Marcin Bielecki, Advanced Macroeconomics IE, Spring 2025
Homework 6 Solutions

Problem 1

Examine the following "congestion" model. Suppose the economy's production function depends positively ($A'(\cdot) > 0$) on the ratio of government expenditures to GDP, denoted with $\omega \equiv G/Y$:

$$Y_t = A(\omega) \cdot K_t$$

Assume no population growth for simplicity. Then the problem of the households can be stated using aggregate variables:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & K_{t+1} = (1 + r_t) K_t - C_t \end{aligned}$$

Assume that there is a firm revenue tax τ^y and the representative firm solves the following profit maximization problem:

$$\begin{aligned} \max \quad & D_t = (1 - \tau^y) Y_t - (r_t + \delta) K_t \\ \text{subject to} \quad & Y_t = A(\omega) \cdot K_t \end{aligned}$$

(a) Find the first order conditions characterizing the optimal choice of the consumer.

Lagrangian

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[(1 + r_t) K_t - C_t - K_{t+1}\right] \\ &= \ldots + \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} + \ldots + \beta^t \lambda_t \left[(1 + r_t) K_t - C_t - K_{t+1}\right] \\ &\quad + \beta^{t+1} \lambda_{t+1} \left[(1 + r_{t+1}) K_{t+1} - C_{t+1} - K_{t+2}\right] + \ldots \end{aligned}$$

First Order Conditions

$$\begin{aligned} C_t : \quad & \beta^t C_t^{-\sigma} + \beta^t \lambda_t [-1] = 0 && \rightarrow && \lambda_t = C_t^{-\sigma} \\ K_{t+1} : \quad & \beta^t \lambda_t [-1] + \beta^{t+1} \lambda_{t+1} (1 + r_{t+1}) = 0 && \rightarrow && \lambda_t = \beta \lambda_{t+1} (1 + r_{t+1}) \end{aligned}$$

Euler equation

$$\begin{aligned} C_t^{-\sigma} &= \beta (1 + r_{t+1}) C_{t+1}^{-\sigma} \\ C_{t+1} &= \left[\beta (1 + r_{t+1})\right]^{1/\sigma} C_t \end{aligned}$$

(b) Find the first order conditions characterizing the optimal behavior of the firm.

Profit maximization problem

$$\max_{K_t} \quad D_t = (1 - \tau^y) A(\omega) K_t - (r_t + \delta) K_t$$

First Order Condition

$$K_t : \quad (1 - \tau^y) A(\omega) - (r_t + \delta) = 0 \quad \rightarrow \quad r_t = (1 - \tau^y) A(\omega) - \delta$$

(c) Describe the general equilibrium in this economy using (a) and (b).

Government budget constraint (assuming only τ^y is used)

$$G_t = \omega Y_t = \tau^y Y_t$$

Resource constraint

$$\begin{aligned} K_{t+1} &= (1 + (1 - \tau^y) A(\omega) - \delta) K_t - C_t \\ &= (1 - \tau^y) A(\omega) K_t + (1 - \delta) K_t - C_t \\ &= Y_t + (1 - \delta) K_t - G_t - C_t \\ &= (1 - \omega) A(\omega) K_t + (1 - \delta) K_t - C_t \end{aligned}$$

Euler equation

$$C_{t+1} = [\beta (1 + (1 - \tau^y) A(\omega) - \delta)]^{1/\sigma} C_t$$

(d) Solve the social planner's problem using the following resource constraint (for simplicity treat ω as exogenous):

$$K_{t+1} = A(G_t/Y_t) \cdot K_t + (1 - \delta) K_t - C_t - G_t \quad \rightarrow \quad K_{t+1} = (1 - \omega) A(\omega) \cdot K_t + (1 - \delta) K_t - C_t$$

Lagrangian

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t [(1 - \omega) A(\omega) \cdot K_t + (1 - \delta) K_t - C_t - K_{t+1}] \\ &= \ldots + \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} + \ldots + \beta^t \lambda_t [(1 - \omega) A(\omega) \cdot K_t + (1 - \delta) K_t - C_t - K_{t+1}] \\ &\quad + \beta^{t+1} \lambda_{t+1} [(1 - \omega) A(\omega) \cdot K_{t+1} + (1 - \delta) K_{t+1} - C_{t+1} - K_{t+2}] + \ldots \end{aligned}$$

First Order Conditions

$$\begin{aligned} C_t : &\quad \beta^t C_t^{-\sigma} + \beta^t \lambda_t [-1] = 0 && \rightarrow \quad \lambda_t = C_t^{-\sigma} \\ K_{t+1} : &\quad \beta^t \lambda_t [-1] + \beta^{t+1} \lambda_{t+1} ((1 - \omega) A(\omega) + (1 - \delta)) = 0 && \rightarrow \quad \lambda_t = \beta \lambda_{t+1} (1 + (1 - \omega) A(\omega) - \delta) \end{aligned}$$

Euler equation

$$\begin{aligned} C_t^{-\sigma} &= \beta (1 + (1 - \omega) A(\omega) - \delta) C_{t+1}^{-\sigma} \\ C_{t+1} &= [\beta (1 + (1 - \omega) A(\omega) - \delta)]^{1/\sigma} C_t \end{aligned}$$

(e) Under which conditions there could be equivalence between the decentralized equilibrium from (c) and the social planner's equilibrium from (d)?

The two equations are identical only if $\tau^y = \omega$. This implies that no other taxes would be used for raising tax revenue. This result is due to negative externality of production: the more one agent produces, the lower is the productivity of others (while holding G constant, think about a traffic jam). The agents need then to receive a signal that makes them internalize the externality. In this model the income tax allows at the same time sending the proper signal and financing the expenditures.

Problem 2

Consider a perfectly competitive economy where individual price taking firms face the following production function:

$$Y_{it} = A_t K_{it}^{\alpha} L_{it}^{1-\alpha}$$

Assume that publicly available technology depends on the average level of capital per worker k:

$$A_t = \left(\frac{\sum_i K_{it}}{\sum_i L_{it}} \right)^{\eta} = k_t^{\eta}$$

where η represents a learning-by-doing externality. The aggregate final goods production is a sum of individual firms' outputs:

$$Y_t = \sum_i Y_{it}$$

Consumers solve the following utility maximization problem:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t + (1+n)\, a_{t+1} = w_t + (1+r_t)\, a_t + d_t \end{aligned}$$

(a) Find the first order conditions characterizing the optimal choice of the consumer.

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t + d_t - c_t - (1+n)\, a_{t+1} \right]$$

First Order Conditions

$$\begin{aligned} c_t: \quad & \beta^t c_t^{-\sigma} + \beta^t \lambda_t [-1] = 0 & \rightarrow \quad & \lambda_t = c_t^{-\sigma} \\ a_{t+1}: \quad & \beta^t \lambda_t [-(1+n)] + \beta^{t+1} \lambda_{t+1} [(1+r_{t+1})] = 0 & \rightarrow \quad & \lambda_t = \frac{\beta (1+r_{t+1})}{1+n} \lambda_{t+1} \end{aligned}$$

(b) Find the first order conditions characterizing the optimal behavior of the firm assuming that there is a constant rate of capital depreciation δ.

Single firm profit maximization

$$\max \quad D_{it} = A_t K_{it}^{\alpha} L_{it}^{1-\alpha} - w_t L_{it} - (r_t + \delta)\, K_{it}$$

First Order Conditions

$$\begin{aligned} K_{it}: \quad & \alpha A_t K_{it}^{\alpha-1} L_{it}^{1-\alpha} - (r_t + \delta) = 0 & \rightarrow \quad & r_t = \alpha A_t k_{it}^{\alpha-1} - \delta \\ L_{it}: \quad & (1-\alpha)\, A_t K_{it}^{\alpha} L_{it}^{-\alpha} - w_t = 0 & \rightarrow \quad & w_t = (1-\alpha)\, A_t k_{it}^{\alpha} \end{aligned}$$

Since all firms optimally choose the same capital-to-worker ratio, we can replace k_{it} with k_t. Firms profits are equal to 0.

(c) Describe the general equilibrium in this economy using (a) and (b).

Resource constraint ($k = a$)

$$\begin{aligned} c_t + (1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t + d_t \\ c_t + (1+n)\, k_{t+1} &= (1-\alpha)\, A_t k_t^{\alpha} + \left(1 + \alpha A_t k_t^{\alpha-1} - \delta\right) k_t + 0 \\ c_t + (1+n)\, k_{t+1} &= (1-\alpha)\, A_t k_t^{\alpha} + \alpha A_t k_t^{\alpha} + (1-\delta)\, k_t \\ c_t + (1+n)\, k_{t+1} &= k_t^{\eta} k_t^{\alpha} + (1-\delta)\, k_t \\ c_t + (1+n)\, k_{t+1} &= k_t^{\alpha+\eta} + (1-\delta)\, k_t \end{aligned}$$

Euler equation

$$\begin{aligned} c_t^{-\sigma} &= \frac{\beta (1+r_{t+1})}{1+n} c_{t+1}^{-\sigma} \\ c_{t+1} &= \left[\frac{1+r_{t+1}}{(1+\rho)(1+n)} \right]^{1/\sigma} c_t \\ c_{t+1} &= \left[\frac{1+\alpha k_t^{\alpha+\eta-1} - \delta}{(1+\rho)(1+n)} \right]^{1/\sigma} c_t \end{aligned}$$

(d) Draw a phase diagram in the (k, c) space; will the long run equilibrium in this economy be stable if $\alpha + \eta < 1$? What about if $\alpha + \eta = 1$?

For the case $\alpha + \eta < 1$ the phase diagram qualitatively is identical to the Ramsey model, the only difference is in a higher value of "α". The economy eventually reaches a steady state.
For the case $\alpha + \eta = 1$ the phase diagram is identical to the AK model. The economy does not have a steady state, only a balanced growth path.

(e) Assuming that the initial level of capital in this economy is below its steady-state value describe the behavior of k, c, y and the growth rate of per capita income over time in the two above mentioned cases.

For the case $\alpha + \eta < 1$ all three variables grow over time as they asymptotically converge to their steady state values. The growth rate of per capita income is initially positive and trends towards 0 as the economy converges to the steady state.
For the case $\alpha + \eta = 1$ all three variables grow at a common, cosnstant growth rate. The growth rate of per capita income is positive and does not diminish over time.

Problem 3

Consider an economy where final goods are produced according to the following production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

where L is the constant labor force, M_t is the number of invented types of intermediate goods and x_{it} denotes usage of intermediate good type i in the final goods production. The inventor of type i holds a perpetual patent that gives exclusive, monopolistic rights to produce this type of an intermediate good, with marginal cost of production equal to 1. Assume that the government taxes the monopolists and each of them pays lump-sum tax T per period.

(a) Solve the profit maximization problem of the final goods producer to obtain the (inverse) demand function for intermediate goods.

The final goods producer solves the following problem

$$\max_{L,\, x_{it}} \quad L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First Order Conditions

$$L: \qquad (1-\alpha) L^{-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t = 0 \qquad \rightarrow \qquad w_t = (1-\alpha) Y_t / L$$

$$x_{it}: \qquad L^{1-\alpha} \cdot \alpha x_{it}^{\alpha-1} - p_{it} = 0 \qquad \rightarrow \qquad p_{it} = \alpha x_{it}^{\alpha-1} L^{1-\alpha}$$

(b) Solve the profit maximization problem of the intermediate goods producers (monopolists). Find the optimal price, quantity produced and maximal after-tax profit per period.

$$\max_{x_{it}} \quad D_{it} = (p_{it} - 1) x_{it} - T = \alpha x_{it}^{\alpha} L^{1-\alpha} - x_{it} - T$$

First Order Condition

$$x_{it}: \qquad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1} L^{1-\alpha}}_{p_{it}} - 1 = 0 \quad \rightarrow \quad \alpha p_{it} - 1 = 0 \quad \rightarrow \quad p_{it} = \frac{1}{\alpha}$$

Optimal quantity produced

$$\begin{aligned} \alpha^2 x_{it}^{\alpha-1} L^{1-\alpha} &= 1 \\ x_{it}^{1-\alpha} &= \alpha^2 L^{1-\alpha} \\ x_{it} &= \alpha^{2/(1-\alpha)} L \end{aligned}$$

Maximal after-tax profit

$$D_{it} = (p_{it} - 1)\, x_{it} - T = \underbrace{\left(\frac{1}{\alpha} - 1\right) \alpha^{2/(1-\alpha)}}_{d} L - T \equiv dL - T$$

(c) Assume that inventing a new type of an intermediate good costs $1/\eta$ units of the final good. Equalize the cost of invention with the discounted after-tax value of profit flows of a new monopolist.

$$\frac{1}{\eta} = V = \frac{D}{r} = \frac{dL - T}{r}$$

(d) Transform the expression from (c) to obtain the real interest rate. Use the Euler equation $g = (r - \rho)/\sigma$ to obtain the equilibrium growth rate of the economy.

$$r = \eta\,(dL - T)$$

$$g = \frac{\eta\,(dL - T) - \rho}{\sigma}$$

(e) Discuss how the growth rate of the economy depends on the level of taxation T. Should the government aim to reduce the after-tax profits of the monopolists to 0?

$$\frac{\partial g}{\partial T} < 0$$

Driving profits to 0 would result in no innovation in equilibrium, and no economic growth in the long run.

Problem 4

Consider a variant of the Romer model where labor is not used in final good production, but is used as the unique input in the intermediate goods production. Assume the final good production function is given by:

$$Y_t = \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Also, suppose that $1/M_t$ units of labor are required to produce one unit of any intermediate good. As in the Romer model, suppose that $\Delta M_{t+1} = \eta M_t L_R$, where L_R is labor allocated to R&D. Define $L_X \equiv L - L_R$ as the amount of labor allocated to production of intermediates.

(a) What is the equilibrium level of intermediate good production?

Final goods producer

$$\max \quad \sum_{i=1}^{M_t} x_{it}^{\alpha} - \sum_{i=1}^{M_t} p_{it} x_{it}$$

First Order Condition

$$x_{it}: \qquad \alpha x_{it}^{\alpha-1} - p_{it} = 0 \qquad \rightarrow \qquad p_{it} = \alpha x_{it}^{\alpha-1}$$

Intermediate goods producers

$$\max_{x_{it}} \quad D_{it} = \left(p_{it} - \frac{w_t}{M_t}\right) x_{it} = \alpha x_{it}^{\alpha} - \frac{w_t}{M_t} x_{it}$$

First Order Condition

$$x_{it} \quad : \quad \alpha \cdot \alpha x_{it}^{\alpha-1} - \frac{w_t}{M_t} = 0$$

$$x_{it}^{\alpha-1} = \frac{w_t}{\alpha^2 M_t}$$

$$x_{it} = \alpha^{2/(1-\alpha)} \left(\frac{M_t}{w_t}\right)^{1/(1-\alpha)}$$

(b) What is the equilibrium price of a unit of any intermediate good?
First Order Condition

$$x_{it} \quad : \quad \alpha \cdot \underbrace{\alpha x_{it}^{\alpha-1}}_{p_{it}} - \frac{w_t}{M_t} = 0 \quad \rightarrow \quad \alpha p_{it} - \frac{w_t}{M_t} = 0 \quad \rightarrow \quad p_{it} = \frac{1}{\alpha}\frac{w_t}{M_t}$$

(c) Looking at a balanced-growth path equilibrium in which wages and number of intermediate varieties grow at the same rate, compute the maximal profit for intermediate producers.
Maximal profit

$$D_{it} = \left(p_{it} - \frac{w_t}{M_t}\right) x_{it} = \left(\frac{1}{\alpha} - 1\right) \frac{w_t}{M_t} \cdot \alpha^{2/(1-\alpha)} \left(\frac{M_t}{w_t}\right)^{1/(1-\alpha)} = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} \left(\frac{w_t}{M_t}\right)^{1-\frac{1}{1-\alpha}}$$

(d) Write down the research arbitrage condition.

$$\frac{w_t}{\eta M_t} = V = \frac{1}{r}\left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} \left(\frac{w_t}{M_t}\right)^{1-\frac{1}{1-\alpha}}$$

$$r = \eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} \left(\frac{w_t}{M_t}\right)^{-\frac{1}{1-\alpha}}$$

(e) Compute the equilibrium growth rate of the economy.

From Euler equation

$$g = \frac{r-\rho}{\sigma} \quad \rightarrow \quad \sigma g = \eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} \left(\frac{w}{M}\right)^{-\frac{1}{1-\alpha}} - \rho$$

Labor division

$$L_R = L - L_X = L - M \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{w}{M}\right)^{-\frac{1}{1-\alpha}} \cdot \frac{1}{M} = L - \alpha^{\frac{2}{1-\alpha}} \left(\frac{w}{M}\right)^{-\frac{1}{1-\alpha}}$$

From research production function

$$g = \frac{\Delta M}{M} = \frac{\eta M L_R}{M} = \eta L_R$$

$$g = \eta \left[L - \alpha^{\frac{2}{1-\alpha}} \left(\frac{w}{M}\right)^{-\frac{1}{1-\alpha}}\right] \quad \rightarrow \quad \alpha^{\frac{2}{1-\alpha}} \left(\frac{w}{M}\right)^{-\frac{1}{1-\alpha}} = L - \frac{g}{\eta}$$

Back to Euler equation

$$\sigma g = \eta \left(\frac{1-\alpha}{\alpha}\right) \left(L - \frac{g}{\eta}\right) - \rho$$

$$\sigma g = \eta \left(\frac{1-\alpha}{\alpha}\right) L - \left(\frac{1-\alpha}{\alpha}\right) g - \rho$$

$$g = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) L - \rho}{\sigma + \left(\frac{1-\alpha}{\alpha}\right)} = \frac{\eta (1-\alpha) L - \alpha\rho}{1 + \alpha(\sigma - 1)}$$