Marcin Bielecki, Advanced Macroeconomics QF, Fall 2024
Homework 3 – deadline: 6th January, 23:59

Problem 1 (2 points)

Consider the following utility maximization problem, where the households' budget constraint is replaced by the resource constraint. This is the "social planner's" version of the problem:

$$\begin{aligned} \max_{\{c_t,\, k_{t+1}\}_{t=0}^{\infty}} \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t + (1+n)\, k_{t+1} = A k_t^{\alpha} + (1-\delta)\, k_t \qquad \text{for all } t = 0,1,2,\ldots,\infty \\ & k_0 > 0 \end{aligned}$$

where $A > 0$ is the level of technology of the economy.

(a) Find the First Order Conditions for the choice of c_t and k_{t+1}. Obtain the Euler equation (use the fact that $\beta = 1/(1+\rho)$). Verify that it's identical to the Euler equation obtained in the lecture (when $A = 1$).

(b) Compute the steady state values of k^* and c^*. Verify that they are identical to the levels obtained in the lecture (when $A = 1$).

(c) How do the results from (b) change in response to changes in A, δ and ρ (hint: establish that $\partial c^* / \partial k^* > 0$)?

(d) Consider now the environment of constant technological progress (where $A_{t+1} = (1+g)\, A_t$). The social planner's problem then becomes

$$\begin{aligned} \max_{\{\hat{c}_t,\, \hat{k}_{t+1}\}_{t=0}^{\infty}} \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{(A_t \hat{c}_t)^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & \hat{c}_t + (1+n)(1+g)\, \hat{k}_{t+1} = \hat{k}_t^{\alpha} + (1-\delta)\, \hat{k}_t \qquad \text{for all } t = 0,1,2,\ldots,\infty \\ & \hat{k}_0 > 0 \end{aligned}$$

Find the First Order Conditions for the choice of $\hat{c}_t$ and $\hat{k}_{t+1}$. Obtain the Euler equation. Verify that it's identical to the Euler equation obtained in the lecture.

Problem 2 (2 points)

Consider a Ramsey economy where for simplicity we assume $n = g = 0$ and $N = A = 1$.
The representative household solves the following utility maximization problem:

$$\begin{aligned} \max_{\{c_t,\, a_{t+1}\}_{t=0}^{\infty}} \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t + a_{t+1} = w_t + (1+r_t)\, a_t + v_t \qquad \text{for all } t = 0,1,2,\ldots,\infty \\ & a_0 > 0 \end{aligned}$$

where v is the lump-sum transfer from the government to households.

The representative firm solves the following profit maximization problem:

$$\begin{aligned} \max_{Y_t,\, K_t,\, L_t} \quad & D_t = (1-\tau^y)\, Y_t - (r_t + \delta)\, K_t - w_t L_t \\ \text{subject to} \quad & Y_t = K_t^{\alpha} L_t^{1-\alpha} \end{aligned}$$

where τ^y is the firm revenue tax, which is in equilibrium equivalent to taxing households' income at the same linear rate, regardless of the source of income (i.e. $\tau^w = \tau^r = \tau^y$).

(a) Derive the First Order Conditions of the household. Obtain the Euler equation.

(b) Derive the First Order Conditions of the firm. Express prices of factors of production as functions of capital per worker k.

(c) Write down the government budget constraint. Using the assumptions of closed economy and balanced government budget, find the conditions for general equilibrium in this economy.

(d) Find the steady state level of capital per worker k^* and consumption per worker c^* in this economy. How they depend on the tax rate τ^y?

Problem 3 (2 points)

In the basic increasing product variety (horizontal innovation) model of endogenous growth we assumed that once invention is made, the monopolist can reap the monopolistic rents forever. This time we will assume that at invention the inventor is granted a patent that guarantees exclusive rightis to produce and sell a given variety, but after the patent expires the good will be produced perfectly competitively. It is a bit more convenient to model patent duration not as a fixed length period, but by assuming that the patent expires with probability $z \in (0,1)$ per period.

(a) Calculate the expected patent duration

(b) Derive the formula for firm value just after the invention is made (and production can start from next period onwards) under the assumption that when the patent stochastically expires, the firm loses all current and future profits

(c) What is the modified formula for the growth rate along the balanced growth path? How does it depend on the probability of patent expiry?

(d) Assuming $L = 1$ for simplicity, calculate the level of final goods production along the balanced growth path, assuming that the ratio of competitively produced good types to all invented good types is $M^c/M = z/(z+g)$

(e) Calculate the level of final goods production under the assumption that $z = 0$ (fully monopolistic economy), and holding the number of all existing good types identical to (d)

(f) Make an argument why there exists some strictly positive z that maximizes household welfare.