Marcin Bielecki, Advanced Macroeconomics IE, Spring 2025
New Keynesian model: problems and solutions

Problem 1

Suppose that consumers have the "love of variety" utility function and would like to consume a bit out of every existing goods variety. It is a bit easier to solve the analogous problem of a (perfectly competitive) firm that buys separate goods and repackages them as a final consumption good, purchased by the consumers. This problem can be stated as:

$$\begin{aligned} \max \quad & Py - \int_0^1 P_i y_i \,\mathrm{d}i \\ \text{subject to} \quad & y = \left(\int_0^1 y_i^{\frac{\epsilon-1}{\epsilon}} \,\mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}} \end{aligned}$$

where y is the aggregate real expenditure, P is the aggregate price index and $\epsilon > 1$ is the elasticity of substitution between varieties.

(a) Set up a Lagrangian and derive the first order conditions with respect to y and y_i. Obtain the demand function for a representative i-th good. Hint: figure out how to use the "production function" to simplify the FOC for y_i.

Lagrangian

$$\mathcal{L} = PY - \int_0^1 P_i Y_i \,\mathrm{d}i + \lambda \left[\left(\int_0^1 Y_i^{\frac{\epsilon-1}{\epsilon}} \,\mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}} - Y \right]$$

FOCs

$$\begin{aligned} Y \quad &: \quad P - \lambda = 0 \\ Y_i \quad &: \quad -P_i + \lambda \left[\frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_i^{\frac{\epsilon-1}{\epsilon}} \,\mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1} - 1} \cdot \frac{\epsilon - 1}{\epsilon} Y_i^{\frac{\epsilon-1}{\epsilon} - 1} \right] \end{aligned}$$

$$\begin{aligned} P_i &= P \left[\left(\int_0^1 Y_i^{\frac{\epsilon-1}{\epsilon}} \,\mathrm{d}i \right)^{\frac{1}{\epsilon-1}} Y_i^{-\frac{1}{\epsilon}} \right] \\ P_i &= P \left[Y^{\frac{1}{\epsilon}} Y_i^{-\frac{1}{\epsilon}} \right] \\ \frac{P_i}{P} Y^{-\frac{1}{\epsilon}} &= Y_i^{-\frac{1}{\epsilon}} \\ Y_i &= (P_i/P)^{-\epsilon} \, Y \end{aligned}$$

(b) Use the condition that the final goods producers' profits are 0 to derive the formula for the aggregate price index P as a function of all P_i.

$$PY = \int_0^1 P_i Y_i \,\mathrm{d}i$$

$$PY = \int_0^1 P_i P_i^{-\epsilon} P^{\epsilon} Y \,\mathrm{d}i$$

$$P^{1-\epsilon} = \int_0^1 P_i^{1-\epsilon} \,\mathrm{d}i$$

$$P = \left(\int_0^1 P_i^{1-\epsilon} \,\mathrm{d}i \right)^{\frac{1}{1-\epsilon}}$$

(c) The profit of the producer of good i is given by $D_i = (P_i - MC_i)\, y_i$. Use the demand function for good i and find the optimal price P_i. Verify that for $\epsilon = \infty$ (perfect competition) the optimal price is equal to the marginal cost of production.

$$\begin{aligned} \max \quad & D_i = (P_i - MC_i)\, Y_i \\ \text{subject to} \quad & Y_i = (P_i/P)^{-\epsilon}\, Y \end{aligned}$$

Lagrangian

$$\mathcal{L} = P_i Y_i - MC_i Y_i + \lambda \left[(P_i/P)^{-\epsilon}\, Y - Y_i \right]$$

FOCs

$$\begin{aligned} P_i \quad &: \quad Y_i + \lambda \left(Y_i \cdot -\frac{\epsilon}{P_i} \right) = 0 \rightarrow \lambda = \frac{P_i}{\epsilon} \\ Y_i \quad &: \quad P_i - MC_i - \lambda = 0 \end{aligned}$$

$$\begin{aligned} P_i - \frac{P_i}{\epsilon} &= MC_i \\ P_i &= \frac{\epsilon}{\epsilon - 1} MC_i \end{aligned}$$

For $\epsilon \rightarrow \infty$ we have $P_i \rightarrow MC_i$. Note that the parameter μ that we used in class is related to ϵ as follows: $1 + \mu = \frac{\epsilon}{\epsilon - 1}$.

Problem 2

Consider the Rotemberg scheme where costs of price changes resulted in the NKPC. Assume that only unexpected price increases are associated with loss of customer loyalty, but that anticipated price changes are considered "fair". Hence the costs of price changes are given by:

$$\phi \left(p_t - p_t^e \right)^2$$

Further, assume that customers have following price expectations:

$$p_t^e = p_{t-1} + \pi_{t-1}$$

The loss function of the firm is given by:

$$L = \sum_{j=0}^{\infty} \beta^j \cdot \mathrm{E}_t \left[\left(p_{t+j} - p_{t+j}^* \right)^2 + \phi \left(p_{t+j} - p_{t+j}^e \right)^2 \right]$$

Derive the modified equation for the NKPC. What are the similarities and differences between the derived equation and the standard NKPC?

Solution
Loss function

$$\begin{aligned} L &= \sum_{j=0}^{\infty} \beta^j \cdot \mathrm{E}_t \left[\left(p_{t+j} - p_{t+j}^* \right)^2 + \phi \left(p_{t+j} - p_{t+j-1} - \pi_{t+j-1} \right)^2 \right] \\ &= \left(p_t - p_t^* \right)^2 + \phi \left(p_t - p_{t-1} - \pi_{t-1} \right)^2 \\ &+ \beta \mathrm{E}_t \left[\left(p_{t+1} - p_{t+1}^* \right)^2 + \phi \left(p_{t+1} - p_t - \pi_t \right)^2 \right] + \ldots \end{aligned}$$

Minimizing the loss function (each firm treats overall inflation rate π as exogenous)

$$\frac{\partial L}{\partial p_t} = 2 \left(p_t - p_t^* \right) + 2\phi \left(p_t - p_{t-1} - \pi_{t-1} \right) + \beta \mathrm{E}_t \left[2\phi \left(p_{t+1} - p_t - \pi_t \right) (-1) \right] = 0$$

$$\frac{1}{\phi}(p_t - p_t^*) + (p_t - p_{t-1} - \pi_{t-1}) = \beta \mathrm{E}_t [p_{t+1} - p_t - \pi_t]$$
$$\frac{1}{\phi}(p_t - p_t^*) + \pi_t - \pi_{t-1} = -\beta\pi_t + \beta \mathrm{E}_t \pi_{t+1}$$

$$(1+\beta)\pi_t = \pi_{t-1} + \beta \mathrm{E}_t \pi_{t+1} - \frac{1}{\phi}(p_t - p_t^*)$$

Modified NKPC

$$\pi_t = \frac{1}{1+\beta}\pi_{t-1} + \frac{\beta}{1+\beta}\mathrm{E}_t \pi_{t+1} + \frac{1}{\phi}(p_t^* - p_t)$$

The second and third term appear also in the standard NKPC, although here the weight of inflation expectations is reduced. The new backward-looking component makes heightened inflation more persistent, implying that reduction of inflation requires a more negative output gap than under a standard NKPC.

Problem 3

Suppose that you have the following simplified New Keynesian model. The two main non-policy equations of the model can be written:

$$x_t = \mathrm{E}_t x_{t+1} - (i_t - \mathrm{E}_t \pi_{t+1})$$
$$\pi_t = x_t + \mathrm{E}_t \pi_{t+1}$$

The central bank obeys a strict inflation targeting rule. In particular, let π_t^* be an exogenous inflation target. The central bank will adjust i_t so that $\pi_t = \pi_t^*$ is consistent with these equations holding. Assume that the inflation target follows an exogenous AR(1) process:

$$\pi_t^* = \rho_\pi \pi_{t-1}^* + \varepsilon_t, \quad \rho_\pi \in [0,1]$$

(a) Derive an analytic expression for i_t as a function of π_t^*. Hint: use the inflation target rule to pin down inflation and output gap expectations.

Strict inflation target

$$\mathrm{E}_t \pi_{t+1} = \mathrm{E}_t [\rho_\pi \pi_t^* + e_{t+1}] = \rho_\pi \pi_t^* = \rho_\pi \pi_t$$

Output gap

$$x_t = \pi_t - \mathrm{E}_t \pi_{t+1} = (1-\rho_\pi)\pi_t$$

Expected output gap

$$\mathrm{E}_t x_{t+1} = (1-\rho_\pi)\mathrm{E}_t \pi_{t+1} = (1-\rho_\pi)\rho_\pi \pi_t$$

NKIS curve

$$x_t = \mathrm{E}_t x_{t+1} - (i_t - \mathrm{E}_t \pi_{t+1})$$
$$i_t = \mathrm{E}_t x_{t+1} - x_t + \mathrm{E}_t \pi_{t+1}$$
$$i_t = (1-\rho_\pi)\rho_\pi \pi_t - (1-\rho_\pi)\pi_t + \rho_\pi \pi_t$$
$$i_t = [\rho_\pi + (1-\rho_\pi)(\rho_\pi - 1)]\pi_t$$

(b) Suppose that ρ_π is 0. In which direction must the central bank adjust i_t in order to achieve an increase in π_t^*?

Desired inflation level

$$\pi_t = e_t$$

This approximates regular monetary policymaking, when the central bank wants to achieve inflation a little higher or lower relative to the time-invariant inflation target (here implicitly equal to 0).

$$i_t = [0 + (1-0)(0-1)]\pi_t = -\pi_t$$

To reduce inflation ($\pi_t < 0$), the central bank needs to raise the nominal interest rate i.

(c) Suppose that ρ_π is 1. In which direction must the central bank adjust i_t in order to achieve an increase in π_t^*?

Desired inflation level

$$\pi_t = \pi_{t-1} + e_t$$

This approximates the situation where the central bank changes the inflation target over time.

$$i_t = [1 + (1-1)(1-1)]\,\pi_t = \pi_t$$

Thanks to the lower average inflation rate, the central bank can maintain a lower nominal interest rate on average.

(d) Provide intuition behind the difference in results in (b) and (c).

The result from (b) is about the short-term movements in inflation rate, which is a negative function of interest rates. The result from (c) states that under unchanged natural real interest rate level, average interest rates are 1:1 related to the inflation target. One can also reaffirm this result via the Fischer equation:

$$(1 + i \downarrow) = (1 + r^*)(1 + \pi \downarrow) \quad \rightarrow \quad i \downarrow \approx r^* + \pi \downarrow$$

Problem 4

Consider a simplified New Keynesian model:

$$x_t = \mathrm{E}_t x_{t+1} - (i_t - \mathrm{E}_t \pi_{t+1} - r^*) + u_t$$
$$\pi_t = \mathrm{E}_t \pi_{t+1} + x_t$$

In time period t the economy is affected by a strong negative demand shock: $u_t = -r^* - v$, which lasts for one period only. The central bank, subject to the zero lower bound constraint, sets $i_t = 0$. After the shock recedes, in time period $t+1$ it will be possible to set $x_{t+1} = \pi_{t+1} = 0$. Additionally, the central bank credibly commits to maintain $x_{t+k} = \pi_{t+k} = 0$ for all $k \geq 2$.

(a) What level of nominal interest rate in $t+1$ will be set by the central bank aiming to minimize $\left(\pi_{t+1}^2 + x_{t+1}^2\right)$?

Since it is assumed to be possible to set $x_{t+k} = \pi_{t+k} = 0$ for all $k \geq 2$, we have

$$\mathrm{E}_{t+1} x_{t+k} = \mathrm{E}_{t+1} \pi_{t+k} = 0 \qquad \forall k \geq 2$$

Therefore, minimizing $\left(\pi_{t+1}^2 + x_{t+1}^2\right)$ means choosing $x_{t+1} = \pi_{t+1} = 0$. After plugging this into the NKIS we get

$$\begin{aligned} x_{t+1} &= \mathrm{E}_{t+1} x_{t+2} - (i_{t+1} - \mathrm{E}_{t+1}\pi_{t+2} - r^*) \\ 0 &= 0 - (i_{t+1} - 0 - r^*) \\ i_{t+1} &= r^* \end{aligned}$$

(b) What will be the levels of output gap and inflation in period t if the agents expect the central bank to act according to (a)?

$$\begin{aligned} &[\text{NKIS}] \qquad && x_t = 0 - (0 - 0 - r^*) - r^* - v = -v \\ &[\text{NKPC}] \qquad && \pi_t = 0 + x_t = -v \end{aligned}$$

(c) Assume the central bank credibly commits to set $i_{t+1} = r^* - a$. What will be the levels of output gap and inflation in periods $t+1$ and t?

In period $t+1$

$$x_{t+1} = 0 - (r^* - a - 0 - r^*) = a$$
$$\pi_{t+1} = 0 + x_{t+1} = a$$

In period t

$$x_t = a - (0 - a - r^*) - r^* - v = 2a - v$$
$$\pi_t = a + 2a - v = 3a - v$$

(d) What is the optimal level of a for a central bank aiming to minimize $\frac{1}{2}\left[\left(\pi_t^2 + x_t^2\right) + \left(\pi_{t+1}^2 + x_{t+1}^2\right)\right]$? Why does the optimal value of a differ from zero?

$$\min_a \quad (3a - v) + (2a - v)^2 + 2a^2$$

First order condition

$$\begin{aligned} 2 \cdot (3a - v)\, 3 + 2 \cdot (2a - v)\, 2 + 2 \cdot 2a &= 0 \quad | \quad : 2 \\ 9a - 3v + 4a - 2v + 2a &= 0 \\ 15a &= 5v \\ a &= \frac{v}{3} \end{aligned}$$

Through forward guidance the central bank commits in period t to set for period $t+1$ a lower interest rate than would be required to maximize welfare from the perspective of period $t+1$. Thanks to that the central bank can still stimulate the economy in t even when constrained by the effective lower bound.