \alpha \ln(\hat{k}_{t+1}/\hat{k}_t) \simeq \alpha g_{\hat{k}}$$

$$g_{\hat{y}} \simeq \alpha \left[s\hat{k}_t^{\alpha-1} - (\delta + n + g)\right]$$

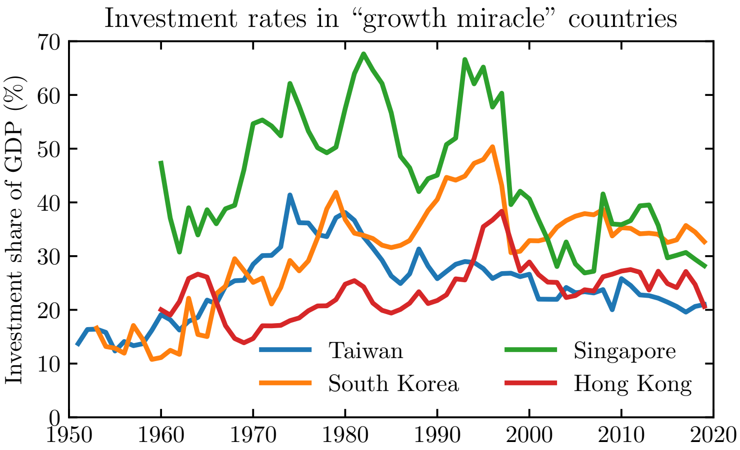
To obtain growth rate of GDP per worker, add the growth rate of technology g:

$$\begin{aligned} g_y &\simeq \alpha \left[s\hat{k}_t^{\alpha-1} - (\delta + n + g)\right] + g \\ &\simeq \alpha \left[s\hat{k}_t^{\alpha-1} - (\delta + n)\right] + (1 - \alpha)\, g \end{aligned}$$

An increase in s or a decrease in n **temporarily** increases the growth rate of y

Note that even if higher g decreases $\hat{k}^*$, it increases the rate of growth of y

Investment share of GDP s in “growth miracle” countries

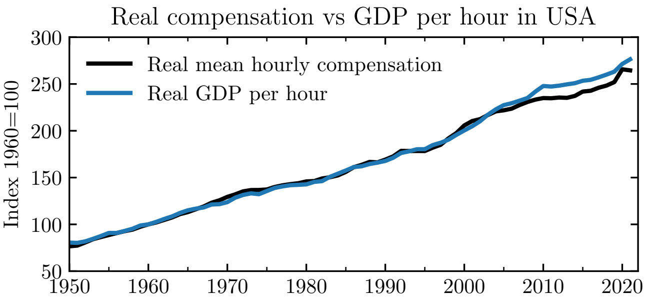


Factor payments once again

Using $\hat{k}^*$ as capital per effective labor along the BGP, let us revisit factor prices:

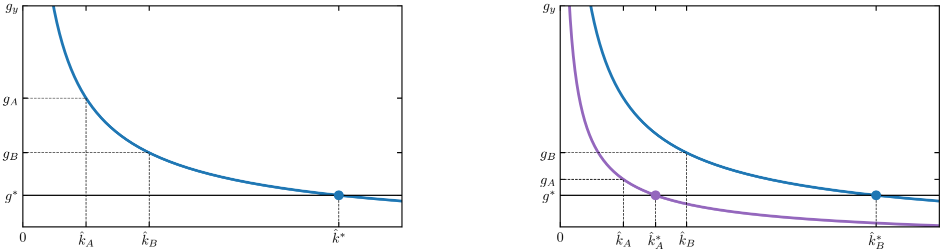
$$r_t^* = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - \delta = \alpha(\hat{k}^*)^{\alpha-1} - \delta$$
$$w_t^* = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} = (1-\alpha) A_t(\hat{k}^*)^{\alpha}$$

The model predicts that along the BGP the interest rates are constant while hourly wages grow at the same rate as GDP per hour:



Convergence

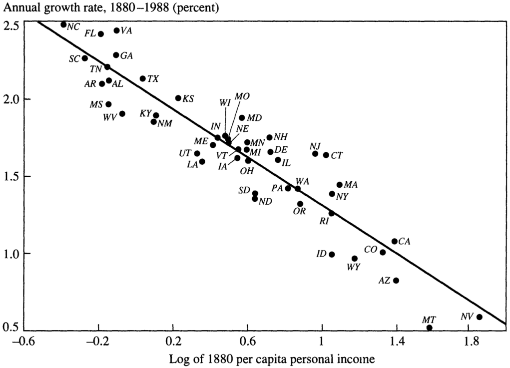
Solow-Swan model predicts that if countries have access to the same technology and share the same steady state, then ones that are poorer should grow faster:



Convergence: USA

We can observe convergence across individual states in USA:

Figure 1. Convergence of Personal Income across U.S. States: 1880 Income and Income Growth from 1880 to 1988

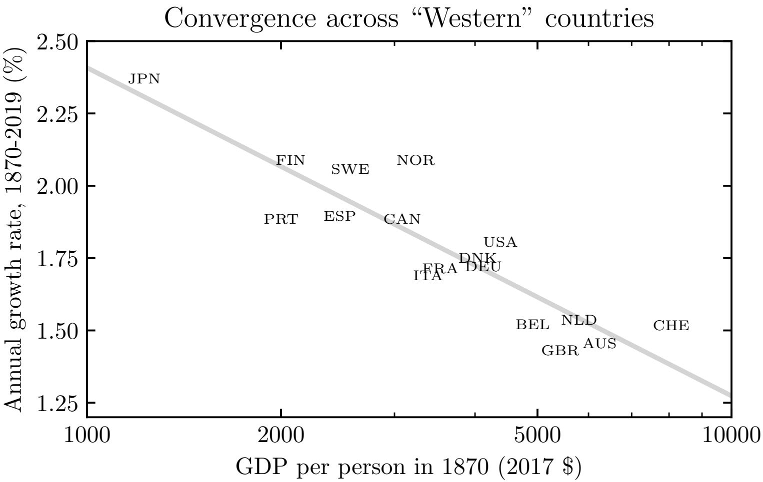


Sources: Bureau of Economic Analysis (1984), Easterlin (1960a, 1960b), and *Survey of Current Business*, various issues. The postal abbreviation for each state is used to plot the figure. Oklahoma, Alaska, and Hawaii are excluded from the analysis.

Barro and Sala-i-Martin (1991) *Convergence across States and Regions*

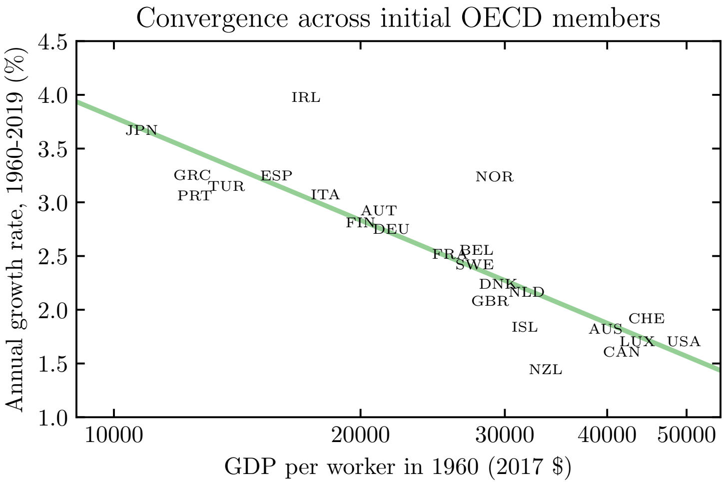
Convergence: “West”

We can observe convergence across “Western” countries (+ Japan):



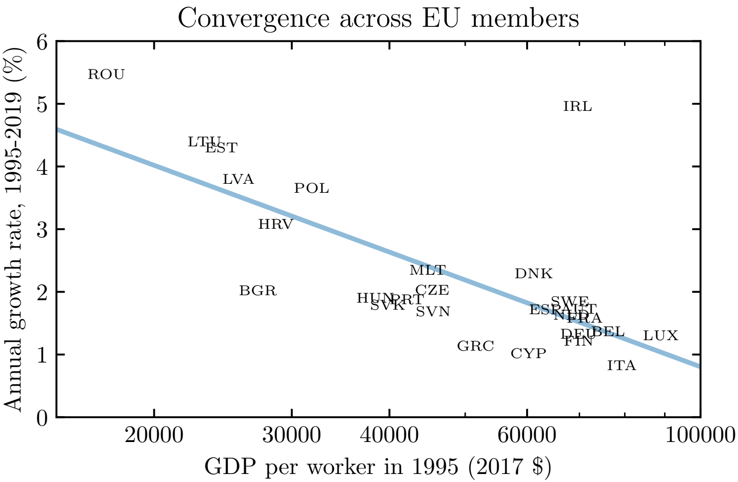
Convergence: OECD

We can observe convergence across initial OECD members:



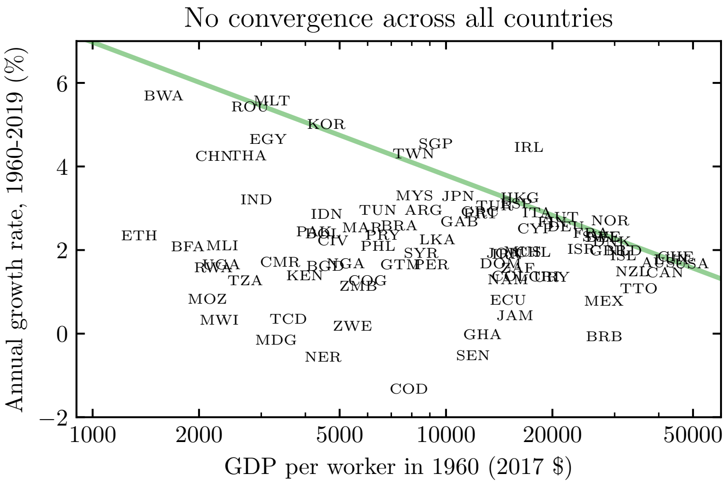
Convergence: EU

We can observe convergence across EU members:

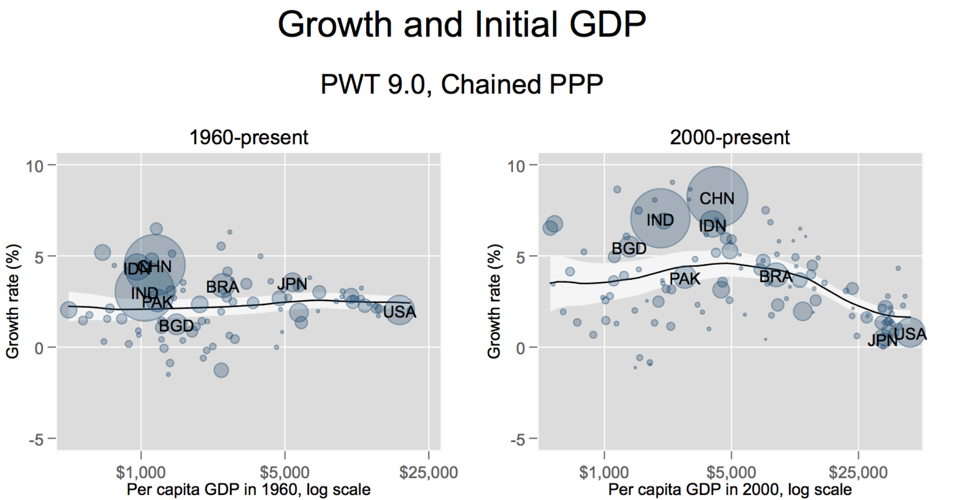


Convergence: conditional/club, but not absolute

In general it is not true that poorer countries grow faster:



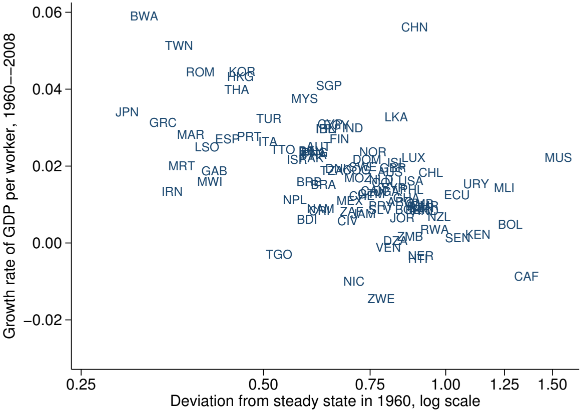
… although trends may have changed recently



Patel et al. (2018) *Everything You Know about Cross-Country Convergence Is (Now) Wrong*

Conditional convergence

Countries grow faster the further away they are from their own BGP / steady state:



Jones and Vollrath (2013) *Introduction to Economic Growth*

Speed of convergence

Relationship between the distance from BGP and the current rate of growth:

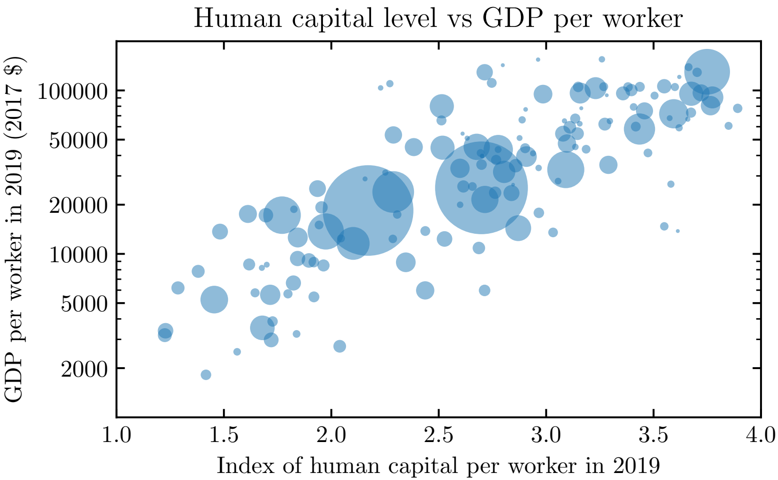
$$g_y \simeq \underbrace{(1-\alpha)(\delta+n+g)}_{\lambda}(\ln y_t^* - \ln y_t)$$

Econometric studies find that $\lambda \approx 0.02$, meaning that it takes 35 years to close half of the gap between the current income and the BGP

Given sensible parameter values: $\alpha = 0.33$, $\delta = 0.05$, $n = 0.01$, $g = 0.02$, the model generates $\lambda \approx 0.053$, implying that it would take about 13 years to close half of the gap

Adding human capital allows the model to assign lower weight to raw labor and be consistent with slow convergence

Human capital per capita h vs real GDP per worker y



Human capital augmented Solow model

Mankiw, Romer and Weil (1992)

The production function that accounts for human capital H:

$$Y_t = K_t^{\alpha} H_t^{\beta} \left(A_t L_t\right)^{1-\alpha-\beta}$$

Physical and human capital accumulation:

$$K_{t+1} = s_k Y_t + (1-\delta) K_t$$
$$H_{t+1} = s_h Y_t + (1-\delta) H_t$$

GDP per worker along the BGP:

$$y_t^* = A_t \left(\frac{s_k^{\frac{\alpha}{\alpha+\beta}} s_h^{\frac{\beta}{\alpha+\beta}}}{\delta + n + g} \right)^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

Human capital augmented Solow model

Econometric model:

$$\ln y_{it} = gt + \frac{\alpha}{1-\alpha-\beta} \ln s_{k,i} + \frac{\beta}{1-\alpha-\beta} \ln s_{h,i} - \frac{\alpha+\beta}{1-\alpha-\beta} \ln(\delta + n_i + g) + \epsilon_i$$

Restricted regression (to easily recover α and β):

$$\ln y_{it} = gt + \frac{\alpha}{1-\alpha-\beta} [\ln s_{k,i} - \ln(\delta + n_i + g)] + \frac{\beta}{1-\alpha-\beta} [\ln s_{h,i} - \ln(\delta + n_i + g)] + \epsilon_i$$

Human capital augmented Solow model: empirical fit

Dependent variable: log GDP per working-age person in 1985

Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\bar{R}^2$	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(I/GDP) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(SCHOOL) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
$\bar{R}^2$	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
p-value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)
Implied β	0.28	0.30	0.37
	(0.03)	(0.04)	(0.12)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Mankiw, Romer and Weil (1992) *A Contribution to the Empirics of Economic Growth*, Table II

Human capital augmented Solow model: convergence rate

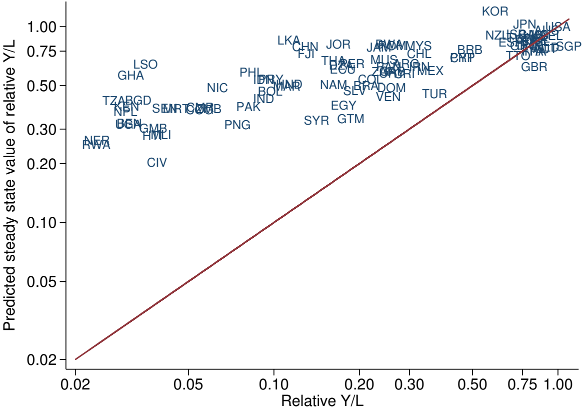
Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	2.46	3.09	3.55
	(0.48)	(0.53)	(0.63)
ln(Y60)	−0.299	−0.372	−0.402
	(0.061)	(0.067)	(0.069)
ln(I/GDP) − ln($n + g + \delta$)	0.500	0.506	0.396
	(0.082)	(0.095)	(0.152)
ln(SCHOOL) − ln($n + g + \delta$)	0.238	0.266	0.236
	(0.060)	(0.080)	(0.141)
$\overline{R}^2$	0.46	0.44	0.66
s.e.e.	0.33	0.30	0.15
Test of restriction:			
p-value	0.40	0.42	0.47
Implied λ	0.0142	0.0186	0.0206
	(0.0019)	(0.0019)	(0.0020)
Implied α	0.48	0.44	0.38
	(0.07)	(0.07)	(0.13)
Implied β	0.23	0.23	0.23
	(0.05)	(0.06)	(0.11)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. ($g + \delta$) is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Mankiw, Romer and Weil (1992) *A Contribution to the Empirics of Economic Growth*, Table VI

Fit of human capital augmented Solow model

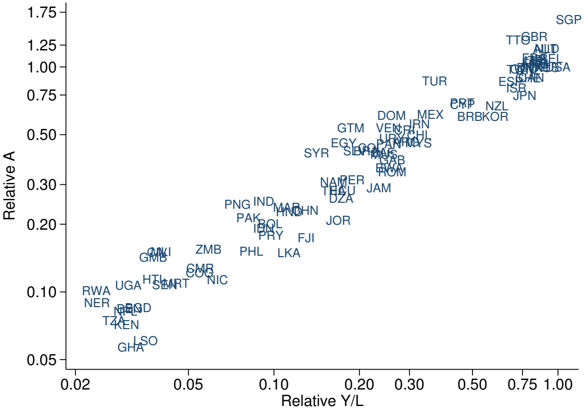
Suggests that poor countries “should” be richer:



Jones and Vollrath (2013) *Introduction to Economic Growth*

Solow residual: accounting for technology differences

There are also significant differences in technology across countries:



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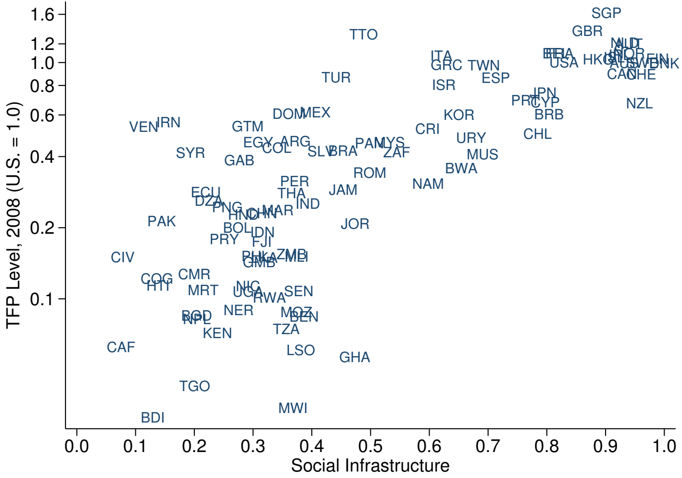
GDP per worker differences decomposition

Country	Y/L	Contribution from $(K/Y)^{\alpha/(1-\alpha)}$	H/L	A
United States	1.000	1.000	1.000	1.000
Canada	0.941	1.002	0.908	1.034
Italy	0.834	1.063	0.650	1.207
West Germany	0.818	1.118	0.802	0.912
France	0.818	1.091	0.666	1.126
United Kingdom	0.727	0.891	0.808	1.011
Hong Kong	0.608	0.741	0.735	1.115
Singapore	0.606	1.031	0.545	1.078
Japan	0.587	1.119	0.797	0.658
Mexico	0.433	0.868	0.538	0.926
Argentina	0.418	0.953	0.676	0.648
U.S.S.R.	0.417	1.231	0.724	0.468
India	0.086	0.709	0.454	0.267
China	0.060	0.891	0.632	0.106
Kenya	0.056	0.747	0.457	0.165
Zaire	0.033	0.499	0.408	0.160
Average, 127 countries:	0.296	0.853	0.565	0.516
Standard deviation:	0.268	0.234	0.168	0.325
Correlation with Y/L (logs)	1.000	0.624	0.798	0.889
Correlation with A (logs)	0.889	0.248	0.522	1.000

Hall and Jones (1999)

Why Do Some Countries Produce So Much More Output Per Worker Than Others?

“Productivity” vs social infrastructure index



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Takeaway

Long run growth stems from improvements in technology

Countries can achieve higher balanced growth paths
if they accumulate more physical and human capital per worker

But even more important than factor accumulation is technology adoption

Did not touch on “deep” causes of growth
– we treated many choice variables as exogenous parameters

Takeaway


Always has been
Wait, it's all PRODUCTIVITY ?
LONG-RUN
ECONOMIC GROWTH
imgflip.com