UNIVERSITY OF WARSAW
Faculty of Economic Sciences

Models of inequality

Advanced Macroeconomics IE

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Spring 2025

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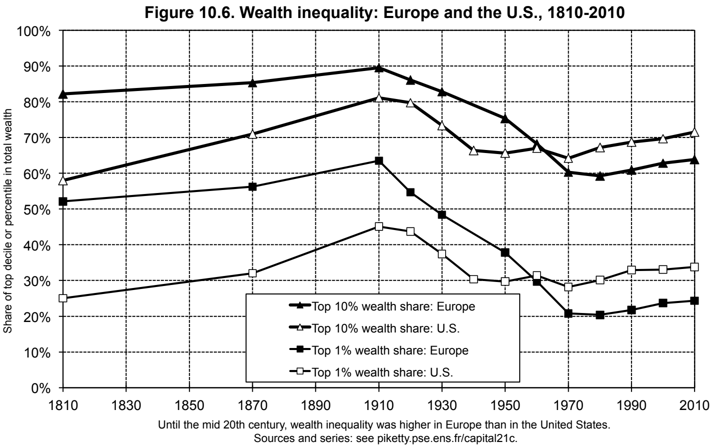
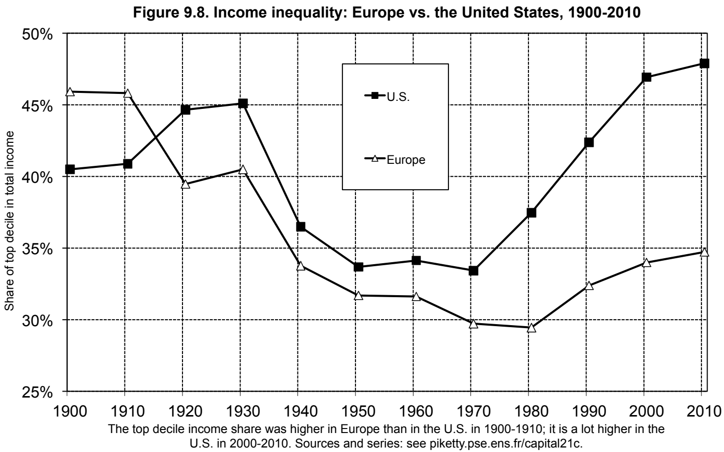
Evolution of top wealth



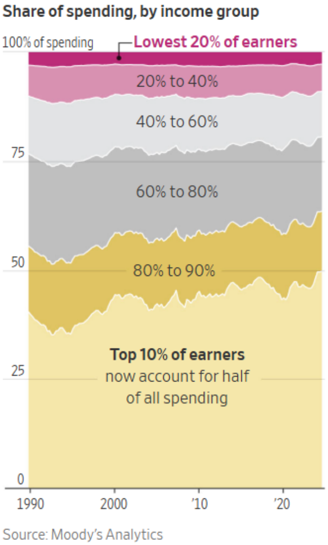
Piketty (2014) Capital in the Twenty-First Century

Evolution of top incomes



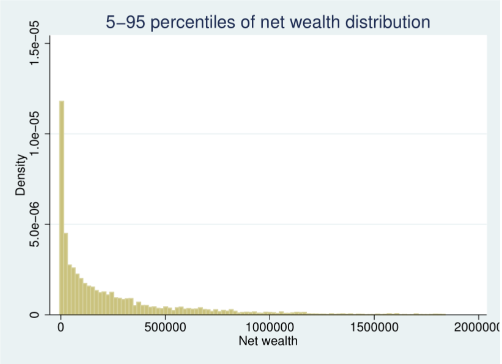
Piketty (2014) Capital in the Twenty-First Century

Share of overall consumption spending by income



Model of (top) wealth inequality: Jones (2015)

US wealth distribution

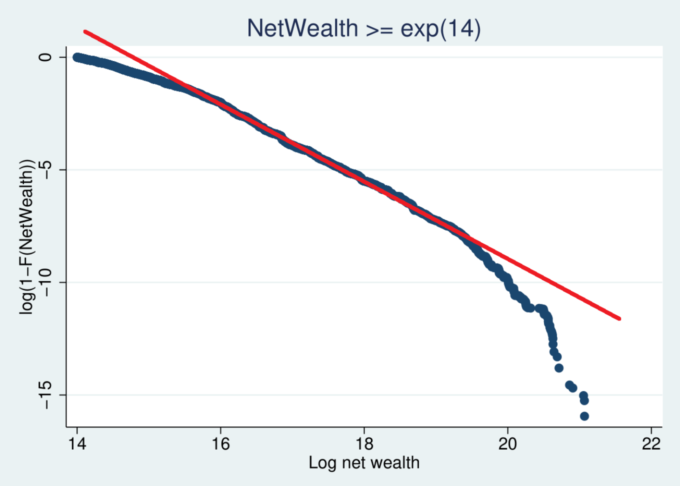


Ben Moll's lecture

Features of US Wealth Distribution:

- right skewness
- heavy upper tail, well approximated by a Pareto distribution

US wealth distribution



Ben Moll's lecture

Properties of Pareto distribution

When a variable (e.g. wealth) is Pareto distributed, it satisfies:

$$\text{Pr}\left[\text{wealth} > a\right] = \left(a/a_{min}\right)^{-1/\eta}$$

which means the fraction of people with wealth greater than some cutoff is proportional to the cutoff raised to some power

Under Pareto distribution, computation of "top shares" is easy. The fraction of wealth going to the top p percentiles is given by:

$$(100/p)^{\eta-1}$$

The higher η is, the more unequal the distribution:

- For $\eta = 0.5$ the top 1% wealth share is $100^{-0.5} = 10\%$
- For $\eta = 0.75$ the top 1% wealth share is $100^{-0.25} \approx 32\%$
- Piketty (2014): in the US the top 1% wealth share $\approx 33\%$, in UK and France between 25% and 30%

Pareto wealth distribution: core intuition

Jones (2015): Assume (for now) that the size of population does not change

Suppose households (or dynasties) face a constant probability of death d

Then the probability that an individual is of at least age x is:

$$\text{Pr}\left[\text{age} > x\right] = (1-d)^x \simeq e^{-dx}$$

Assume (for now) that everyone receives at birth the same initial wealth $= 1$

Let the wealth of households (dynasties) increase with age at rate μ:

$$a\left(x\right) = (1+\mu)^x \simeq e^{\mu x} \quad \rightarrow \quad x\left(a\right) = (1/\mu) \cdot \ln a$$

Then we can easily map the probability of holding at least
some amount of wealth to the probability of being old enough:

$$\text{Pr}\left[\text{wealth} > a\right] = \text{Pr}\left[\text{age} > x\left(a\right)\right] = \exp\left(-\left(d/\mu\right) \cdot \ln a\right) = a^{-d/\mu}$$

Wealth is Pareto distributed with $\eta = \mu/d$

Demographics

Maintain the assumption of constant death probability

Allow population size to change over time

Define a (crude) birth rate $b_t \equiv B_t / N_t$ and assume it's constant

Population growth rate n is the difference
between crude birth and death rates:

$$n = b - d \quad \rightarrow \quad b = n + d$$

Share of people aged x in the population is given by:

$$sh\left(x\right) = b\left(\frac{1-d}{1+n}\right)^x \simeq b\left(1-d-n\right)^x = b\left(1-b\right)^x \simeq be^{-bx}$$

Probability that a person is at least of age x:

$$\Pr\left[\text{age} > x\right] = \int_x^\infty be^{-bt}\,\mathrm{d}t = e^{-bx}$$

Households' choice

Households solve the following utility maximization problem:

$$\max \quad U = \sum_{t=0}^{\infty} \left[\beta\left(1-d\right)\right]^t \ln c_t$$

$$\text{subject to} \quad a_{t+1} = \left(1+r-\tau\right) a_t - c_t$$

where households do not receive any labor income and τ is a tax on wealth

Euler equation:

$$c_{t+1} = \beta\left(1-d\right)\left(1+r-\tau\right) c_t$$

Guess-and-verify that households consume a fixed fraction α of their wealth:

$$\begin{aligned} \alpha a_{t+1} &= \beta\left(1-d\right)\left(1+r-\tau\right)\alpha a_t \\ \alpha\left[\left(1+r-\tau\right)a_t - \alpha a_t\right] &= \beta\left(1-d\right)\left(1+r-\tau\right)\alpha a_t \\ \left(1+r-\tau\right) - \alpha &= \beta\left(1-d\right)\left(1+r-\tau\right) \end{aligned}$$

Wealth dynamics

Budget constraint then determines the dynamics of wealth:

$$a_{t+1} = (1 + r - \tau - \alpha)\, a_t \equiv (1 + \mu)\, a_t \quad \rightarrow \quad a_t = (1 + \mu)^t\, a_0$$

Let $a_t(x)$ denote the wealth of a person aged x at time period t:

$$a_t(x) = (1 + \mu)^x\, a_{t-x}(0)$$

Assume that newly born agents inherit wealth of the deceased:

$$a_t(0) = \frac{dK_t}{B_t} = \frac{dK_t}{bN_t} = \frac{d}{b} k_t$$

Assume the BGP economy with exogenous technological progress:

$$k_t = (1 + g)^t\, k_0 \quad \rightarrow \quad k_t = (1 + g)^x\, k_{t-x}$$

Wealth inherited by newborns in period $t - x$:

$$a_{t-x}(0) = \frac{d}{b} k_{t-x} = \frac{d}{b} (1 + g)^{-x}\, k_t$$

Wealth distribution

Wealth of people aged x at time period t:

$$a_t\left(x\right) = \left(1+\mu\right)^x \cdot \frac{d}{b}\left(1+g\right)^{-x} k_t \simeq \frac{d}{b} k_t \cdot e^{(\mu-g)x}$$

Age x needed to accumulate wealth a:

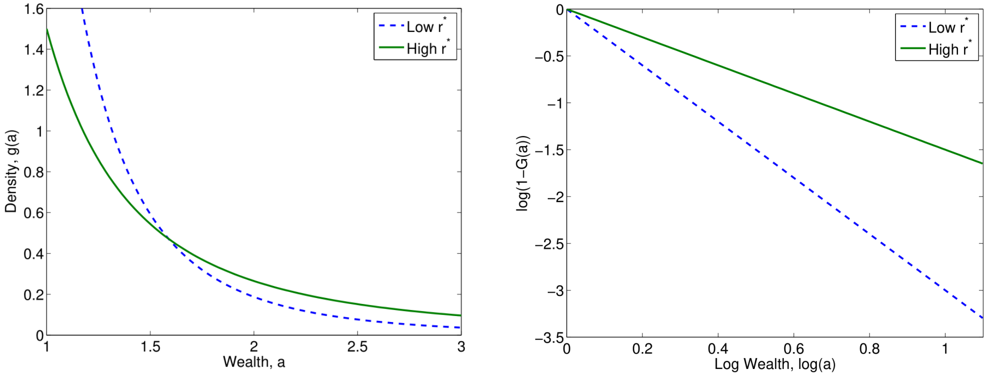
$$x\left(a_t\right) = \frac{1}{\mu-g} \cdot \ln\left[\frac{a_t}{\left(d/b\right) k_t}\right]$$

Probability of holding wealth of at least a is then given by:

$$\begin{aligned}\text{Pr}\left[\text{wealth} > a\right] &= \text{Pr}\left[\text{age} > x\left(a\right)\right] = e^{-bx(a)} \\ &= \exp\left[-\frac{b}{\mu-g} \cdot \ln\left[\frac{a_t}{\left(d/b\right) k_t}\right]\right] = \left[\frac{a_t}{\left(d/b\right) k_t}\right]^{-\frac{b}{\mu-g}}\end{aligned}$$

Wealth is Pareto distributed with $\eta = \dfrac{\mu-g}{b} = \dfrac{r-\tau-\alpha-g}{n+d}$

Wealth distribution (Partial Equilibrium)



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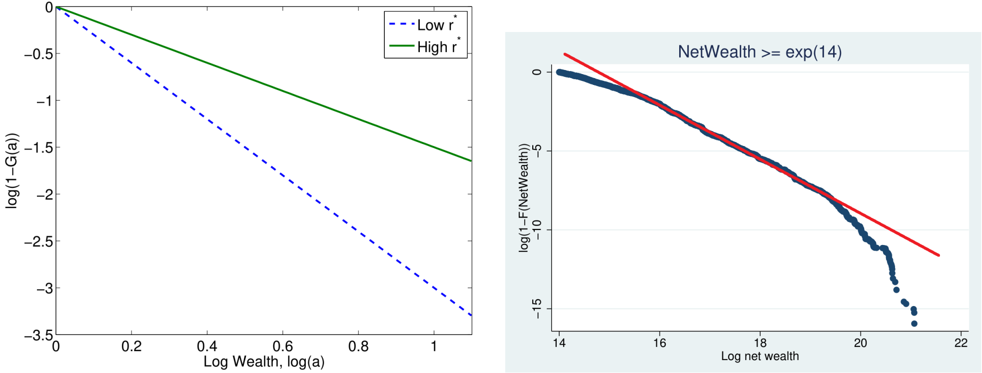
Wealth distribution (Partial Equilibrium)



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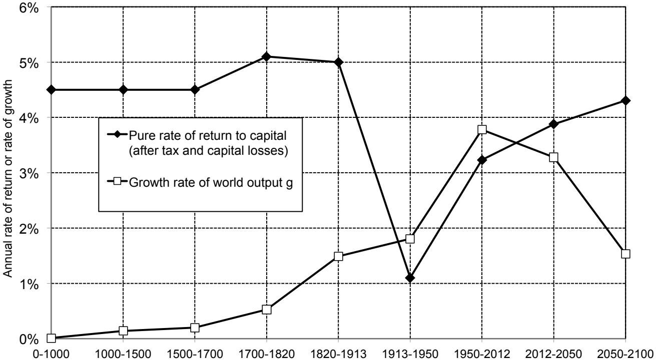
Piketty (2014): importance of $r\ (-\tau) - g\ (-n)$

Figure 10.10. After tax rate of return vs. growth rate at the world level, from Antiquity until 2100



The rate of return to capital (after tax and capital losses) fell below the growth rate during the 20th century, and may again surpass it in the 21st century. Sources and series : see piketty.pse.ens.fr/capital21c

Piketty (2014) Capital in the Twenty-First Century

Wealth inequality (Partial Equilibrium)

Wealth is Pareto distributed with $\eta = \dfrac{r - \tau - \alpha - g}{n + d}$

Those lucky to live a long life (members of long-lived dynasties) will accumulate greater stocks of wealth

Recall that the higher η is, the more unequal the distribution

Piketty (2014): increase in $r - g\,(-n)$ increases wealth inequality

19th century: low g and low $n \rightarrow$ high inequality

Middle 20th century: high g and $n \rightarrow$ low inequality

21st century: declining g and $n \rightarrow$ back to 19th century (?)

Piketty's prescription: increase τ to counteract g and n

Wealth inequality (General Equilibrium)

Relationship between aggregate capital and individual wealth:

$$K_t = \sum_{x=0}^{\infty} sh\left(x\right) N_t \cdot a_t\left(x\right) = \sum_{x=0}^{\infty} b\left(1-b\right)^x N_t \cdot \frac{dk_t}{b}\left(1+\mu-g\right)^x$$

$$\simeq dK_t \sum_{x=0}^{\infty}\left(1+\mu-g-b\right)^x = \frac{dK_t}{1-\left(1+\mu-g-b\right)}$$

Real interest rate under General Equilibrium is given by:

$$d = -\left(\mu-g-b\right) = -\left(r-\tau-\alpha-g-d-n\right)$$

$$r = n+g+\tau+\alpha$$

Wealth inequality coefficient under General Equilibrium:

$$\eta = \frac{r-\tau-\alpha-g}{n+d} = \frac{n+g+\tau+\alpha-g-\tau-\alpha}{n+d} = \frac{n}{n+d}$$

Wealth inequality is determined purely by demography!

Takeaway

$$\eta^{PE} = \frac{r - g - \tau - \alpha}{n + d} \quad \text{vs} \quad \eta^{GE} = \frac{n}{n + d}$$

If wealth tax is redistributed in lump-sum, then $\eta^{GE} = \frac{n - \tau}{n + d}$

Piketty is right to highlight the link between $r - g$, population growth, taxes and top wealth inequality (under PE)

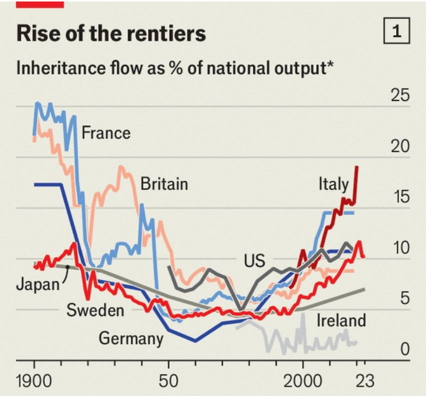
But these results are fragile and can disappear under GE

All above results hinge on the assumptions regarding inheritance

Need for richer framework, including bequests, social mobility, progressive taxation, micro- and macroeconomic shocks, and multiple risk-return asset portfolios

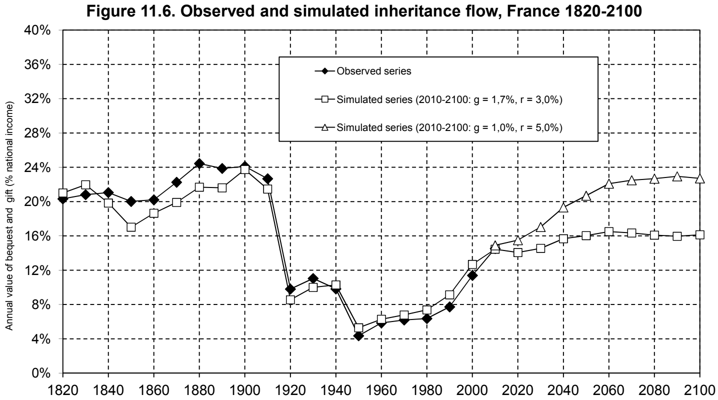
More research needed (empirics & theory)

Inheritance flow is key



*Measure of output varies by country
Sources: Acciari & Morelli, 2020; Alvaredo, Garbinti & Piketty, 2017; Arrigoni, Boyd & McIndoe-Calder, 2024; Bauluz & Meyer, 2024; Gabbuti & Morelli, 2025; Guzzardi & Morelli, work in progress; OECD; Ohlsson, Roine & Waldenström, 2020; Sánchez-Romero, Ogawa & Matsukura, 2013; Schinke, 2012; *The Economist*

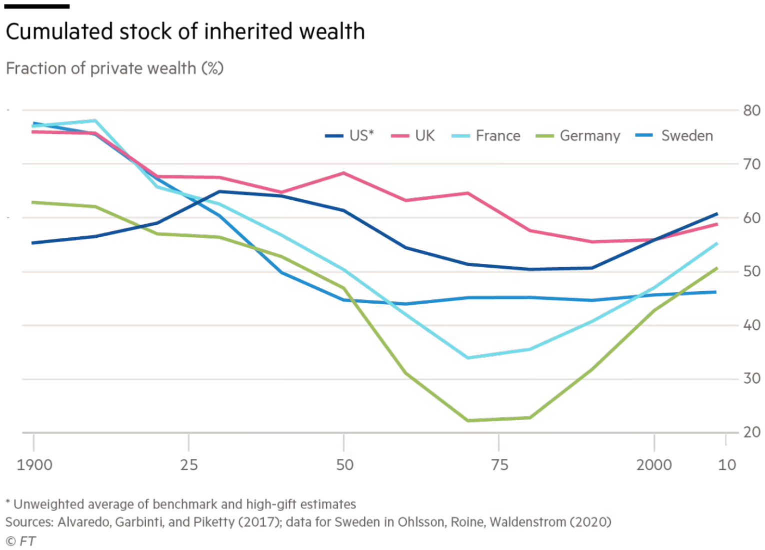
Inheritance flow is key



Simulations based upon the theoretical model indicate that the level of the inheritance flow in the 21st century will depend upon the growth rate and the net rate of return to capital. Sources and series: see piketty.pse.ens.fr/capital21c.

Piketty (2014) Capital in the Twenty-First Century

Inheritance flow is key



Financial Times

Income and wealth inequality: De Nardi (2015)

Basic infinitely-lived Bewley model

Framework proposed by Bewley (1977)

Labor market status z_t (e.g. $z_t = \{0, 1\}$) evolves according to the transition matrix P (with stationary distribution $\bar{P}$)

Households want to maximize lifetime expected utility:

$$
\begin{aligned}
\max_{\{c_t\}_{t=0}^{\infty}} \quad & U = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right] \\
\text{subject to} \quad & c_t + a_{t+1} = z_t w + \left(1 + r\right) a_t \\
& a_{t+1} \geq \underline{a} \\
& z_{t+1} \sim P\left(z_t\right)
\end{aligned}
$$

Solution: infinite sequence of consumption plans $\{c_t\}_{t=0}^{\infty}$

Can rewrite this problem as: choosing today's consumption and tomorrow's assets only, conditional on today's assets and labor market status

Recursive formulation of household's problem

We can rewrite the utility function into the value function:

$$V\left(a_t, z_t\right) = \max_{c_t,\, a_{t+1}} \left\{u\left(c_t\right) + \beta \mathrm{E}_t\left[V\left(a_{t+1}, z_{t+1}\right)|z_t\right]\right\}$$

$$\text{subject to} \quad c_t + a_{t+1} = z_t w + \left(1+r\right) a_t$$

$$a_{t+1} \geq \underline{a}$$

Even more compactly:

$$V\left(a_t, z_t\right) = \max_{a_{t+1} \geq \underline{a}} \left\{u\left(z_t w + \left(1+r\right) a_t - a_{t+1}\right) + \beta \mathrm{E}_t\left[V\left(a_{t+1}, z_{t+1}\right)|z_t\right]\right\}$$

Solution is the policy function A which maps from $\left(a_t, z_t\right)$ to a_{t+1}:

$$a_{t+1} = A\left(a_t, z_t\right)$$

We can also use the budget constraint to obtain
the policy function C which maps from $\left(a_t, z_t\right)$ to c_t:

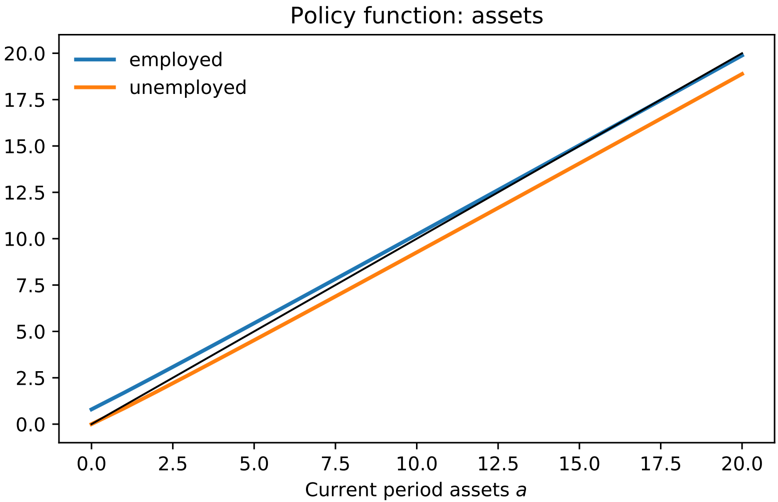
$$c_t = C\left(a_t, z_t\right) = z_t w + \left(1+r\right) a_t - A\left(a_t, z_t\right)$$

Numerical example

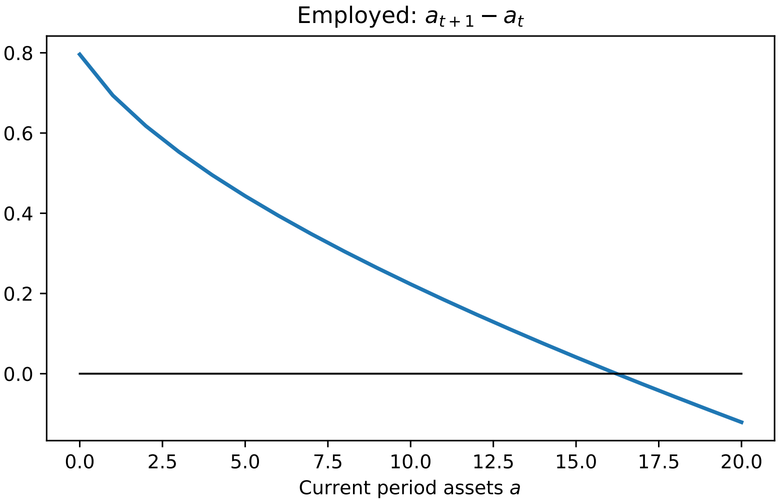
Example model:

- households have low ("unemployed") or high ("employed") labor productivity
- low productivity is 10% of high productivity
- $z = [0.1,\ 1]$, $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$, $\bar{P} = [0.5,\ 0.5]$
- borrowing constraint $\underline{a} = 0$
- $u\left(c\right) = \ln c$, $\beta = 0.96$
- $r = 2\%$ (Partial Equilibrium interest rate)

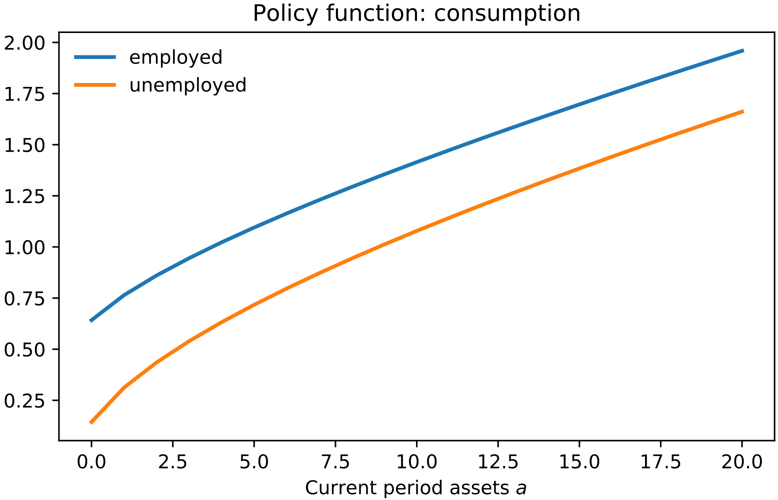
Policy functions (Partial Equilibrium)



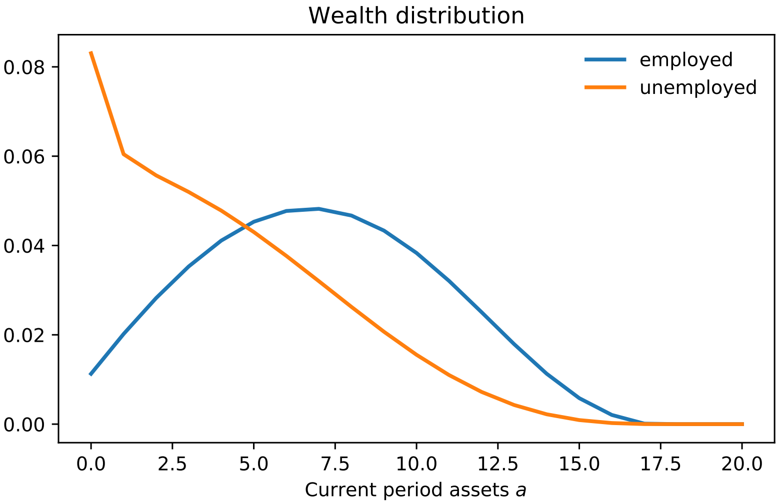
Policy functions (Partial Equilibrium)



Policy functions (Partial Equilibrium)



Wealth distribution (Partial Equilibrium)



General Equilibrium

Households and firms take prices w and r as given

Assume standard production function:

$$Y = K^{\alpha} L^{1-\alpha}$$

Prices depend on the supply of factors of production:

$$L = N \cdot z\bar{P}$$

$$K = \int_{\underline{a}}^{\infty} a \, dg\left(a\right)$$

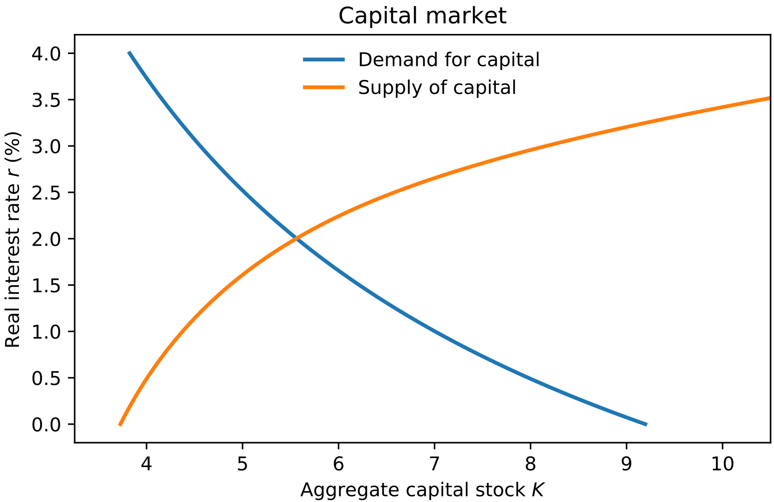
Expressions for prices:

$$w = \alpha K^{\alpha} L^{-\alpha}$$

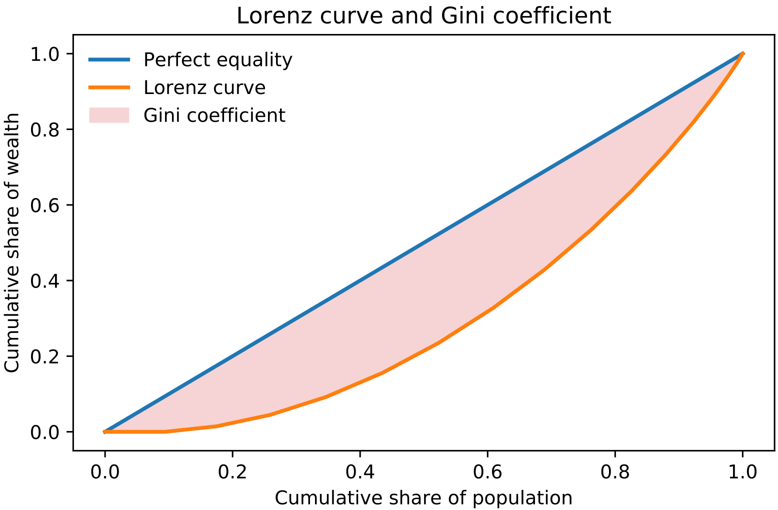
$$r = \left(1-\alpha\right) K^{\alpha-1} L^{1-\alpha} - \delta$$

Market clearing: equalize capital supply (households) with capital demand (firms)

Capital market equilibrium



Lorenz curve and Gini coefficient for wealth



Some issues

Since under General Equilibrium $\beta(1+r) < 1$, households do not want to save without bound (good for computational reasons)

Households are willing to hold positive assets because:

- there is a borrowing constraint
- they don't want to have low assets when unemployed $\rightarrow$ very low consumption
- no reason to increase assets if this possibility is small and in distant future

Hard to generate households with very high wealth

Inequality does not matter much for aggregate outcomes:

- policy functions close to linear
- households with low assets have low consumption
  $\rightarrow$ impact on aggregate consumption small
- not that many of borrowing-constrained households

Aiyagari (1994)

Aiyagari (1994) approximates the earnings of US workers by an AR(1) process:

$$\ln z_t = \rho \ln z_{t-1} + \varepsilon_t$$

Autocorrelation $\rho = 0.6$ and standard deviation $\sigma_\varepsilon = 0.2$

Earning levels discretized to 7 possible values

	% wealth in top		
Gini	1%	5%	20%
U.S. data, 1989 SCF			
.78	29	53	80
Aiyagari Baseline			
.38	3.2	12.2	41.0
Aiyagari higher variability			
.41	4.0	15.6	44.6

Huggett (1996)

Huggett (1996): overlapping generations variant of the Bewley model

Households can live for up to T periods and face age-dependent survival probability ω

Value function is age-dependent:

$$V_t(a_t, z_t) = \max_{c_t,\, a_{t+1}} \{u(c_t) + \beta\omega_{t+1}\mathrm{E}_t[V_{t+1}(a_{t+1}, z_{t+1})\,|z_t]\}$$

$$\text{subject to} \quad c_t + a_{t+1} = e_t(z_t)\,w + (1+r)\,a_t + b_t$$

$$a_{t+1} \geq \underline{a}$$

where b_t are bequest from the deceased (redistributed equally)
plus Social Security payments to retirees

Partial Equilibrium very easy to solve for since V_T is known

De Nardi (2004)

De Nardi (2004): Huggett model with intergenerational links

- voluntary bequests from parents to children (utility from giving)
- transmission of labor productivity from parents to children

Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	40%	60%	Percentage with negative or zero wealth
			U.S. data, 1989 SCF				
.60	.78	29	53	80	93	98	5.8–15.0
			Equal bequests to all (Huggett)				
.67	.67	7	27	69	90	98	17
			Unequal bequests to children (unintentional)				
.38	.68	7	27	69	91	99	17
			Parent's bequest motive				
.55	.74	14	37	76	95	100	19
			Parent's bequest motive and productivity inheritance				
.60	.76	18	42	79	95	100	19

Lifetime wealth profiles

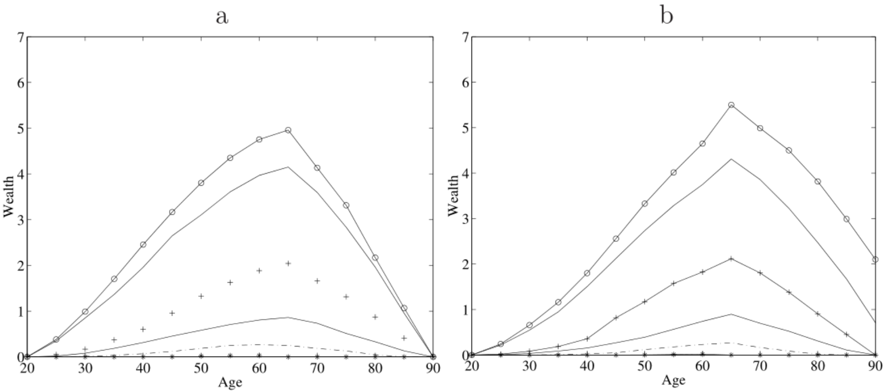


Figure 2: Wealth .1, .3, .5, .7, .9, .95 quantiles. No links, equal bequests to all, panel a, and Bequest motive, panel b .

De Nardi (2015)

Cagetti and De Nardi (2006)

Entrepreneurs: households who declare being self-employed, own a privately held business (or a share of one), and have an active management role in it

Small fraction of the population, but hold a large share of wealth

Top %	1	5	10	20
Whole population				
percentage of total net worth held	30	54	67	81
Entrepreneurs				
percentage of households in a given percentile	63	49	39	28
percentage of net worth held in a given percentile	68	58	53	47

Cagetti and De Nardi (2006)

Altruistic agents care about their children

Agents decide whether to run a business or work for a wage

Entrepreneurial production function depends
on entrepreneurial ability and working capital

Borrowing for working capital is constrained by agents' assets

Rationale for holding high levels of wealth

Wealth	Fraction of	Percentage wealth in the top			
Gini	entrepreneurs	1%	5%	20%	40%
U.S. data					
0.78	7.55%	30	54	81	94
Baseline model with entrepreneurs					
0.78	7.50%	31	60	83	94

Takeaway

De Nardi (2015): Bequests and entrepreneurship generate observed wealth inequality

Changes in some assumptions can yield vastly different welfare effects of policies!

Ahn et al. (2017): To get macroeconomic effects of inequality, two assets are needed:

- "wealthy hand-to-mouth" agents: e.g. low liquid assets and a (mortgaged) house
- consumption choices of these agents matter for aggregate consumption, as they consume a lot and are a significant fraction of the population