Marcin Bielecki, Advanced Macroeconomics QF, Fall 2024
Homework 1, deadline: 3rd November, 23:59

Problem 1 (2 points)

Consider the following two-period utility maximization problem. This utility function belongs to the Constant Relative Risk Aversion (CRRA) class of functions that will be often used throughout our course.[1] An agent lives for two periods and in both receives some positive income. Solve for the optimal consumption values.

$$\max_{c_1,\, c_2,\, a} \quad U = \frac{c_1^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1-\sigma}$$
$$\text{subject to} \quad c_1 + a = y_1$$
$$c_2 = y_2 + (1+r)\, a$$

where $\sigma \geq 0$, $\beta \in [0,1]$, $y_1, y_2 \geq 0$ and $r \geq -1$.

(a) Rewrite the budget constraints into a single lifetime budget constraint and set up the Lagrangian.

(b) Obtain the first order conditions for c_1 and c_2. Express c_2 as a function of c_1. Obtain the formulas for optimal c_1, c_2 and a.

(c) Until the end of the problem assume that: $\sigma = 1$, $y_1 = y$, $y_2 = y/2$, $\beta = 1$ and $r = 0$.
Are the assets at end of period 1 positive or negative?

(d) What are the second-best levels of c_1, c_2 and a when the agent is unable to borrow ($a \geq 0$)?

(e) Suppose that the government arranges a transfer v to this agent by issuing bonds. In the future, the government will tax the agent to be able to buy back the bonds. The new constraints of the agent are stated below. What is the impact of the government transfer on the agent's first period consumption? Does it depend on the size of the transfer v?

$$c_1 + a = y/2 + v$$
$$c_2 = y + (1+r)\, a - (1+r)\, v$$
$$a \geq 0$$

Problem 2 (2 points)

Consider the following two-period model with two assets (bonds and physical capital):

$$\max_{c_1, c_2, b, k} \quad U = \ln c_1 + \beta \ln c_2$$
$$\text{subject to} \quad c_1 + b + k = y_1$$
$$c_2 = y_2 + (1+r)\, b + (1-\delta)\, k$$
$$y_2 = k^{\alpha}$$

where b denotes bonds, $\delta \in (0,1)$ stands for capital depreciation rate and $\alpha \in (0,1)$ is the elasticity of production with respect to capital.

(a) Write down the problem in the form of a Lagrangian.

(b) Find the optimal value of k.

(c) Treating optimal k from (b) as a parameter, find the optimal values of c_1, c_2 and b.

(d) Calculate the derivative of optimal k with respect to r. Provide intuition for this result. *Hint: you can assume $\alpha = 1/2$ to simplify the calculations.*

(e) In equilibrium the rental rate for capital r^k is equal to the marginal product of capital $\partial y_2 / \partial k$. Show that $r^k - \delta = r$.

[1] The CRRA function can be thought of as a generalized logarithmic function. For $\sigma = 1$ the CRRA function becomes logarithmic, which can be easily proven by using the L'Hôpital's rule to compute the following limit: $\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma}$

Problem 3 (2+1 points)

In this economy households enter period 1 with a unit share of firm stock each ($\tilde{s} = 1$) and receive endowment $y_1 = 1$. They can then use these resources to either consume or use to purchase stocks s and bonds b, at their respective prices p^s and p^b. Each unit of a bond will pay a unit of consumption in period 2 with certainty, while both labor income and stock payoff are subject to uncertainty. The firms are going to generate revenue $y_2 = \{y^l, y^h\}$ where $y^l < y^h$, with the probability of the low state denoted by q. The firm (stock) owners will receive a fraction $\alpha \in (0,1)$ of firm's revenue, while workers will receive a fraction $1-\alpha$ of firm's revenue as their labor income. The problem of the household is then ($\beta = 1$ for simplicity):

$$\begin{aligned}
\max_{c_1,\, c_2,\, s,\, b} \quad & U = \frac{c_1^{1-\sigma}}{1-\sigma} + \mathrm{E}\left[\frac{c_2^{1-\sigma}}{1-\sigma}\right] \\
\text{subject to} \quad & c_1 + p^s s + p^b b = y_1 + p^s \tilde{s} \\
& c_2 = (1-\alpha)\, y_2 + d_2 \cdot s + b \\
& d_2 = \alpha y_2
\end{aligned}$$

(a) Using the Lagrangian approach, derive the first order conditions of the households with respect to c_t, c_{t+1}, s_{t+1} and b_{t+1}.

(b) Combine the FOCs with respect to c_1, c_2 and s to obtain the "stock" Euler equation. Combine the FOCs with respect to c_1, c_2 and b to obtain the "bond" Euler equation.

(c) Examine the equilibrium where $s = 1$ (households are on average satisfied with their current holdings of stocks and there are no splits or mergers) and $b = 0$ (net issuance of bonds cancels out). Using the budget constraints and properties of the expected value find the expressions for asset prices.

(d) Assume that: $\sigma = 1$, $q = 1/2$, $y^l = 1 - e$, $y^h = 1 + e$, with $e \in [0, 1)$. Find the asset prices and their expected returns. Calculate the equity risk premium. When would it be equal to 0? Why? *Hint: the resulting stock price will be independent of e, which is an artifact due to our simplifying assumptions.*

(e) **Extra point:** Suppose now that during the low state both labor and asset income are subject to additional risk, and are given respectively by $(1 \pm z)(1-\alpha)\, y_2^l$ and $(1 \pm z)\, \alpha y_2^l$. Each household randomly draws either positive or negative $z \in [0, 1)$ with 50-50% probability. Assume the same numerical values as in (d). Calculate asset prices and expected returns. How does the equity risk premium depend on z? Why? *Hint: the resulting stock price will be independent of z, which is an artifact due to our simplifying assumptions.*