Marcin Bielecki, Advanced Macroeconomics IE, Spring 2025 Endogenous Growth Models (second generation)[1]

1 Expanding product variety

Based on Romer (1990) *Endogenous Technological Change.*

We will first consider a simpler version of the model where final goods are used as R&D input.

1.1 Households

As usual, households want to maximize their utility, subject to the budget constraint:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t + (1+n)\, a_{t+1} = w_t + (1+r_t)\, a_t + d_t \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{c_t}^{1-\sigma}}{1-\sigma} + \lambda_t \left[w_t + (1+r_t)\, \boldsymbol{a_t} + d_t - \boldsymbol{c_t} - (1+n)\, \boldsymbol{a_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} c_t \ &: \quad \beta^t \left\{ c_t^{-\sigma} + \lambda_t [-1] \right\} = 0 && \rightarrow \quad \lambda_t = c_t^{-\sigma} \\ a_{t+1} \ &: \quad \beta^t \lambda_t [-(1+n)] + \beta^{t+1} \lambda_{t+1} (1+r_{t+1}) = 0 && \rightarrow \quad \lambda_t = \beta \lambda_{t+1} \frac{(1+r_{t+1})}{(1+n)} \end{aligned}$$

Resulting Euler equation:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} \frac{(1+r_{t+1})}{(1+n)} = c_{t+1}^{-\sigma} \frac{(1+r_{t+1})}{(1+\rho)(1+n)} \quad \Big| \quad \cdot c_{t+1}^{\sigma} \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \frac{(1+r_{t+1})}{(1+\rho)(1+n)} \quad \Big| \quad (\cdot)^{1/\sigma} \\ \frac{c_{t+1}}{c_t} &= \left[\frac{(1+r_{t+1})}{(1+\rho)(1+n)} \right]^{1/\sigma} \end{aligned}$$

Rate of growth of consumption per worker:

$$\begin{aligned} g_c &= \frac{c_{t+1} - c_t}{c_t} = \frac{c_{t+1}}{c_t} - 1 \\ g_c &\approx \ln(1+g_c) = \ln\left(\frac{c_{t+1}}{c_t} \right) = \frac{1}{\sigma} \left[\ln(1+r_{t+1}) - \ln(1+\rho) - \ln(1+n) \right] \approx \frac{r_{t+1} - \rho - n}{\sigma} \end{aligned}$$

Along the Balanced Growth Path:

$$g_c^* = \frac{r - \rho - n}{\sigma}$$

If there is no population growth ($n = 0$):

$$g_c^* = \frac{r - \rho}{\sigma}$$

At first we'll assume that population is at some constant level L.

[1] This set of lecture notes is based on chapters 3–5 and 7 from Aghion and Howitt (2009) *The Economics of Growth.*

1.2 Producers

Two types of goods:

- homogenous final goods Y_t produced by perfectly competitive, representative firm
- M_t varieties of differentiated intermediate goods (machines) x_{it} produced by monopolists

1.2.1 Final goods

Production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Profit maximization problem:

$$\max \quad L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First order conditions:

$$\begin{aligned}
L \quad &: \quad (1-\alpha) L^{-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha) \frac{Y_t}{L} \\
x_{it} \quad &: \quad L^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha}
\end{aligned}$$

Demand for intermediate good of type i:

$$x_{it} = \left(\frac{\alpha}{p_{it}} \right)^{\frac{1}{1-\alpha}} L$$

1.2.2 Intermediate goods

One unit of intermediate good is produced from one unit of final good. Hence the marginal cost of production in the intermediate goods sector is equal to 1.

Profit maximization problem:

$$\begin{aligned}
\max \quad & D_{it} = (p_{it} - 1) x_{it} = p_{it} x_{it} - x_{it} \\
\text{subject to} \quad & p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha}
\end{aligned}$$

Incorporate the demand schedule into the profit function:

$$\max \quad D_{it} = \alpha x_{it}^{\alpha} L^{1-\alpha} - x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1} L^{1-\alpha}}_{p_{it}} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} = 1$$

Optimal price:

$$p_{it} = \frac{1}{\alpha}$$

Optimal level of production:

$$x_{it} = \left(\frac{\alpha}{p_{it}} \right)^{\frac{1}{1-\alpha}} L = \left(\alpha^2 \right)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{2}{1-\alpha}} L$$

The level of production of all intermediate goods will be the same and constant over time. We can drop subscripts i and t. Maximal profit of the intermediate goods producer is given by:

$$D = (p-1) x = \left(\frac{1}{\alpha} - 1 \right) \alpha^{\frac{2}{1-\alpha}} L = \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{2}{1-\alpha}} L$$

1.3 Research and Development

Developing a new type of intermediate good requires sacrificing $1/\eta$ units of final good. Parameter η measures the productivity of the R&D sector. Let R_t denote the amount of resources devoted to R&D. Then the number of varieties will increase by:

$$\Delta M_{t+1} = \eta R_t$$

Assume that the research sector is perfectly competitive. Then the cost of invention $1/\eta$ will have to be equal to the present discounted value of profit flows of a new intermediate good producing monopolist:[2]

$$\frac{1}{\eta} = V = \sum_{t=1}^{\infty} \frac{1}{(1+r)^t} D = D \cdot \frac{\frac{1}{1+r}}{1 - \frac{1}{1+r}} = D \cdot \frac{\frac{1}{1+r}}{\frac{r}{1+r}} = \frac{D}{r}$$

And the interest rate will have to satisfy:

$$r = \eta D$$

1.4 General Equilibrium

We can now plug the above interest rate into the Euler equation:

$$g_c = \frac{\eta D - \rho}{\sigma} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

The last formality is to show that the rates of growth of consumption, output and product variety are identical. Start with the output accounting identity:

$$Y_t = Lc_t + M_t x + R_t$$
$$Lc_t = Y_t - M_t x - R_t$$

If we are able to show that the RHS grows at the rate of product variety growth, then consumption will also grow at that rate. Consider final goods output:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_i^\alpha = L^{1-\alpha} M_t \left(\alpha^{\frac{2}{1-\alpha}} L\right)^\alpha = M_t \alpha^{\frac{2\alpha}{1-\alpha}} L \quad \rightarrow \quad g_Y = g_M$$

Now take a look at the R&D sector:

$$\Delta M_{t+1} = \eta R_t \quad \rightarrow \quad g_M = \frac{\Delta M_{t+1}}{M_t} = \eta \frac{R_t}{M_t}$$

This implies that if we are on the Balanced Growth Path and variables grow at constant rates, then:

$$g_R = g_M$$

because otherwise g_M would not be constant. Since both Y_t and R_t grow at the same rate as M_t, then consumption c_t also grows at that rate and the rate of growth of the economy is equal to:

$$g = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

Growth rate increases with the productivity of R&D as measured by the parameter η and with the size of the economy as measured by labor supply L, and decreases with the rate of time preference ρ and degree of risk aversion σ.

The prediction that g should increase with L was first seen as a virtue of the model, suggesting that larger countries or larger free-trade zones should grow faster. However, Jones (1995) pointed out that this prediction is counterfactual. On the other hand, Kremer (1993) argued that the above equation approximates well the growth experience of the world economy treated as a whole.

[2] The formula below is valid for constant r and Π, as is the case along the BGP.

1.5 Socially optimal rate of growth

Set up the social planner's problem. For simplicity, normalize $L = 1$, so that $c_t \equiv C_t$:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & Y_t = M_t x_t^\alpha \\ & C_t + M_t x_t + R_t = Y_t \\ & M_{t+1} = \eta R_t + M_t \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\boldsymbol{C_t^{1-\sigma}}}{1-\sigma} + \lambda_t \left[\boldsymbol{M_t x_t^\alpha} - \boldsymbol{C_t} - \boldsymbol{M_t x_t} - \boldsymbol{R_t} \right] + \mu_t \left[\eta \boldsymbol{R_t} + \boldsymbol{M_t} - \boldsymbol{M_{t+1}} \right] \right\}$$

First order conditions:

$$\begin{aligned} C_t &: \quad \beta^t \left\{ C_t^{-\sigma} + \lambda_t [-1] \right\} = 0 && \rightarrow \quad \lambda_t = C_t^{-\sigma} \quad \rightarrow \quad C_t = \lambda_t^{-1/\sigma} \\ x_t &: \quad \beta^t \left\{ \alpha M_t x_t^{\alpha-1} - M_t \right\} = 0 && \rightarrow \quad x^{sp} = \alpha^{\frac{1}{1-\alpha}} \\ R_t &: \quad \beta^t \left\{ \lambda_t [-1] + \mu_t \eta \right\} = 0 && \rightarrow \quad \lambda_t = \mu_t \eta \\ M_{t+1} &: \quad \beta^t \mu_t [-1] + \beta^{t+1} \left\{ \lambda_{t+1} \left[x_{t+1}^\alpha - x_{t+1} \right] + \mu_{t+1} \right\} = 0 && \rightarrow \quad \mu_t = \beta \left[\lambda_{t+1} \left(x_{t+1}^\alpha - x_{t+1} \right) + \mu_{t+1} \right] \end{aligned}$$

Optimal quantity of intermediates produced:

$$x_t^{sp} = \alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}} = x_t^{dec}$$

Express FOC for number of varieties in terms of λ:

$$\frac{\lambda_t}{\eta} = \beta \left[\lambda_{t+1} \left(x_{t+1}^\alpha - x_{t+1} \right) + \frac{\lambda_{t+1}}{\eta} \right] \quad \rightarrow \quad \lambda_t = \frac{\lambda_{t+1} \left[\eta \left(x_{t+1}^\alpha - x_{t+1} \right) + 1 \right]}{1+\rho}$$

Rate of growth of consumption:

$$g_c \approx \ln\left(\frac{C_{t+1}}{C_t}\right) = \ln\left[\left(\frac{\lambda_t}{\lambda_{t+1}}\right)^{1/\sigma}\right] = \frac{1}{\sigma} \ln\left\{ \frac{1 + \eta \left(x_{t+1}^\alpha - x_{t+1} \right)}{1+\rho} \right\} = \frac{1}{\sigma} \left\{ \ln\left[1 + \eta \left(x_{t+1}^\alpha - x_{t+1} \right)\right] - \ln(1+\rho) \right\}$$
$$\approx \frac{\eta \left(x_{t+1}^\alpha - x_{t+1} \right) - \rho}{\sigma} = \frac{\eta x_{t+1} \left(x_{t+1}^{\alpha-1} - 1 \right) - \rho}{\sigma} = \frac{\eta \alpha^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) - \rho}{\sigma} = \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} - \rho}{\sigma}$$

Compare with the growth rate in the decentralized economy:

$$g^{sp} = \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} - \rho}{\sigma} > \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{2}{1-\alpha}} - \rho}{\sigma} = g^{dec}$$

Because intermediate goods producers do not internalize their contribution to product diversity and because researchers do not internalize research spillovers, the growth rate in the decentralized economy is always lower than is socially optimal.[3] Quite surprisingly, it turns out that if we would want to bring the decentralized economy to the socially optimal solution, we would need to subsidize the monopolists!

[3]Note however that Benassy (1998) shows that with a slight modification of the production function the decentralized economy's rate of growth might exceed socially optimal rate of growth.

1.6 Patent duration

Until now we have assumed that the monopolist is able to enjoy monopolistic profits forever. Here we take a look at the case where the monopolistic position might be terminated, e.g. due to the fixed term of patent protection. Formally, we would then model the value of a newly emerged monopolist as:

$$V = \sum_{t=1}^{T} \frac{D}{(1+r)^t}$$

where T is the length of patent duration. An approximated, but simpler expression assumes that instead of patents binding for a fixed term, in each period there is a probability z of a monopolist losing the patent protection. Consequently, the expected length of patent duration is equal to $1/z$. Then the value of a newly emerged monopolist is equal to:

$$V = \sum_{t=1}^{\infty} \frac{(1-z)^{t-1}}{(1+r)^t} D = D \cdot \frac{\frac{1}{1+r}}{1 - \frac{1-z}{1+r}} = D \cdot \frac{\frac{1}{1+r}}{\frac{r+z}{1+r}} = \frac{D}{r+z}$$

and the research-arbitrage condition yields:

$$\frac{1}{\eta} = V = \frac{D}{r+z} \quad \rightarrow \quad r = \eta D - z$$

The growth rate of the decentralized economy is then equal to:

$$g = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - z - \rho}{\sigma}$$

and decreases with z (increases with the length of patent protection). The intuition behind this result is simple: if the expected value of a new firm is lower, the incentives for performing R&D are also lower.

The finite patent duration does however have some benefits, since some intermediate goods are produced by competitive firms, and not the monopolists. Perfectly competitive firms set the price of their intermediate goods varieties to 1:

$$p^c = 1 < \frac{1}{\alpha} = p^m$$

and produce them at higher quantities:

$$x^c = \alpha^{\frac{1}{1-\alpha}} L > \alpha^{\frac{2}{1-\alpha}} L = x^m$$

The BGP share of competitively produced intermediates is given by:

$$\frac{M^c}{M} = \frac{z}{g+z}$$

The level of ouput is higher than in the case of infinite patent protection and is given by:

$$\begin{aligned} Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_i^\alpha &= L^{1-\alpha} \left[M_t^c \left(\alpha^{\frac{1}{1-\alpha}} L \right)^\alpha + (M_t - M_t^c) \left(\alpha^{\frac{2}{1-\alpha}} L \right)^\alpha \right] \\ &= L \alpha^{\frac{2\alpha}{1-\alpha}} M_t \left[\frac{M^c}{M} \alpha^{\frac{-\alpha}{1-\alpha}} + \left(1 - \frac{M^c}{M} \right) \right] = L \alpha^{\frac{2\alpha}{1-\alpha}} M_t \left[1 + \frac{z}{g+z} \left((1/\alpha)^{\frac{\alpha}{1-\alpha}} - 1 \right) \right] > Y_t^m \end{aligned}$$

There is a trade-off between the rate of economic growth along the BGP and the level of output along the BGP. Since the households are impatient, they prefer to live in a world with limited monopoly power.

This also creates a time-inconsistency problem. On the one hand, we would like for all already invented varieties to be produced competitively so that the level of output (and consumption) today would be higher. But on the other hand we would like to incentivize the R&D sector to invent new varieties, promising them long patent duration. Such promises tend, of course, not to be credible. One possible way to proceed is to assume that the government commits itself not to change the probability z for existing products but can choose this probability for goods that are yet to be invented.

1.7 Labor as R&D Input

Romer (1990) actually assumed that labor, not final goods, was the R&D input. Accordingly, suppose now that labor can be either used to produce final goods (L_Y) or to perform R&D activities (L_R):

$$L = L_Y + L_R$$

We can transfer some results from the previous sections, we just need to be careful in replacing L with L_Y.

Output is given by:

$$Y_t = L_Y^{1-\alpha} M_t x^{\alpha}$$

The optimal level of intermediate goods production equals:

$$x = \alpha^{\frac{2}{1-\alpha}} L_Y$$

And the profit flow of a monopolist is given by:

$$D = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L_Y$$

Wages are equal to:

$$w_t = (1-\alpha)\frac{Y_t}{L_Y} = (1-\alpha) L_Y^{-\alpha} M_t x^{\alpha} = (1-\alpha) L_Y^{-\alpha} M_t \left(\alpha^{\frac{2}{1-\alpha}} L_Y\right)^{\alpha} = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} M_t$$

Now we are going to assume that the number of varieties M_t grows at a rate that depends on the amount of labor devoted to research L_R:

$$\Delta M_{t+1} = \eta M_t L_R$$

This equation reflects the existence of spillovers in research activities; that is, all researchers can make use of the accumulated knowledge M_t embodied in existing designs.[4]

Research arbitrage condition now implies:

$$\frac{w_t}{\eta M_t} = V = \frac{D}{r}$$

$$r = \frac{\eta M_t D}{w_t} = \frac{\eta M_t \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L_Y}{(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} M_t} = \eta L_Y \alpha^{-1} \alpha^{\frac{2}{1-\alpha}} \alpha^{-\frac{2\alpha}{1-\alpha}} = \alpha\eta L_Y$$

Now we need to find the actual division of labor between production and research. Let's look at research:

$$g_M = \frac{\Delta M_{t+1}}{M_t} = \eta L_R = \eta (L - L_Y)$$

Then:

$$L_Y = L - \frac{g_M}{\eta} \quad \rightarrow \quad r = \alpha(\eta L - g_M)$$

Plug into the Euler equation:

$$g_c = \frac{\alpha(\eta L - g_M) - \rho}{\sigma}$$

By assuming the two rates of growth are equal (and denoting them both with g) we get:

$$\sigma g = \alpha\eta L - \alpha g - \rho \quad \rightarrow \quad g = \frac{\alpha\eta L - \rho}{\alpha + \sigma}$$

And the proof that the two rates of growth are indeed equal is trivial since now output is split only between consumption and intermediate goods:

$$Y_t = Lc_t + M_t x \quad \rightarrow \quad Lc_t = L_Y^{1-\alpha} M_t x^{\alpha} + M_t x$$

It is now obvious that both consumption and the number of varieties grow at exactly the same rates.

[4] Jones (1995) proposes a variant of the Romer (1990) model where $\Delta M_{t+1} = \eta M_t^{\phi} L_R^{\lambda}$. Assuming $\phi < 1$ implies that the BGP growth rate is a positive function of the population growth rate and is in equilibrium given by $g = \frac{\lambda n}{1-\phi}$.

2 Increasing product quality (Schumpeterian growth)

Based on Aghion and Howitt (1992) *A Model of Growth Through Creative Destruction.*

Assume constant population L for simplicity. Analyze the case of a single intermediate good first.

2.1 Producers

2.1.1 Final goods

Production function:

$$Y_t = L^{1-\alpha} A_t^{1-\alpha} x_t^{\alpha}$$

where A_t is the level of quality/productivity of the intermediate good at time t.

Profit maximization problem:

$$\max \quad (A_t L)^{1-\alpha} x_t^{\alpha} - w_t L - p_t x_t$$

First order condition:

$$x_t \quad : \quad \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1} - p_t = 0 \quad \rightarrow \quad p_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1}$$

2.1.2 Intermediate goods

Same as before, one unit of intermediate good is produced from one unit of final good.

Profit maximization problem:

$$\begin{aligned} \max \quad & D_t = p_t x_t - x_t \\ \text{subject to} \quad & p_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1} \end{aligned}$$

Incoroprate the demand schedule:

$$\max \quad D_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha} - x_t$$

First order condition:

$$x_t \quad : \quad \alpha \cdot \underbrace{\alpha (A_t L)^{1-\alpha} x_t^{\alpha-1}}_{p_t} - 1 = 0 \quad \rightarrow \quad \alpha p_t - 1 = 0$$

Optimal price:

$$p_t = \frac{1}{\alpha}$$

Optimal level of production:

$$x_t = \alpha^{\frac{2}{1-\alpha}} A_t L$$

Maximal profit:

$$D(A_t) = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A_t L \equiv d A_t L$$

Value of the firm with quality level A_t:

$$V(A_t) = \sum_{t=1}^{\infty} \frac{(1-z)^{t-1}}{(1+r)^t} D(A_t) = \frac{D(A_t)}{r+z} = \frac{d A_t L}{r+z}$$

where z is the probability of being replaced by a successful innovator. Note here that I assume that both the real interest rate r and the probability z are constant in equilibrium, which is indeed the case.

Final goods output:

$$Y_t = (A_t L)^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_t L\right)^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}} A_t L$$

Output grows at a rate of growth of quality/productivity level A_t.

2.2 Research and Development

A successful innovator invents a better version of the product with quality level $A'_{t+1} \equiv (1+q)\,A_t$ (where $q > 0$) and replaces the previous monopolist. The change in product's quality is then:

$$\Delta A_{t+1} \equiv A'_{t+1} - A_t = (1+q)\,A_t - A_t = qA_t$$

Success probability z_t depends on the amount of devoted R&D resources R_t, adjusted by the target quality level A^*_t, reflecting the notion that as technology advances it becomes harder to improve upon:

$$z_t = \eta \frac{R_t}{A'_t}$$

with parameter η reflecting the productivity of the R&D sector.

If successful, the innovator will gain ownership of a firm with quality level A'_{t+1}:

$$V\left(A'_{t+1}\right) = \frac{dA'_{t+1}L}{r+z}$$

The expected net benefit of R&D activity is:

$$z_t \cdot V\left(A'_{t+1}\right) - R_t = \eta \frac{R_t}{A'_{t+1}} \cdot \frac{dA'_{t+1}L}{r+z} - R_t = \frac{\eta R_t dL}{r+z} - R_t$$

Free entry into the R&D sector makes the expected net benefit equal to 0:

$$R_t \left(\frac{\eta dL}{r+z} - 1\right) = 0 \quad \rightarrow \quad z = \eta dL - r \quad \rightarrow \quad r = \eta dL - z$$

2.3 General Equilibrium

Solving the standard utility maximization problem of the consumer results in the Euler equation:

$$g_c = \frac{r-\rho}{\sigma}$$

The expected rate of productivity growth driven by innovation is given by:

$$\mathrm{E}\left[g_A\right] = \mathrm{E}\left[\frac{\Delta A_{t+1}}{A_t}\right] = \frac{(1+zq)\,A_t - A_t}{A_t} = \frac{zqA_t}{A_t} = zq$$

By assuming that along the Balanced Growh Path rates of growth of consumption and productivity are equal, as is indeed the case, we get the following system of three equations linking real interest rate r, innovative success probability z and rate of growth of the economy g:

$$\begin{cases} \sigma g = r - \rho & \text{Euler equation} \\ r = \eta dL - z & \text{R\&D free entry} \\ z = g/q & \text{Expected productivity dynamics} \end{cases}$$

Solving the system:

$$\sigma g = \eta dL - z - \rho$$
$$\sigma g = \eta dL - g/q - \rho$$

$$g = \frac{\eta dL - \rho}{\sigma + 1/q} = \frac{\eta\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}L - \rho}{\sigma + 1/q}$$

$$z = \frac{\eta dL - \rho}{\sigma q + 1} \quad \text{and} \quad r = \frac{\rho + \sigma q \eta dL}{\sigma q + 1}$$

Growth rate g increases with the productivity of R&D η, the size of innovative step q and with the size of the economy as measured by labor supply L, and decreases with the rate of time preference ρ and degree of risk aversion σ.

2.4 Eliminating Scale Effects

Consider now the case of M distinct intermediate goods, produced by their respective monopolists. We will also allow for population growth at rate n.

Final goods

Production function:

$$Y_t = (L_t/M_t)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

This production function is the same as the one assumed in the expanding product variety model, except that (a) each product has its own unique productivity parameter A_{it} instead of having $A_{it} = 1$ for all products, and (b) we assume that what matters is not the absolute input L_t of labor but the input per product L_t/M_t.[5]

Now we have to specify the process by which product variety increases. The simplest scheme is to suppose that each person has a probability ψ of inventing a new intermediate product, with no expenditure at all on research. Suppose also that the exogenous fraction ϵ of products disappears each period. The number of intermediate products will stabilize at a level proportional to population:

$$\begin{aligned}
M_{t+1} &= (1-\epsilon) M_t + \psi L_t \quad | \quad : L_t \\
\frac{M_{t+1}}{L_t} &= (1-\epsilon) \frac{M_t}{L_t} + \psi \\
\frac{M_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t} &= (1-\epsilon) \frac{M_t}{L_t} + \psi \\
\frac{M_{t+1}}{L_{t+1}} (1+n) &= (1-\epsilon) \frac{M_t}{L_t} + \psi
\end{aligned}$$

Balanced Growth Path:

$$\frac{M_{t+1}}{L_{t+1}} = \frac{M_t}{L_t} = \left(\frac{M}{L}\right)^* \quad \rightarrow \quad \left(\frac{M}{L}\right)^* (n+\epsilon) = \psi \quad \rightarrow \quad \left(\frac{M}{L}\right)^* = \frac{\psi}{n+\epsilon} \quad \rightarrow \quad \left(\frac{L}{M}\right)^* = \frac{\epsilon+n}{\psi} \equiv \ell$$

where ℓ denotes workers per product line. Assume we have reached the Balanced Growth Path.

Rewritten profit maximization problem:

$$\max \quad \ell^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L_t - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \ell^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1}$$

Intermediate goods

Profit maximization problem:

$$\begin{aligned}
\max \quad & D_{it} = p_{it} x_{it} - x_{it} \\
\text{subject to} \quad & p_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1}
\end{aligned}$$

Incoroprate the demand schedule:

$$\max \quad D_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha} - x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} - 1 = 0$$

[5] This production function is a special case of the one that Benassy (1998) showed does not necessarily yield a positive productivity effect of product variety.

Optimal price:

$$p_{it} = \frac{1}{\alpha}$$

Optimal level of production:

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} A_{it} \ell$$

Maximal profit:

$$D_{it} = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A_{it} \ell \equiv d A_{it} \ell$$

Value of the firm with quality level A_{it} (where ϵ appears as the probability of "design disappearing"):

$$V\left(A_{it}\right) = \frac{d A_{it} \ell}{r + z + \epsilon}$$

Final goods output:

$$Y_t = \left(L_t / M_t\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_{it} L_t / M_t\right)^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}} L_t \left(\frac{1}{M_t} \sum_{i=1}^{M_t} A_{it}\right) = \alpha^{\frac{2\alpha}{1-\alpha}} A_t L_t$$

where the aggregate productivity is the simple average of all the individual productivities:

$$A_t \equiv \frac{1}{M_t} \sum_{i=1}^{M_t} A_{it}$$

Research and Development

As before, a successful innovation will increase the productivity of an intermediate good by $1+q$:

$$A'_{i,t+1} = (1+q) A_{it}$$

and the success probability depends on R&D resources adjusted for target productivity:

$$z_{it} = \eta \frac{R_{it}}{A'_{i,t+1}}$$

The expected net benefit of R&D activity is:

$$z_{it} \cdot V\left(A'_{i,t+1}\right) - R_{it} = \eta \frac{R_{it}}{A'_{i,t+1}} \cdot \frac{d A'_{i,t+1} \ell}{r + z_{i,t+1} + \epsilon} - R_{it} = \frac{\eta R_{it} d \ell}{r + z_{i,t+1} + \epsilon} - R_{it}$$

Free entry condition:

$$R_{it}\left(\frac{\eta d \ell}{r + z_{i,t+1} + \epsilon} - 1\right) = 0 \quad \rightarrow \quad z_{i,t+1} = \eta d \ell - r - \epsilon$$

Note that in equlibrium the probability of a successful innovation will be the same for all intermediates.

General Equilibrium

Because the number of intermediate good types is large, the growth rate of the economy will be "smooth":

$$\Delta A_{t+1} = \frac{1}{M_t} \sum_{i=1}^{M_t} \Delta A_{i,t+1} = \frac{1}{M_t} \sum_{i=1}^{M_t} \left[(1+zq) A_{it} - A_{it}\right] = zq \cdot \frac{1}{M_t} \sum_{i=1}^{M_t} A_{it} = zq A_t$$

$$g_A = \frac{\Delta A_{t+1}}{A_t} = zq$$

The growth rate of the economy is given by the expression similar to one for the single-good case, however this time population size plays no role. Growth rate now depends ambiguously on the rate of population growth n:

$$g = \frac{\eta d \ell - \epsilon - \rho - n}{\sigma + 1/q} = \frac{\eta\left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} (\epsilon + n) / \psi - \epsilon - \rho - n}{\sigma + 1/q}$$

3 Innovation and capital accumulation

Suppose now that intermediate goods are produced using capital, which can accumulate over time.

Final goods

Production function:

$$Y_t = \left(\frac{L_t}{M_t}\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} = \ell_t^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

where as before $\ell_t \equiv L_t/M_t$ denotes workers per product line.

Profit maximization problem:

$$\max \quad \ell_t^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L_t - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \ell_t^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1} - p_{it} = 0 \quad \rightarrow \quad p_{it} = \alpha \left(A_{it}\ell_t\right)^{1-\alpha} x_{it}^{\alpha-1} \quad \rightarrow \quad x_{it} = \left(\alpha/p_{it}\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t$$

Intermediate goods

Now in order to produce one unit of an intermediate good, its producer needs to rent one unit of capital at capital rental rate r_t^k. Profit maximization problem:

$$\begin{aligned} \max \quad & D_{it} = p_{it} x_{it} - r_t^k x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it}\ell_t\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Incorporate the demand schedule:

$$\max \quad D_{it} = \alpha \left(A_{it}\ell_t\right)^{1-\alpha} x_{it}^{\alpha} - r_t^k x_{it}$$

First order condition:

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha \left(A_{it}\ell_t\right)^{1-\alpha} x_{it}^{\alpha-1}}_{p_{it}} - r_t^k = 0 \quad \rightarrow \quad \alpha p_{it} = r_t^k \quad \rightarrow \quad p_{it} = \frac{r_t^k}{\alpha}$$

Optimal level of production:

$$x_{it} = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t$$

The capital rental rate is determined in the market for capital, where the supply is the historically predetermined capital stock K_t and the demand is the sum of all intermediate goods demands:

$$K_t = \sum_{i=1}^{M_t} x_{it} = \sum_{i=1}^{M_t} \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} L_t \cdot \frac{1}{M_t} \sum_{i=1}^{M_t} A_{it} = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_t L_t$$

where $A_t \equiv \frac{1}{M_t} \sum_{i=1}^{M_t} A_{it}$ is the average productivity.

Denote with $\hat{k}$ the level of capital per effective labor:

$$\hat{k}_t \equiv \frac{K_t}{A_t L}$$

The level of capital rental rate depends negatively on the level of capital per effective labor:

$$K_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} A_t L \quad \rightarrow \quad \hat{k}_t = \left(\alpha^2/r_t^k\right)^{\frac{1}{1-\alpha}} \quad \rightarrow \quad r_t^k = \alpha^2 \hat{k}_t^{\alpha-1}$$

The optimal level of intermediate goods production can be expressed also as:

$$x_{it} = \left(\frac{\alpha^2}{\alpha^2 \hat{k}_t^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t = \left(\hat{k}_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}} A_{it}\ell_t = A_{it}\ell_t\hat{k}_t$$

Maximal profit:

$$D_{it} = \left(p_{it} - r_t^k\right) x_{it} = \left(\frac{1-\alpha}{\alpha}\right) r_t^k x_{it} = \left(\frac{1-\alpha}{\alpha}\right) \alpha^2 \hat{k}_t^{\alpha-1} \cdot A_{it}\ell_t\hat{k}_t = (1-\alpha)\,\alpha\hat{k}_t^{\alpha} \cdot A_{it}\ell_t \equiv d(\hat{k}_t)A_{it}\ell_t$$

Profits increase with capital per effective labor, because an increase in $\hat{k}_t$ reduces the monopolist's per-unit cost of production equal to the rental rate of capital r_t^k.

Final goods output is then given by a familiar Cobb-Douglas production function:

$$Y_t = \ell_t^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha} = \left(\frac{L_t}{M_t}\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} \left(A_{it}\ell_t\hat{k}_t\right)^{\alpha} = \hat{k}_t^{\alpha} L_t \cdot \frac{1}{M_t} \sum_{i=1}^{M_t} A_{it} = \hat{k}_t^{\alpha} L_t A_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

Innovation and Growth

As before, a successful innovation will increase the productivity of an intermediate good by $1+q$ and the success probability depends on R&D resources adjusted for target productivity:

$$z_{it} = \eta \frac{R_{it}}{A'_{i,t+1}}$$

Technically speaking, the following expression for the value of the firm is incorrect when the capital per effective labor changes over time, but we'll focus on the BGP anyway:

$$V\left(A_{it}\right) = \frac{d(\hat{k}^*)A_{it}\ell^*}{r(\hat{k}^*) + z^* + \epsilon}$$

The interest rate and capital rental rate are related by:

$$r(\hat{k}^*) = r^k(\hat{k}^*) - \delta = \alpha^2(\hat{k}^*)^{\alpha-1} - \delta$$

The expected net benefit of R&D activity is:

$$\eta \frac{R_{it}}{A'_{i,t+1}} \cdot V\left(A'_{i,t+1}\right) - R_{it} = \eta \frac{R_{it}}{A'_{i,t+1}} \cdot \frac{d(\hat{k}^*)A'_{i,t+1}\ell^*}{r(\hat{k}^*) + z^* + \epsilon} - R_{it} = \frac{\eta R_{it} d(\hat{k}^*)\ell^*}{r(\hat{k}^*) + z^* + \epsilon} - R_{it}$$

Free entry condition:

$$R_{it}\left(\frac{\eta d(\hat{k}^*)\ell^*}{r(\hat{k}^*) + z^* + \epsilon} - 1\right) = 0 \quad \rightarrow \quad z^* = \eta d(\hat{k}^*)\ell^* - r(\hat{k}^*) - \epsilon$$

The growth rate along the BGP now depends positively on capital per effective labor, since higher capital per effective labor means higher profits and lower interest rates:

$$g^* = z^* q = q\left[\eta d(\hat{k}^*)\ell^* - r(\hat{k}^*) - \epsilon\right] = q\left[\eta\,(1-\alpha)\,\alpha(\hat{k}^*)^{\alpha}\ell^* - \left(\alpha^2(\hat{k}^*)^{\alpha-1} - \delta\right) - \epsilon\right]$$

General Equilibrium

We will now refer to the Euler equation to obtain the second expression governing the growth rate:

$$g^* = g_c^* = \frac{r(\hat{k}^*) - \rho - n}{\sigma} = \frac{\alpha^2(\hat{k}^*)^{\alpha-1} - \delta - \rho - n}{\sigma}$$

The BGP level of capital per effective labor is then given by:

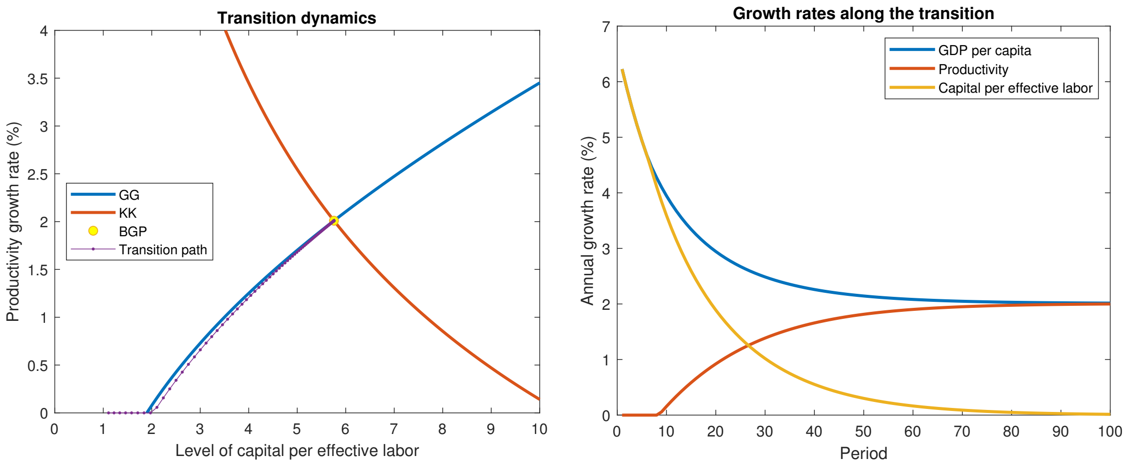
$$\hat{k}^* = \left(\frac{\alpha^2}{\rho + \delta + n + \sigma g^*}\right)^{1/(1-\alpha)}$$

Note that the BGP level of capital per effective labor depends negatively on the growth rate.

The BGP level of capital per effective labor and the BGP productivity growth rate are jointly determined and influence each other:

$$\begin{cases} g^* = q\left[\eta d(\hat{k}^*)\ell^* - r(\hat{k}^*) - \epsilon\right] & \text{BGP productivity growth rate (GG curve)} \\ \hat{k}^* = \left(\dfrac{\alpha^2}{\rho + \delta + n + \sigma g^*}\right)^{1/(1-\alpha)} & \text{BGP level of capital per effective labor (KK curve)} \end{cases}$$

Although this system does not have a closed-form solution, we can solve it numerically and produce a graphical illustration:[6]



When the initial level of capital per effective labor is below its BGP level, the growth rate of productivity is also lower than along the BGP as intermediate goods producers face higher production costs and the gains from engaging in R&D are smaller. Also initially the rate of growth of capital per effective labor contributes more to the growth of GDP per capita than productivity growth, but eventually productivity growth becomes the sole driver of growth in GDP per capita in the long run.

[6]The following parameter values were assumed: $\alpha = 0.33$, $\delta = 0.08$, $n = 0.01$, $s = 0.2$, $\ell = 15$ (average employment per firm), $q = 0.05$. To get $\ell = 15$, the rate of firm destruction was assumed at $\varepsilon = 0.12$ annually and $\psi = 0.00865$. Finally, $\eta = 0.085$ was chosen to match the 2% growth rate observed on average in the United States.

4 Technology Transfer and Cross-Country Convergence

We can also analyze the issue of international transfer of technology. Suppose that there exist two groups of countries: technology leaders and technology adopters.[7] The behavior of technology leaders is the same as described in the previous sections. Technology adopters enjoy an "advantage of backwardness" and can increase their productivity by adopting technologies developed in other countries. However, if a country does not innovate at all, then it will stagnate while the rest of the world continues to advance.

Productivity and Distance to Frontier

We will assume that the number of intermediate good types M is constant over time and the same in all countries, but each country might have access to different productivity / quality levels of these intermediate goods. A successful innovator / imitator in any sector gets to implement a technology with a productivity parameter equal to a level $\bar{A}_{it}$, which represents the world technology frontier in this sector and which grows at a rate $\bar{g}$ determined outside the country. Each sector's productivity parameter A_i will evolve according to:

$$A_{i,t+1} = \begin{cases} \bar{A}_{it} & \text{with probability } z \\ A_{it} & \text{with probability } 1-z \end{cases}$$

That is, in the fraction of sectors z productivity increases from A_{it} to $\bar{A}_{it}$, whereas in the remaining fraction productivity remains unchanged. The country's aggregate productivity evolves according to:

$$A_{t+1} = \frac{1}{M}\sum_{i=1}^{M} A_{i,t+1} = \frac{1}{M}\sum_{i=1}^{M}\left[z\bar{A}_{it} + (1-z)\,A_{it}\right] = z\sum_{i=1}^{M}\bar{A}_{it} + (1-z)\,\frac{1}{M}\sum_{i=1}^{M} A_{it} = z\bar{A}_t + (1-z)\,A_t$$

That is, in the fraction of sectors z that manage to innovate / imitate productivity jumps to level $\bar{A}_t$, whereas in the remaining fraction productivity remains the same as in period t.

The country's "proximity" to the world technology frontier is the ratio of its aggregate productivity to the global productivity frontier:

$$a_t = A_t/\bar{A}_t$$

and evolves according to:

$$\begin{aligned} A_{t+1} &= z\bar{A}_t + (1-z)\,A_t \quad | \quad : \bar{A}_t \\ \frac{A_{t+1}}{\bar{A}_t} = \frac{A_{t+1}}{\bar{A}_{t+1}}\frac{\bar{A}_{t+1}}{\bar{A}_t} &= z + (1-z)\,\frac{A_t}{\bar{A}_t} = z + (1-z)\,a_t \\ a_{t+1}\,(1+\bar{g}) &= z + (1-z)\,a_t \\ a_{t+1} &= \frac{z + (1-z)\,a_t}{1+\bar{g}} \end{aligned}$$

There is a unique steady-state proximity a^*, which can be found by setting $a_{t+1} = a_t = a^*$:

$$\begin{aligned} a^*\,(1+\bar{g}) &= z + (1-z)\,a^* \\ a^*\,(\bar{g}+z) &= z \\ a^* &= \frac{z}{\bar{g}+z} < 1 \end{aligned}$$

Once the steady-state proximity is reached, the country's productivity growth rate is given by:

$$g = \frac{A_{t+1} - A_t}{A_t} - 1 = \frac{z\bar{A}_t + (1-z)\,A_t}{A_t} - 1 = \frac{z}{a^*} + (1-z) - 1 = \bar{g} + z - z = \bar{g}$$

Therefore, all technology adopters that innovate ($z > 0$) will converge to the same growth rate, although their steady-state proximity to the technology frontier may differ due to different z.

[7] This is a simplifying assumption. In reality this distinction is not strict, as even highly developed countries are technology adopters in some industries.

Convergence and Divergence

Recall the formula for the probability of innovating / imitating z (whenever the formula would yield negative values, a country does not innovate at all and sets $z = 0$):

$$z = \eta\Pi(\hat{k}) - r(\hat{k})$$

Let us focus first on those technology adopters that innovate / imitate ($z > 0$). Another "advantage of backwardness" is that the growth rate of productivity is faster the further behind the technology frontier a country is. The average innovation size is given by:

$$q_{t+1} \equiv \frac{\bar{A}_t - A_t}{A_t} = \frac{1}{a_t} - 1$$

And the country's productivity growth rate is:

$$g_t = zq_t = z\left(\frac{1}{a_{t-1}} - 1\right)$$

Therefore, the further behind the frontier the country is, the higher its productivity growth rate will be, conditional on $z > 0$. This fact limits how far behind the frontier a country can fall, because eventually it will get so far behind that its growth rate will be just as large as the growth rate of the frontier, at which point the gap will stop increasing.

However, if countries do not innovate at all ($z = 0$), maybe due to poor macroeconomic conditions, legal environment, education system, or credit markets, or simply due to a low level of capital per effective labor, they will not partake in technology transfers, but will instead stagnate. If this situation persists, their productivity level will remain constant and they will diverge from the club of innovating countries.

Together these two results help to explain the empirical fact that there is a group of countries that are converging to parallel growth paths (i.e., with identical long-run growth rates) and another group of countries that are falling further and further behind. Notice that even countries that are converging to parallel growth paths are not necessarily converging in levels. That is, one country's steady-state proximity to the frontier a^* can differ from another's if they have different values of the critical parameters governing the intensity of R&D.

This result helps us to account for the fact that there are systematic and persistent differences across countries in the level of productivity. That is, convergence in levels is not absolute but conditional. In our model, two countries will end up with the same productivity levels in the long run if they share identical parameter values, but not otherwise.