Marcin Bielecki, Advanced Macroeconomics IE, Spring 2025
Homework 4, deadline: April 8, 4:45 PM

Problem 1

Let's examine the role of taxes in the Solow-Swan model. Suppose that the behavior of an economy may be summarized by the following four equations:

$$K_{t+1} = I_t + (1-\delta) K_t$$
$$I_t = s(1-\tau) Y_t$$
$$C_t = (1-s)(1-\tau) Y_t$$
$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

Assume that population grows at rate n and technology at rate g, so that $L_{t+1}/L_t = (1+n)$ and $A_{t+1}/A_t = (1+g)$, respectively. Income in this economy is taxed with rate τ and the tax revenues are used for government consumption which is useless from the point of view of households.

(a) Transform the four equations into per effective labor form, i.e. divide them by $A_t L_t$. Make use of notational convention $\hat{x}_t \equiv X_t / (A_t L_t)$.

(b) Find the steady state level of capital per effective labor $\hat{k}^*$ in this economy.

(c) Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's steady state level of capital per effective labor $\hat{k}^*$.

(d) Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's steady state level of consumption per effective labor $\hat{c}^*$.

(e) Households care about the level of consumption per capita, i.e. c_t. This variable grows at rate g once the economy reaches its balanced growth path. Discuss whether low g or high g is better from the point of view of households.

Problem 2

Robert Solow in his 1956 article "A Contribution to the Theory of Economic Growth" considered the behavior of economy when output was produced according to various production functions. One of them was the following Constant Elasticity of Substitution (CES) function:

$$Y_t = \left[a K_t^{\rho} + (1-a) L_t^{\rho}\right]^{1/\rho}$$

where $a \in (0,1)$, $\rho \leq 1$ and for simplicity technology growth rate is set to 0. The CES function reduces to the Cobb-Douglas function in the limit of $\rho \to 0$.

Saving and investment behaviour of the economy are described respectively as:

$$i_t = s \cdot y_t$$
$$(1+n) k_{t+1} = i_t + (1-\delta) k_t$$

where lower case letters i_t, y_t, k_t denote per worker quantities, n denotes population growth, and δ denotes the depreciation rate.

(a) Transform the production function into per worker form (divide it by L_t).

(b) Find the steady state value for capital per worker k^*.

(c) Show how an increase in the saving rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).

(d) Show how an increase in the population growth rate affects the steady state level of capital per worker (you can use graphical analysis instead of algebra).

Problem 3

Consider an "endowment" version of the two-period OLG model. N_t^y agents are born in time t, where $N_t^y = (1+n)^t N_0^y$. Normalize $N_0^y = 1$ and let $n > 0$. Preferences of a young agent born in time t are:

$$U_t = \ln c_t^y + \beta \ln c_{t+1}^o$$

The initial old agents want to consume as much as possible. Each young agent is endowed with y units of the consumption good. The old have no endowment whatsoever. There is a storage technology that allows to convert one unit of period t goods into $1+r$ units of period $t+1$ goods. There is a social security system that is "pay-as-you-go". In each period t the government taxes the young and uses the receipts to make transfers to the old. We consider a per capita tax on the young that is constant over time, i.e. $\tau_t = \tau$ for all $t = 0, 1, 2, \ldots$

(a) What is the government budget constraint in period t? Write it both in aggregate and in per person terms.

(b) Solve for the competitive equilibrium consumption levels. Find the savings of the representative young agent a_{t+1}.

(c) What is the effect of an increase in τ on savings of the representative young agent?

(d) What is the optimal tax rate τ if $n < r$? Explain why.

Problem 4

Consider the standard OLG model. For simplicity assume no technological progress: $g = 0$, $A = 1$ and $\delta = 1$.

(a) Solve the following utility maximization problem of the household for the CRRA utility function:

$$\begin{aligned} \max \quad & U = \frac{(c_t^y)^{1-\sigma}}{1-\sigma} + \beta \frac{\left(c_{t+1}^o\right)^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & c_t^y + a_{t+1} = w_t \\ & c_{t+1}^o = (1 + r_{t+1})\, a_{t+1} \end{aligned}$$

(b) Verify that for $\sigma = 1$ the solution from (a) reduces to expressions obtained in the lectures. How the consumption of young and their accumulated assets prior to retirement depend on the expected rate of return on assets (r) for different σ?

(c) Obtain equilibrium factor prices by solving the profit maximization problem of the firms:

$$\max_{K,L} \quad D_t = K_t^\alpha L_t^{1-\alpha} - (r_t + \delta) K_t - w_t L_t$$

(d) Show that in the OLG model the goods market is in equilibrium if the labor and asset markets are in equilibrium.

(e) Assume $\sigma = 1$ for simplicity and derive the steady state level of capital per worker k^*.