University of Warsaw
Faculty of Economic Sciences

Endogenous Growth (second generation)

Advanced Macroeconomics IE: Lecture 15

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Endogenous growth

Endogenous growth

In exogenous growth models only technology A drives growth in the long run

We assumed that A grows at some rate g over time, where does it come from?

Think of A as ideas of how to combine inputs more efficiently:
better ideas, higher output per worker

Economics of ideas are different from economics of goods and services:

- Ideas are non-rival: I can use the idea of calculus at the same time as you
- Ideas are (generally) non-exclusive: I cannot stop you from using calculus

That sounds like public goods! But someone needs to invent them in the first place:

- Ideas have high fixed costs: it took a lot of effort to invent calculus or a new drug
- Ideas have low (zero) marginal costs: it costs nothing for you to use calculus now, it costs very little to produce one more pill of a drug

This implies that ideas have increasing returns to scale

Endogenous growth

Increasing returns to scale implies that the average cost of the idea (or good that embodies the idea) is higher than the marginal cost of reproducing the idea (or good that embodies the idea)

So ideas (or the goods embodying them) will only be produced if someone can charge more than their marginal cost

There must be imperfect competition

We'll learn the following (second generation) endogenous growth models:

- Increasing product variety (horizontal innovation)
- Increasing product quality (vertical innovation)
- Capital accumulation and innovation
- International technology transfer

Endogenous growth

All models will share the same structure of the economy:

- Households
- Firms
    - Perfectly competitive final goods producers
    - Monopolistic intermediate goods producers (M such firms)
    - Research and development with free entry

Households

Households' utility maximization problem

$$\max_{\{c_t,\, a_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$\text{subject to} \quad c_t + (1+n)\, a_{t+1} = w_t + (1+r_t)\, a_t + d_t$$

Construct the Lagrangian and expand it around the choice variables in t, c_t and a_{t+1}

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t + d_t - c_t - (1+n)\, a_{t+1}\right]$$
$$= \ldots + \beta^t \frac{\boldsymbol{c_t}^{1-\sigma}}{1-\sigma} + \ldots + \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t + d_t - \boldsymbol{c_t} - (1+n)\, \boldsymbol{a_{t+1}}\right]$$
$$+ \beta^{t+1} \lambda_{t+1} \left[w_{t+1} + (1+r_{t+1})\, \boldsymbol{a_{t+1}} + d_t - c_{t+1} - (1+n)\, a_{t+2}\right] + \ldots$$

Expanded Lagranian

$$\mathcal{L} = \ldots + \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \ldots + \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t + d_t - c_t - (1+n)\, a_{t+1} \right]$$
$$+ \beta^{t+1} \lambda_{t+1} \left[w_{t+1} + (1+r_{t+1})\, a_{t+1} + d_t - c_{t+1} - (1+n)\, a_{t+2} \right] + \ldots$$

First Order Conditions

$$c_t : \quad \beta^t c_t^{-\sigma} - \beta^t \lambda_t = 0 \quad \rightarrow \quad \lambda_t = c_t^{-\sigma}$$
$$a_{t+1} : \quad \beta^t \lambda_t \left[-(1+n) \right] + \beta^{t+1} \lambda_{t+1} (1+r_{t+1}) = 0 \quad \rightarrow \quad \lambda_t = \frac{\beta\,(1+r_{t+1})}{1+n} \lambda_{t+1}$$

Resulting Euler equation ($\beta = \frac{1}{1+\rho}$)

$$c_t^{-\sigma} = \frac{\beta\,(1+r_{t+1})}{1+n} c_{t+1}^{-\sigma} \quad \rightarrow \quad \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} = \frac{1+r_{t+1}}{(1+\rho)(1+n)} \quad \rightarrow \quad \frac{c_{t+1}}{c_t} = \left[\frac{1+r_{t+1}}{(1+\rho)(1+n)} \right]^{1/\sigma}$$

Households

Euler equation

$$\frac{c_{t+1}}{c_t} = \left[\frac{1 + r_{t+1}}{(1+\rho)(1+n)}\right]^{1/\sigma}$$

Rate of growth of consumption

$$g_c \equiv \frac{\Delta c_{t+1}}{c_t} = \frac{c_{t+1} - c_t}{c_t} = \frac{c_{t+1}}{c_t} - 1$$

$$g_c \approx \ln(1 + g_c) = \ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\sigma}\left[\ln(1 + r_{t+1}) - \ln(1+\rho) - \ln(1+n)\right] \approx \frac{r_{t+1} - \rho - n}{\sigma}$$

Along the Balanced Growth Path (BGP)

$$g_c^* = \frac{r - \rho - n}{\sigma}$$

If there is no population growth ($n = 0$)

$$g_c^* = \frac{r - \rho}{\sigma}$$

Increasing product variety (horizontal innovation)

Increasing product variety (horizontal innovation)

Based on Romer (1990) *Endogenous Technological Change*

Assume constant population / number of workers L

The number of intermediate good types M grows over time

Production function

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Profit maximization problem

$$\max_{L,\, \{x_{it}\}_{i=1}^{M_t}} \quad 1 \cdot L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First Order Conditions

$$L \;:\quad (1-\alpha)\, L^{-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha)\, \frac{Y_t}{L}$$

$$x_{it} \;:\quad L^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha} \quad \rightarrow \quad x_{it} = (\alpha / p_{it})^{\frac{1}{1-\alpha}}\, L$$

Intermediate goods producers (monopolists)

One unit of intermediate good is produced from one unit of final good

The marginal cost of production in the intermediate goods sector is equal to 1

Profit maximization problem

$$\max_{p_{it},\, x_{it}} \quad D_{it} = (p_{it} - 1)\, x_{it} = p_{it} x_{it} - x_{it}$$

$$\text{subject to} \quad p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha}$$

Plug in the inverse demand function

$$\max_{x_{it}} \quad D_{it} = \alpha x_{it}^{\alpha} L^{1-\alpha} - x_{it}$$

First Order Condition

$$x_{it} \; : \quad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1} L^{1-\alpha}}_{p_{it}} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} = 1 \quad \rightarrow \quad p_{it} = \frac{1}{\alpha}$$

Intermediate goods producers (monopolists)

Optimal price and production level

$$p_{it} = \frac{1}{\alpha} > 1 \quad \text{and} \quad x_{it} = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} L = \left(\alpha^2\right)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{2}{1-\alpha}} L$$

Maximal profit is constant in time and common for all producers

$$D = (p-1)\,x = \left(\tfrac{1}{\alpha} - 1\right) \alpha^{\frac{2}{1-\alpha}} L = \left(\tfrac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L$$

Value of the monopolistic firm (along the BGP with constant r)

$$V = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} D = D \cdot \frac{\frac{1}{1+r}}{1 - \frac{1}{1+r}} = D \cdot \frac{\frac{1}{1+r}}{\frac{r}{1+r}} = \frac{D}{r}$$

Research and development (R&D)

Developing a new variety of intermediate good requires using $1/\eta$ units of final good

Parameter η measures the productivity of the R&D sector

Let R denote the amount of resources devoted to R&D,
then the number of varieties increases by

$$\Delta M_{t+1} = M_{t+1} - M_t = \eta R_t$$

Free entry condition results in equalization of R&D cost of inventing a single intermediate type with the benefits of "selling" a patent for V

$$\frac{1}{\eta} = V = \frac{D}{r} \quad \rightarrow \quad r = \eta D$$

General Equilibrium and the BGP growth rate

We can now plug the interest rate into the Euler equation to get the BGP growth rate

$$g^* = \frac{r - \rho}{\sigma} = \frac{\eta D - \rho}{\sigma} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

Growth rate increases with

- the productivity of the R&D sector as measured by the parameter η
- the size of the economy as measured by labor supply L

and decreases with

- the rate of time preference ρ
- the degree of risk aversion σ

Increasing product quality (vertical innovation)

Increasing product quality (vertical innovation)

Based on Aghion and Howitt (1992) *A Model of Growth Through Creative Destruction*

This time the number of intermediate good types M is constant,
but their quality increases over time. Again assume constant population L

Final goods production function

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M} \left(A_{it}^{1-\alpha} x_{it}^{\alpha} \right)$$

where A_{it} is the quality level of i-th itermediate good at period t

Profit maximization problem

$$\max_{\ell,\, x_{it}} \quad 1 \cdot L^{1-\alpha} \sum_{i=1}^{M} \left(A_{it}^{1-\alpha} x_{it}^{\alpha} \right) - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First Order Condition with respect to x_{it}

$$x_{it} \;:\quad L^{1-\alpha} A_{it}^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha x_{it}^{\alpha-1} \left(A_{it} L \right)^{1-\alpha}$$

Intermediate goods producers (monopolists)

Profit maximization problem

$$\max_{p_{it},\, x_{it}} \quad D_{it} = (p_{it} - 1)\, x_{it} = p_{it} x_{it} - x_{it}$$

$$\text{subject to} \quad p_{it} = \alpha x_{it}^{\alpha-1} (A_{it} L)^{1-\alpha}$$

Plug in the inverse demand function

$$\max_{x_{it}} \quad D_{it} = \alpha x_{it}^{\alpha} (A_{it} L)^{1-\alpha} - x_{it}$$

First Order Condition

$$x_{it} \; : \quad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1} (A_{it} L)^{1-\alpha}}_{p_{it}} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} = 1 \quad \rightarrow \quad p_{it} = \frac{1}{\alpha}$$

Optimal production, maximal profit and firm value

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} A_{it} L, \quad D_{it} = \left(\tfrac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A_{it} L \equiv d A_{it} \quad \text{and} \quad V(A_{it}) = \frac{d A_{it}}{r + z_{it}}$$

where z_{it} is the probability of being replaced by a successful innovator

Research and development (R&D)

A successful innovator replaces the monopolist in an industry i
and increases the quality of the intermediate good by $1+q$, where $q>0$

$$A'_{i,t+1} = (1+q)\,A_{it}$$

Success probability z_{it} depends on R&D resources R_{it}, adjusted by the target quality

$$z_{it} = \eta R_{it}/A'_{i,t+1}$$

If successful, the innovator will gain ownership of a firm with quality level $A'_{i,t+1}$

$$V\left(A'_{i,t+1}\right) = \frac{dA'_{i,t+1}}{r+z_{i,t+1}}$$

Expected net benefit of R&D

$$z_{it}V\left(A'_{i,t+1}\right) - R_{it} = \frac{\eta R_{it}}{A'_{i,t+1}} \cdot \frac{dA'_{i,t+1}}{r+z_{i,t+1}} - R_{it}$$

Free entry makes the expected net benefit equal to 0

$$R_{it}\left(\frac{\eta d}{r+z_{i,t+1}} - 1\right) = 0 \quad \rightarrow \quad z_{i,t+1} = z = \eta d - r$$

General Equilibrium

Final goods production function

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M} \left(A_{it}^{1-\alpha} x_{it}^{\alpha}\right) = L^{1-\alpha} \sum_{i=1}^{M} \left(A_{it}^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_{it} L\right)^{\alpha}\right) = \alpha^{\frac{2\alpha}{1-\alpha}} L \sum_{i=1}^{M} A_{it} \equiv \alpha^{\frac{2\alpha}{1-\alpha}} L A_t$$

where A is the aggregate productivity (in increasing variety model $A = M$)

Dynamics of aggregate quality / productivity (in expectation)

$$\mathrm{E}\left[A_{t+1}\right] = \sum_{i=1}^{M} \mathrm{E}\left[A_{i,t+1}\right] = \sum_{i=1}^{M} \left[z\left(1+q\right) A_{it} + \left(1-z\right) A_{it}\right] = \left(1+zq\right) \sum_{i=1}^{M} A_{it} = \left(1+zq\right) A_t$$

$$\mathrm{E}\left[g_A\right] = \frac{\mathrm{E}\left[\Delta A_{t+1}\right]}{A_t} = \frac{\left(1+zq\right) A_t - A_t}{A_t} = zq$$

Solve the system of equations

$$\begin{cases} \sigma g = r - \rho & \text{Euler equation} \\ r = \eta d - z & \text{R\&D free entry} \\ z = g/q & \text{Expected productivity dynamics} \end{cases}$$

Solution

$$\sigma g = \eta d - g/q - \rho \quad \rightarrow \quad (\sigma + 1/q)\, g = \eta d - \rho$$

$$g = \frac{\eta d - \rho}{\sigma + 1/q} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma + 1/q}$$

$$z = \frac{\eta d - \rho}{\sigma q + 1} \quad \text{and} \quad r = \frac{\rho + \sigma q \eta d}{\sigma q + 1}$$

BGP growth rate

$$g^* = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma + 1/q}$$

Growth rate increases with

- the productivity of the R&D sector as measured by the parameter η
- the size of the economy as measured by labor supply L
- the size of the innovative step q

and decreases with

- the rate of time preference ρ
- the degree of risk aversion σ

Since in the data we do not observe “strong” scale effects (economies with larger L don’t grow faster), we will eliminate them going forward

Capital accumulation and innovation

Capital accumulation and innovation

Based on Aghion and Howitt (1999) *The Economics of Growth*, chapter 5.4

This time we allow for population change at some rate n

Intermediate goods are interpreted as capital transformed into particular “machines”

We allow for both (simplified) horizontal and vertical innovation

Horizontal innovation is random: each worker invents a new machine with probablity ψ, and existing machine types stop being produced with probability ϵ

The resulting BGP level of employment per machine type is

$$\ell^* = \frac{\epsilon + n}{\psi}$$

Final goods producers (perfectly competitive)

Production function

$$Y_t = (L_t/M_t)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} \equiv \ell_t^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

where $\ell \equiv L/M$ denotes employment per "machine" type

Profit maximization problem

$$\max_{\ell_t,\, x_{it}} \quad 1 \cdot \ell_t^{1-\alpha} \sum_{i=1}^{M} \left(A_{it}^{1-\alpha} x_{it}^{\alpha} \right) - w_t \ell_t M_t - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First Order Condition with respect to x_{it}

$$x_{it} \;:\; \quad \ell_t^{1-\alpha} A_{it}^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha \left(A_{it} \ell_t \right)^{1-\alpha} x_{it}^{\alpha-1}$$

Demand function for i-th "machine" $x_{it} = (\alpha/p_{it})^{1/(1-\alpha)} A_{it} \ell_t$

Intermediate goods producers (monopolists)

This time the marginal cost of production is given by the rental rate of capital r_t^k

Profit maximization problem

$$\begin{aligned} \max_{p_{it},\, x_{it}} \quad & D_{it} = \left(p_{it} - r_t^k\right) x_{it} = p_{it} x_{it} - r_t^k x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it} \ell_t\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Plug in the inverse demand function

$$\max_{x_{it}} \quad D_{it} = \alpha \left(A_{it} \ell_t\right)^{1-\alpha} x_{it}^{\alpha} - r_t^k x_{it}$$

First Order Conditions

$$x_{it} \;:\quad \alpha \cdot \alpha \left(A_{it} \ell_t\right)^{1-\alpha} x_{it}^{\alpha-1} - r_t^k = 0 \quad \rightarrow \quad \alpha p_{it} = r_t^k \quad \rightarrow \quad p_{it} = \frac{r_t^k}{\alpha}$$

Optimal level of production

$$x_{it} = \left(\frac{\alpha}{r_t^k / \alpha}\right)^{1/(1-\alpha)} A_{it} \ell_t = \left(\alpha^2 / r_t^k\right)^{1/(1-\alpha)} A_{it} \ell_t$$

Capital market equilibrium

Aggregate production of "machines" cannot exceed the accumulated capital K

$$K_t = \sum_{i=1}^{M_t} x_{it} = \sum_{i=1}^{M_t} \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} L_t \cdot \frac{1}{M_t}\sum_{i=1}^{M_t} A_{it} = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_t L_t$$

This time A is the simple average of industry-level product quality

Capital rental rate depends on the level of capital per effective labor $\hat{k} \equiv K/(AL)$

$$K_t = \left(\alpha^2/r_t^k\right)^{1/(1-\alpha)} A_t L_t \quad \to \quad \hat{k}_t = \left(\alpha^2/r_t^k\right)^{1/(1-\alpha)} \quad \to \quad r_t^k = \alpha^2 \hat{k}_t^{\alpha-1}$$

Optimal production level can be expressed as

$$x_{it} = \left[\alpha^2/(\alpha^2\hat{k}_t^{\alpha-1})\right]^{1/(1-\alpha)} A_{it}\ell_t = (\hat{k}_t^{1-\alpha})^{1/(1-\alpha)} A_{it}\ell_t = A_{it}\ell_t\hat{k}_t$$

Final goods production function becomes the familiar Cobb-Douglas one

$$Y_t = \left(\frac{L_t}{M_t}\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha}\left(A_{it}\frac{L_t}{M_t}\hat{k}_t\right)^{\alpha} = \hat{k}_t^{\alpha} L_t \cdot \frac{1}{M_t}\sum_{i=1}^{M_t} A_{it} = \hat{k}_t^{\alpha} L_t A_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$$

Research and development (R&D)

Maximal profit depends positively on $\hat{k}$, via reduction of costs of production

$$D_{it} = \left(\frac{1-\alpha}{\alpha}\right) r_t^k x_{it} = \left(\frac{1-\alpha}{\alpha}\right) \alpha^2 \hat{k}_t^{\alpha-1} \cdot A_{it} \ell_t \hat{k}_t = (1-\alpha)\, \alpha \hat{k}_t^{\alpha} \cdot A_{it} \ell_t \equiv d(\hat{k}_t) A_{it}$$

Value of the firm producing the i-th machine along the BGP

$$V^*\left(A_{it}\right) = \frac{d(\hat{k}^*) A_{it}}{r(\hat{k}^*) + z^* + \epsilon} = \frac{d(\hat{k}^*) A_{it}}{r^k(\hat{k}^*) - \delta + z^* + \epsilon}$$

Vertical innovations lead to improvements of "machine" quality by q and their probability depends on R&D expenditure adjusted by target quality: $z_{it} = \eta R_{it} / A'_{i,t+1}$

Free entry condition equalizes the expected net benefits of R&D to 0

$$z_{it} V^*\left(A'_{i,t+1}\right) - R_{it} = \frac{\eta R_{it}}{A'_{i,t+1}} \cdot \frac{d(\hat{k}^*) A'_{i,t+1}}{r(\hat{k}^*) + z^* + \epsilon} - R_{it} = R_{it}\left[\frac{\eta d(\hat{k}^*)}{r(\hat{k}^*) + z^* + \epsilon} - 1\right] = 0$$

$$z^* = \eta d(\hat{k}^*) - r(\hat{k}^*) - \epsilon$$

Balanced Growth Path General Equilibrium

BGP growth rate g^* depends positively on $\hat{k}^*$ (**GG** curve)

$$g^* = z^* q = q\left[\eta d(\hat{k}^*)\ell^* - r(\hat{k}^*) - \epsilon\right] = q\left[\eta\,(1-\alpha)\,\alpha(\hat{k}^*)^{\alpha}\frac{\epsilon+n}{\psi} - \left(\alpha^2(\hat{k}^*)^{\alpha-1} - \delta\right) - \epsilon\right]$$

Higher capital per effective labor $\hat{k}^*$ means higher profits and lower interest rates

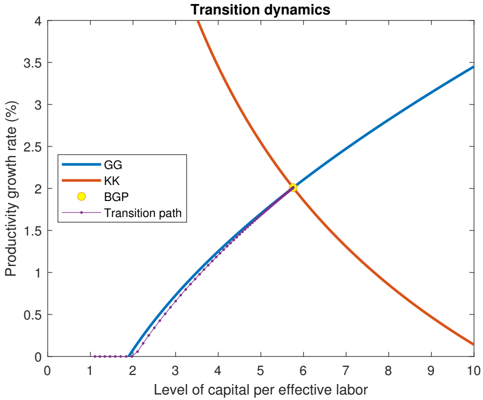
In turn $\hat{k}^*$ depends negatively on g^*: from the Euler equation we get

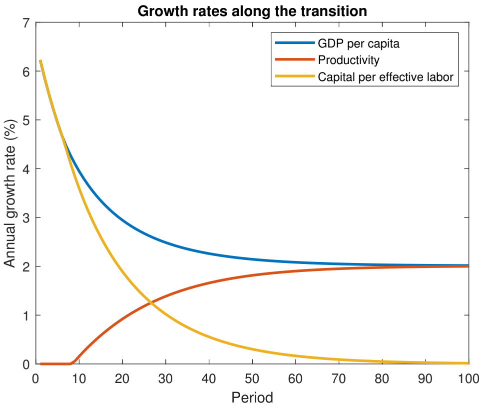
$$g_c^* = g^* = \frac{r-\rho-n}{\sigma} = \frac{\alpha^2(\hat{k}^*)^{\alpha-1} - \delta - \rho - n}{\sigma}$$

KK curve

$$\hat{k}^* = \left(\frac{\alpha^2}{\rho+\delta+n+\sigma g^*}\right)^{1/(1-\alpha)}$$

Model dynamics



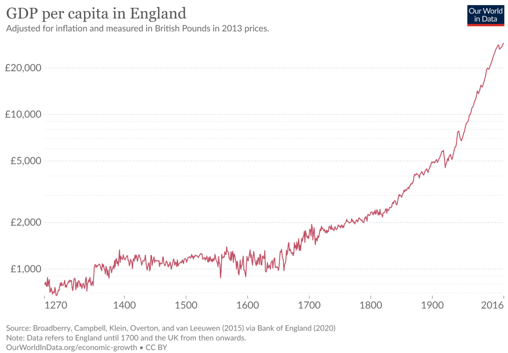


Whenever $\hat{k} < \hat{k}^*$, the rate of growth of productivity is $0 \leq g < g^*$

Initially capital accumulation is the main driver of growth

Improvements in productivity become increasingly more important and in the long run are the sole driver of GDP per worker growth

Economic growth in the very long run



Adding the "minimal" consumption (Stone-Geary) makes convergence to BGP slow:

- slow capital accumulation
- slow increase in productivity (even stagnation), growing number of "brains"
- initial innovation "accidental" (scientific revolution: $A\uparrow$ and $\eta\uparrow$),
  only later we get industrial innovation (industrial revolution), 20th century is BGP

James Watt's steam engine patent from 1769

A.D. 1769 . . . . N° 913.

Steam Engines, &c.

WATT'S SPECIFICATION.

TO ALL TO WHOM THESE PRESENTS SHALL COME, I, JAMES WATT, of Glasgow, in Scotland, Merchant, send greeting.

WHEREAS His most Excellent Majesty King George the Third, by His Letters Patent under the Great Seal of Great Britain, bearing date the Fifth day of January, in the ninth year of His said Majesty's reign, did give and grant unto me, the said James Watt, His special licence, full power, sole priviledge and authority, that I, the said James Watt, my exors, admors, and assigns, should and lawfully might, during the term of years therein expressed, use, exercise, and vend, throughout that part of His Majesty's Kingdom of Great Britain called England, the Dominion of Wales, and Town of Berwick upon Tweed, and also in His Majesty's Colonies and Plantations abroad, my "NEW INVENTED METHOD OF LESSENING THE CONSUMPTION OF STEAM AND FUEL IN FIRE ENGINES;" in which said recited Letters Patent is contained a proviso obliging me, the said James Watt, by writing under my hand and seal, to cause a particular description of the nature of the said Invention to be inrolled in His Majesties High Court of Chancery within four calendar months after the date of the said recited Letters Patent, as in and by the said Letters Patent, and the Statute in that behalf made, relation being thereunto respectively had, may more at large appear.

NOW KNOW YE, that in compliance with the said provisoe, and in pursuance of the said Statute, I, the said James Watt, do hereby declare that the

following is a particular description of the nature of my said Invention, and of the manner in which the same is to be performed (that is to say) :—

My method of lessening the consumption of steam, and consequently fuel, in fire engines consists of the following principles :—

First, that vessell in which the powers of steam are to be employed to work the engine, which is called the cylinder in common fire engines, and which I call the steam vessell, must during the whole time the engine is at work be kept as hot as the steam that enters it, first, by enclosing it in a case of wood or any other materials that transmit heat slowly; secondly, by surrounding it with steam or other heated bodies; and, thirdly, by suffering neither water or any other substance colder than the steam to enter or touch it during that time.

Secondly, in engines that are to be worked wholly or partially by condensation of steam, the steam is to be condensed in vessells distinct from the steam vessells or cylinders, although occasionally communicating with them. These vessells I call condensers, and whilst the engines are working, these condensers ought at least to be kept as cold as the air in the neighbourhood of the engines by application of water or other cold bodies.

Thirdly, whatever air or other elastic vapour is not condensed by the cold of the condenser, and may impede the working of the engine, is to be drawn out of the steam vessells or condensers, by means of pumps wrought by the engines themselves, or otherwise.

Fourthly, I intend in many cases to employ the expansive force of steam to press on the pistons, or whatever may be used instead of them, in the same manner as the pressure of the atmosphere is now employed in common fire engines. In cases where cold water cannot be had in plenty, the engines may be wrought by this force of steam only, by discharging the steam into the open air after it has done its office.

Fifthly, where motions round an axis are required, I make the steam vessells in form of hollow rings or circular channels, with proper inletts and outletts for the steam, mounted on horizontal axles like the wheels of a water mill; within them are placed a number of valves that suffer any body to go round the channell in one direction only. In these steam vessells are placed weights, so fitted to them as intirely to fill up a part or portion of their channels, yet rendered capable of moving freely in them by the means herein-after mentioned or specified. When the steam is admitted in these engines between these weights and the valves, it acts equally on both, so as to raise the weight to one side of the wheel, and by the reaction on the valves successively to give a circular motion to the wheel, the valves opening in the direction in which the

weights are pressed, but not in the contrary. As the steam vessel moves round it is supplied with steam from the boiler, and that which has performed its office may either be discharged by means of condensers, or into the open air.

Sixthly, I intend in some cases to apply a degree of cold not capable of reducing the steam to water, but of contracting it considerably, so that the engines shall be worked by the alternate expansion and constraction of the steam.

Lastly, instead of using water to render the piston or other parts of the engines air and steam tight, I employ oils, wax, rosinous bodies, fat of animals, quicksilver and other metalls, in their fluid state.

In witness whereof, I have hereunto set my hand and seal, this Twenty-fifth day of April, in the year of our Lord One thousand seven hundred and sixty-nine.

JAMES WATT. (L.S.)

Sealed and delivered in the presence of

COLL. WILKIE.
GEO. JARDINE.
JOHN ROEBUCK.

Be it remembered, that the said James Watt doth not intend that any thing in the fourth article shall be understood to extend to any engine where the water to be raised enters the steam vessell itself, or any vessell having an open communication with it.

JAMES WATT.

Witnesses,

COLL. WILKIE.
GEO. JARDINE.

AND BE IT REMEMBERED, that on the Twenty-fifth day of April, in the year of our Lord 1769, the aforesaid James Watt came before our said Lord the King in His Chancery, and acknowledged the Specification aforesaid, and all and every thing therein contained and specified, in form above written. And also the Specification aforesaid was stampt according to the tenor of the Statute made in the sixth year of the reign of the late King and Queen William and Mary of England, and so forth.

Wilkie, Extr.

Inrolled the Twenty-ninth day of April, in the year of our Lord One thousand seven hundred and sixty-nine.

LONDON:
Printed by GEORGE EDWARD EYRE and WILLIAM SPOTTISWOODE,
Printers to the Queen's most Excellent Majesty. 1855.

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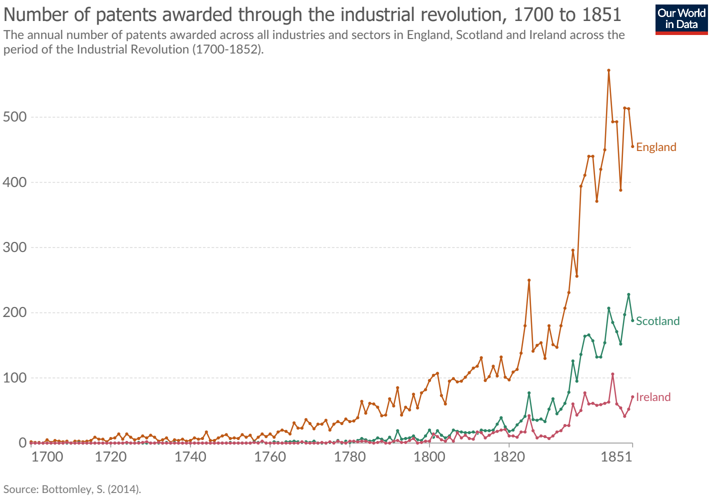
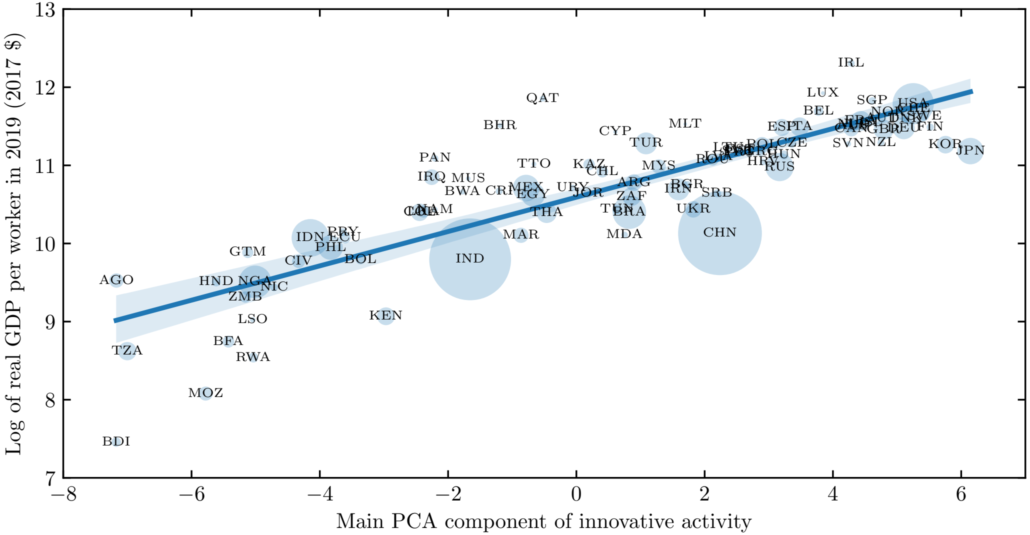
Patents awarder during the industrial revolution



International technology transfer

Advanced economies are innovation leaders



International technology transfer

Based on Aghion and Howitt (1999) *The Economics of Growth*, chapter 7.2

Assume two groups of countries: technology leaders and technology followers

- Technology leaders invent new technologies: their rate of growth is explained by the previous model and is denoted with $\bar{g}$
- Technology followers adopt / imitate the leading technologies

Probability of successful adoption / imitation of leading technology $\bar{A}$ is z

$$A_{i,t+1} = \begin{cases} \bar{A}_{it} & \text{with probability} \quad z \\ A_{it} & \text{with probability} \quad 1-z \end{cases}$$

Higher z translates to higher productivity growth

$$A_{t+1} = \frac{1}{M}\sum_{i=1}^{M} A_{i,t+1} = \frac{1}{M}\sum_{i=1}^{M}\left[z\bar{A}_{it} + (1-z)\,A_{it}\right] = z\frac{1}{M}\sum_{i=1}^{M}\bar{A}_{it} + (1-z)\,\frac{1}{M}\sum_{i=1}^{M} A_{it}$$

$$A_{t+1} = z\bar{A}_t + (1-z)\,A_t$$

Proximity to technology frontier

Proximity to technology frontier $a_t \equiv A_t / \bar{A}_t$

Dynamics of proximity

$$A_{t+1} = z\bar{A}_t + (1-z)\,A_t \quad \Big| \quad : \bar{A}_t$$

$$\frac{A_{t+1}}{\bar{A}_t} = \frac{A_{t+1}}{\bar{A}_{t+1}}\frac{\bar{A}_{t+1}}{\bar{A}_t} = z + (1-z)\,\frac{A_t}{\bar{A}_t} = z + (1-z)\,a_t$$

$$a_{t+1}\,(1+\bar{g}) = z + (1-z)\,a_t$$

$$a_{t+1} = \frac{z + (1-z)\,a_t}{1+\bar{g}}$$

BGP proximity

$$a^*\,(1+\bar{g}) = z + (1-z)\,a^*$$

$$a^*\,(\bar{g}+z) = z$$

$$a^* = \frac{z}{z+\bar{g}} < 1$$

Proximity to technology frontier and growth

BGP growth rate of technology followers is also $\bar{g}$

$$g = \frac{A_{t+1} - A_t}{A_t} - 1 = \frac{z\bar{A}_t + (1-z)\,A_t}{A_t} - 1 = \frac{z}{a^*} + (1-z) - 1 = \bar{g} + z - z = \bar{g}$$

Before they converge to BGP, tech followers grow faster ("advantage of backwardness")

$$q_t \equiv \frac{\bar{A}_t - A_t}{A_t} = \frac{1}{a_t} - 1 \geq \frac{1}{a^*} - 1$$

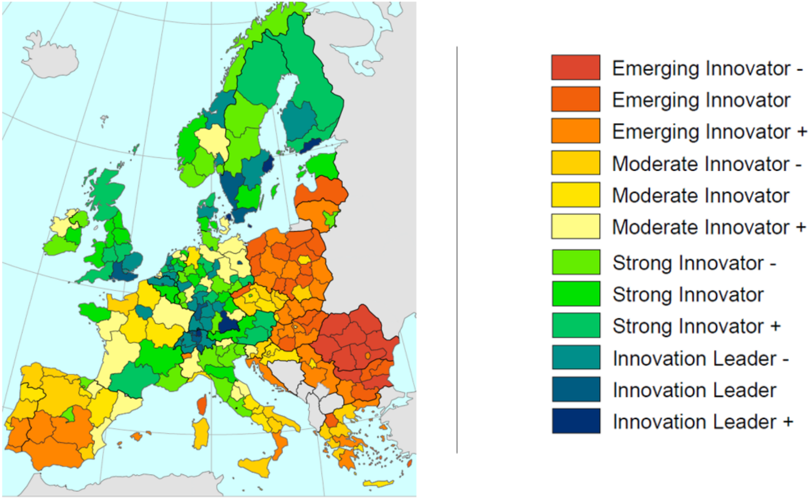
$$g_t = zq_t = z\left(\frac{1}{a_t} - 1\right) \geq \bar{g}$$

Domestic $z = \eta d(\hat{k}) - r(\hat{k})$ does not determine growth rate, but **relative GDP per worker**

Determinants of country "rank" in world GDP per worker distribution

- R&D productivity η (quantity and quality of human capital, top universities)
- Firm profitability d (efficient bureaucratic and legal system, no corruption)
- Financing conditions r (efficient equity markets, access to venture capital funds)

Innovation activity in the European Union



Regional Innovation Scoreboard (2021)

Advanced economies grow together, but persistently differ in GDP levels