Marcin Bielecki, Advanced Macroeconomics QF, Fall 2024
Homework 2 – deadline: 9th December, 23:59

Problem 1 (2 points)

Consider a two-period problem of choosing the level of capital stock by two firms: A and B. For simplicity we will assume that production requires capital only. We will follow the convention that in time period t the level of capital K_t is predetermined, but the firm can choose its future level of capital, K_{t+1}.

Firm A does not own its capital stock, but instead rents it at price r^K, the rental cost of capital. Future profit flows are discounted with the real interest rate r. The problem of maximizing the value of firm A is given by:

$$\max_{K_{t+1}} \quad V^A = F\left(K_t\right) - r^K K_t + \frac{1}{1+r}\left[F\left(K_{t+1}\right) - r^K K_{t+1}\right]$$

Firm B owns its capital stock, and can adjust its level via investment. After production will take place in period $t+1$, the firm will sell the undepreciated capital. The problem of maximizing the value of firm B is given by:

$$\begin{aligned} \max_{I_t,\, K_{t+1}} \quad & V^B = F\left(K_t\right) - I_t + \frac{1}{1+r}\left[F\left(K_{t+1}\right) + (1-\delta) K_{t+1}\right] \\ \text{subject to} \quad & K_{t+1} = (1-\delta) K_t + I_t \end{aligned}$$

(a) Derive the first order condition of firm A.

(b) Derive the first order conditions of firm B.

(c) What condition has to be satisfied for both firms to choose the same level of K_{t+1}?

(d) Imagine you are the owner of firm C, which rents capital goods to firm A. What would be the maximal level of r^K that you could charge this firm?

(e) What would happen if you demanded higher rental rate than one found in (d)?

Problem 2 (2 points)

Consider the following problem of a manager maximizing the value of the firm:

$$\begin{aligned} \max_{\left\{L_t,\, I_t^n,\, K_{t+1}\right\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}\left[(1-\tau)\left(K_t^\alpha L_t^{1-\alpha} - w_t L_t - \delta K_t - I_t^n\right) - \frac{\chi}{2}\frac{\left(I_t^n\right)^2}{K_t}\right] \\ \text{subject to} \quad & K_{t+1} = I_t^n + K_t \qquad \forall t = 0, 1, \ldots, \infty \end{aligned}$$

where τ is a tax levied on firm's profits, K is firm's capital stock, L are firm's employees, $\alpha \in (0,1)$ is output elasticity w.r.t. capital, I^n is net investment and $\delta \in (0,1)$ stands for capital depreciation. Parameter χ describes the magnitude of capital installation costs. Note that the tax code does not treat installation costs as tax deductible.

(a) Write down the problem in the Lagrangian form and derive the first order conditions.

(b) Find the steady state level of q (the Lagrange multiplier). Is it equal to 1? *Hint: by definition in the steady state firm's capital stock is constant and net investment is 0.*

(c) Find the desired level of firm's capital stock per employee, $k \equiv K/L$, treating interest rate r as given.

(d) Suppose the tax on firm's profits is reduced. What happens with the firm's investment if its level of capital stock per employee was at the level from (c) prior to the tax change?

(e) What happens with the firm's investment if its level of capital stock per employee was lower than the level from (c) prior to the tax change?

Problem 3 (2 points)

Robert Solow in his 1956 article "A Contribution to the Theory of Economic Growth" considered the behavior of economy when output was produced according to various production functions. One of them was the following Constant Elasticity of Substitution (CES) function:

$$Y_t = \left[aK_t^{\rho} + (1-a)\,L_t^{\rho}\right]^{1/\rho}$$

where $a \in (0,1)$, $\rho \leq 1$ and for simplicity technology growth rate is set to 0. The CES function reduces to the Cobb-Douglas function in the limit of $\rho \to 0$.

Saving and investment behaviour of the economy are described respectively as:

$$i_t = s \cdot y_t$$
$$(1+n)\,k_{t+1} = i_t + (1-\delta)\,k_t$$

where lower case letters i_t, y_t, k_t denote per worker quantities, n denotes population growth, and δ denotes the depreciation rate.

(a) Transform the production function into per worker form (divide it by L_t).

(b) Find the steady state value for capital per worker k^*.

(c) Show how an increase in the saving rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).

(d) Show how an increase in the population growth rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).