# Chapter 3

# Investment

We already discussed the model of optimal decision-making of the household, both in terms of allocating consumption and savings over time, as well as supplying labor.

Today we focus a bit more on the firm side: we derive the formula for fundamental pricing of the firm and then discuss the dynamics of investment through the lens of the $q$ theory of investment.

This is also the first time when we encounter a phase diagram: a schematic representation of the dynamics of a multi-variable system of equations.

## 3.1 Neoclassical investment theory

### Households

The economy is populated by $N$ identical, infinitely lived households that solve the following problem:

$$\max_{\{c_t,\, b_{t+1},\, s_{t+1}\}_{t=0}^{\infty}} \quad U = \sum_{t=0}^{\infty} \beta^t \ln c_t$$
$$\text{subject to} \quad c_t + b_{t+1} + p_t s_{t+1} = w_t + (1+r)\, b_t + (d_t + p_t)\, s_t \quad \text{for all } t = 0, 1, 2, \ldots, \infty$$

where $b$ denotes holding of corporate bonds that pay real interest rate $r$, $w$ denotes the wage that the household receives for supplying labor,[1] $p$ is the price of a single share of a representative firm, $s$ is the number of shares owned by the household and $d$ is the dividend per share from the firm. We assume that the dividends are paid at the end of a time period and only after that the trade in firm shares takes place.

Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \lambda_t \left[w_t + (1+r)\, b_t + (d_t + p_t)\, s_t - c_t - b_{t+1} - p_t s_{t+1}\right]\right]$$

Let us expand the Lagrangian so that taking FOCs will be easier:

$$\begin{aligned} \mathcal{L} = \ldots &+ \beta^t \left[\ln c_t + \lambda_t \left[w_t + (1+r)\, b_t + (d_t + p_t)\, s_t - c_t - b_{t+1} - p_t s_{t+1}\right]\right] \\ &+ \beta^{t+1} \left[\ln c_{t+1} + \lambda_{t+1} \left[w_{t+1} + (1+r)\, b_{t+1} + (d_{t+1} + p_{t+1})\, s_{t+1} - c_{t+1} - b_{t+2} - p_{t+1} s_{t+2}\right]\right] + \ldots \end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned} c_t &: \quad \beta^t \left[\frac{1}{c_t} - \lambda_t\right] = 0 && \rightarrow \quad \lambda_t = \frac{1}{c_t} \\ b_{t+1} &: \quad \beta^t \left[-\lambda_t\right] + \beta^{t+1} \left[\lambda_{t+1} (1+r)\right] = 0 && \rightarrow \quad \lambda_t = \beta \lambda_{t+1} (1+r) \\ s_{t+1} &: \quad \beta^t \left[-\lambda_t p_t\right] + \beta^{t+1} \left[\lambda_{t+1} (d_{t+1} + p_{t+1})\right] = 0 && \rightarrow \quad \lambda_t p_t = \beta \lambda_{t+1} (d_{t+1} + p_{t+1}) \end{aligned}$$

Combining the FOCs for consumption and bonds, we get the usual Euler equation:

$$\frac{1}{c_t} = \beta (1+r) \frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta (1+r)\, c_t$$

If we divide the FOC for shares by the FOC for bonds, we get the fundamental pricing equation:[2]

$$p_t = \frac{d_{t+1} + p_{t+1}}{1+r}$$

Denote with $S$ the entire stock of firm shares. Then, total dividend $D = d \cdot S$ and total market value of the firm $V = p \cdot S$. The fundamental pricing equation can be rewritten as:

$$V_t = p_t S = \frac{d_{t+1} S + p_{t+1} S}{1+r} = \frac{D_{t+1} + V_{t+1}}{1+r} = \frac{D_{t+1} + \frac{D_{t+2} + V_{t+2}}{1+r}}{1+r} = \frac{D_{t+1}}{1+r} + \frac{D_{t+2} + V_{t+2}}{(1+r)^2} = \ldots$$

By iterating the above formula ad infinitum one concludes that the fundamental value of the firm can be expressed as the present discounted value (PDV) sum of the future dividend flows:[3]

$$V_t = \sum_{j=1}^{\infty} \frac{D_{t+j}}{(1+r)^j}$$

[1] For simplicity it is assumed here that labor supply is perfectly inelastic.

[2] We saw earlier that uncertainty surrounding future outcomes will make the formula slightly different: $p_t = \mathrm{E}_t \left[d_{t+1} + p_{t+1}\right] / (1 + r + rp_t)$ (where $rp_t$ is the risk premium adjustment).

[3] Provided that the value of the firm does not increase faster than the discount factor: $\lim_{i \to \infty} V_{t+i} / (1+r)^i = 0$.

## Firms

A single representative firm converts inputs (capital $K$ and labor $L$) into output $Y$ according to a neoclassical production function $F$:

$$Y_t = F\left(K_t, L_t\right)$$

The firm buys investment goods to increase its future capital stock:

$$K_{t+1} = I_t + \left(1-\delta\right) K_t$$

where $I$ is gross investment and $\delta$ denotes the rate of depreciation of physical capital.

The firm's dividend flow can be expressed as:

$$D_t = P_t^Y \cdot F\left(K_t, L_t\right) - w_t L_t - P_t^I \cdot I_t + B_{t+1} - \left(1+r\right) B_t$$

where $P^Y$ and $P^I$ are the prices of firm's output and investment goods, and $B$ is the stock of corporate bonds issued by the firm that yield real interest rate $r$. For simplicity we assume here that $P_t^Y = P_t^I = 1$.

Assume that firm managers want to maximize the sum of current dividend flow and value of the firm, which is consistent with shareholders' preferences. Since the current value of the firm is the PDV sum of future dividend flows, the objective function is then the PDV sum of current and future profit flows:

$$\begin{aligned} \max_{\{L_t, I_t, B_{t+1}, K_{t+1}\}_{t=0}^{\infty}} \quad & \left(D_0 + V_0\right) = \sum_{t=0}^{\infty} \frac{1}{\left(1+r\right)^t} \left[F\left(K_t, L_t\right) - w_t L_t - I_t + B_{t+1} - \left(1+r\right) B_t\right] \\ \text{subject to} \quad & K_{t+1} = I_t + \left(1-\delta\right) K_t \quad \text{for all } t = 0, 1, 2, \ldots, \infty \end{aligned}$$

Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{\left(1+r\right)^t} \left[F\left(K_t, L_t\right) - w_t L_t - I_t + B_{t+1} - \left(1+r\right) B_t + q_t \left[I_t + \left(1-\delta\right) K_t - K_{t+1}\right]\right]$$

where $q$ is the Lagrange multiplier associated with the capital accumulation equation.

The choice variables of the firm at time period $t$ are: employment $L_t$, investment $I_t$, bond issuance $B_{t+1}$ and next period capital stock $K_{t+1}$. The expanded Lagrangian is:

$$\begin{aligned} \mathcal{L} = \ldots &+ \frac{1}{\left(1+r\right)^t} \left[F\left(K_t, \boldsymbol{L_t}\right) - w_t \boldsymbol{L_t} - \boldsymbol{I_t} + \boldsymbol{B_{t+1}} - \left(1+r\right) B_t + q_t \left[\boldsymbol{I_t} + \left(1-\delta\right) K_t - \boldsymbol{K_{t+1}}\right]\right] \\ &+ \frac{1}{\left(1+r\right)^{t+1}} \left[\begin{array}{c} F\left(\boldsymbol{K_{t+1}}, L_{t+1}\right) - w_{t+1} L_{t+1} - I_{t+1} + B_{t+2} - \left(1+r\right) \boldsymbol{B_{t+1}} \\ + q_{t+1} \left[I_{t+1} + \left(1-\delta\right) \boldsymbol{K_{t+1}} - K_{t+2}\right] \end{array}\right] + \ldots \end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned} L_t :& \quad \frac{1}{\left(1+r\right)^t} \left[\frac{\partial F\left(K_t, L_t\right)}{\partial K_t} - w_t\right] = 0 && \rightarrow \quad w_t = F_{L,t} \\ I_t :& \quad \frac{1}{\left(1+r\right)^t} \left[-1 + q_t\right] = 0 && \rightarrow \quad q_t = 1 \\ B_{t+1} :& \quad \frac{1}{\left(1+r\right)^t} + \frac{1}{\left(1+r\right)^{t+1}} \left[-\left(1+r\right)\right] = 0 && \rightarrow \quad 1 = \frac{1+r}{1+r} \\ K_{t+1} :& \quad \frac{1}{\left(1+r\right)^t} \left[-q_t\right] + \frac{1}{\left(1+r\right)^{t+1}} \left[\frac{\partial F\left(K_{t+1}, L_{t+1}\right)}{\partial K_{t+1}} + \left(1-\delta\right) q_{t+1}\right] = 0 \\ & \hookrightarrow \quad q_t = \frac{F_{K,t+1} + \left(1-\delta\right) q_{t+1}}{1+r} \end{aligned}$$

where $F_{L,t} \equiv \frac{\partial F(K_t, L_t)}{L_t}$ and $F_{K,t} \equiv \frac{\partial F(K_t, L_t)}{K_t}$ denote, respectively, the derivatives of the production function with respect to employment and capital in time period $t$.

The FOC for employment says that in optimum the firm should keep hiring employees up to a point where the marginal product of labor becomes equal with the market wage. In equilibrium labor demand and labor supply are equal, which in our context translates to $L_t = N$ and since we have assumed that labor supply is perfectly inelastic (constant), we can ignore this variable in further analysis. In this model the real wage $w_t$ will flexibly adjust to always clear the labor market in each time period.

The FOC for net investment says that the Lagrange multiplier $q$ is always equal to 1, which can be interpreted that the marginal cost of a unit of investment (equal to 1 by our simplifying assumption) is equal to its marginal benefit $q$, being the PDV sum of additional future profits generated by this extra investment.

The FOC for bonds is satisfied always, independently from the level of $B$. That means that any amount of firm debt is consistent with profit-maximizing behavior (i.e. leverage does not matter) and the firm can finance investment equally well either through debt or through retained earnings, because the internal and external cost of capital are equal. This is a version of the Modigliani-Miller (1958) theorem.

Since $q_t = q_{t+1} = 1$, we can rewrite the FOC for capital as:[4]

$$1 = \frac{F_{K,t+1} + 1 - \delta}{1 + r} \quad \rightarrow \quad F_{K,t+1} = r + \delta$$

which says that the future marginal product of capital $F_{K,t+1}$ has to equal its acquisition cost (real interest rate) plus the depreciation rate.

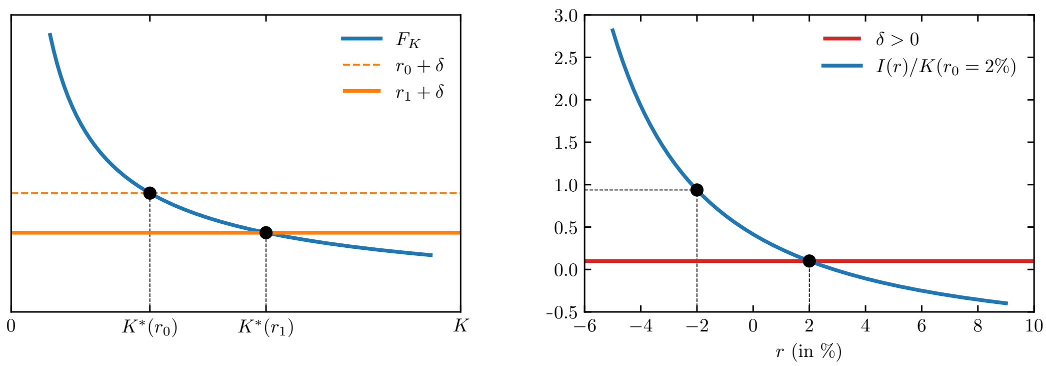
One property of the neoclassical production function is that the marginal product of capital is positive, but declining. The desired capital stock can then be expressed as a function of the real interest rate:[5]

$$K^* = K^*(r) = F_K^{-1}(r + \delta)$$

and investment is given by:[6]

$$I_t = K^*(r) - (1 - \delta) K_t \quad \rightarrow \quad \frac{I_t}{K_t} = \frac{K^*(r)}{K_t} - 1 + \delta$$

Thus, if the interest rate decreases, desired capital increases and firms want to immediately close the gap between actual and desired level of capital through positive investment:




The simple neoclassical setup predicts implausibly large fluctuations of investment in response to interest rate changes. In the above example, following a decrease in the real interest rate from 2% to −2%, the firms would like to immediately almost double their capital stock. To amend this, we turn to the $q$ theory of investment.

[4]In the general case where we allow prices of output and capital goods to fluctuate, this expression is given by: $F_K(K_{t+1}) = \left[iP_t^I + \delta P_{t+1}^I - (P_{t+1}^I - P_t^I)\right] / P_{t+1}^Y$ where $i$ is the nominal interest rate on corporate debt.

[5]The function $F_K^{-1}$ is the inverse function to $F_K$.

[6]When $K_t = K^*$, the firms simply replace depreciated capital units and gross investment equals $\delta K_t$.

## 3.2 Capital adjustment costs: $q$ theory of investment

Previously we have assumed that the only cost related with new capital is its acquisition cost. Here we will assume that there is also an adjustment cost, which will be proportional to the net investment to capital stock ratio, with $\chi \geq 0$ capturing the additional costs of installing new capital goods. We will also assume that depreciated capital stock $\delta K$ can be replaced without any additional costs.

Since leverage will not matter in this case as well, we can drop the corporate bonds $B$ from the problem. We will also introduce a separate variable for net investment $I^n$:

$$I_t^n \equiv I_t - \delta K_t$$
$$K_{t+1} = I_t + (1-\delta) K_t = I_t^n + K_t$$

The dividend flow is now given by:

$$D_t = F(K_t, L_t) - w_t L_t - \delta K_t - I_t^n \left(1 + \frac{\chi}{2} \frac{I_t^n}{K_t}\right)$$

Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[F(K_t, L_t) - w_t L_t - \delta K_t - I_t^n \left(1 + \frac{\chi}{2} \frac{I_t^n}{K_t}\right) + q_t \left[I_t^n + K_t - K_{t+1}\right]\right]$$

The expanded Lagrangian is:

$$\mathcal{L} = \ldots + \frac{1}{(1+r)^t} \left[F(K_t, \boldsymbol{L_t}) - w_t \boldsymbol{L_t} - \delta K_t - \boldsymbol{I_t^n} - \frac{\chi}{2} \frac{(\boldsymbol{I_t^n})^2}{K_t} + q_t \left[\boldsymbol{I_t^n} + K_t - \boldsymbol{K_{t+1}}\right]\right]$$
$$+ \frac{1}{(1+r)^{t+1}} \left[\begin{array}{c} F(\boldsymbol{K_{t+1}}, L_{t+1}) - w_{t+1} L_{t+1} - \delta \boldsymbol{K_{t+1}} - I_{t+1}^n - \frac{\chi}{2} \frac{(I_{t+1}^n)^2}{\boldsymbol{K_{t+1}}} \\ + q_{t+1} \left[I_{t+1}^n + \boldsymbol{K_{t+1}} - K_{t+2}\right] \end{array}\right] + \ldots$$

First order conditions (FOCs):

$$L_t: \quad \frac{1}{(1+r)^t} \left[F_{L,t} - w_t\right] = 0 \qquad \rightarrow \qquad w_t = F_{L,t}$$

$$I_t^n: \quad \frac{1}{(1+r)^t} \left[-1 - \chi \frac{I_t^n}{K_t} + q_t\right] = 0 \qquad \rightarrow \qquad q_t = 1 + \chi \frac{I_t^n}{K_t}$$

$$K_{t+1}: \quad \frac{1}{(1+r)^t} \left[-q_t\right] + \frac{1}{(1+r)^{t+1}} \left[F_{K,t+1} - \delta + \frac{\chi}{2} \left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2 + q_{t+1}\right] = 0$$

$$\hookrightarrow \quad q_t = \frac{1}{1+r} \left[q_{t+1} + F_{K,t+1} - \delta + \frac{\chi}{2} \left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2\right]$$

We can see now that $q$ is not always equal to 1 but is related to the net investment/capital ratio. We can iterate on the expression from capital FOC and express $q$ as the PDV sum of future marginal products of capital net of depreciation as well as gains from increased capital stock which makes future investment less costly:

$$q_t = \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \left[F_{K,t+j} - \delta + \frac{\chi}{2} \left(\frac{I_{t+j}^n}{K_{t+j}}\right)^2\right]$$

The FOC for net investment dictates now how much a firm should optimally invest given $q$ and $K$:

$$I_t^n = \frac{q_t - 1}{\chi} \cdot K_t \qquad \rightarrow \qquad I_t = \left(\frac{q_t - 1}{\chi} + \delta\right) K_t$$

The firm will have positive net investment and increase its capital stock only when $q > 1$. If $q = 1$, the firm maintains its capital stock at a constant level and only replaces depreciated capital. If $q < 1$, the firm liquidates a portion of its capital stock.

## Steady state

Let us now find the steady state of the system: the situation where the firm has already reached its desired level of capital $K^*$ and does not wish to change it, with net investment being equal to 0:[7]

$$0 = I^{n*} = \frac{q^* - 1}{\chi} \cdot K^* \quad \rightarrow \quad q^* = 1$$

Note well that even when net investment is 0, the firm still has positive gross investment as it needs to continually replace deprecating capital:

$$I^* = \delta K^*$$

To find the steady state level of capital stock we need to refer to the FOC for capital and use the conditions that $q^* = 1$ and $I^{n*} = 0$:

$$q^* = \frac{1}{1+r}\left[q^* + F_K^* - \delta + \frac{\chi}{2}\left(\frac{I^{n*}}{K^*}\right)^2\right] = 0 \quad \rightarrow \quad 1 + r = 1 + F_K^* - \delta$$

We get exactly the same expression as for the model without the adjustment costs, but this time such condition applies only to the steady state situation:

$$F_K^* = r + \delta \quad \rightarrow \quad K^* = F_K^{-1}(r + \delta)$$

## Transitional dynamics

To see what happens when the firm's capital stock is not equal to its desired level $K^*$ we need to analyze the dynamics of the system outside of the steady state. First transform the following system of three dynamic equations in three unknowns:

$$\begin{aligned} I_t^n &= \frac{q_t - 1}{\chi} \cdot K_t \\ q_t &= \frac{1}{1+r}\left[q_{t+1} + F_K(K_{t+1}) - \delta + \frac{\chi}{2}\left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2\right] \\ K_{t+1} &= I_t + (1-\delta)K_t = I_t^n + K_t \end{aligned}$$

into a system of two equations and two unknowns, by eliminating $I^n$:

$$\begin{aligned} K_{t+1} &= \frac{q_t - 1}{\chi} \cdot K_t + K_t \\ q_t &= \frac{1}{1+r}\left[q_{t+1} + F_K(K_{t+1}) - \delta + \frac{\chi}{2}\left(\frac{q_{t+1} - 1}{\chi}\right)^2\right] \end{aligned}$$

Next rewrite the equations into their difference form:

$$\begin{aligned} \Delta K_{t+1} \equiv K_{t+1} - K_t &= \frac{q_t - 1}{\chi} \cdot K_t \\ q_t + rq_t &= q_{t+1} + F_K(K_{t+1}) - \delta + \frac{(q_{t+1} - 1)^2}{2\chi} \\ \Delta q_{t+1} \equiv q_{t+1} - q_t &= rq_t - F_K(K_{t+1}) + \delta - \frac{(q_{t+1} - 1)^2}{2\chi} \end{aligned}$$

Now find such pairs of $q$ and $K$ for which $\Delta K = 0$ and $\Delta q = 0$ (we can drop the time subscripts):[8]

$$0 = \Delta K = \frac{q - 1}{\chi} \cdot K \quad \rightarrow \quad q = 1$$

$$0 = \Delta q = rq - F_K + \delta - \frac{(q - 1)^2}{2\chi}$$

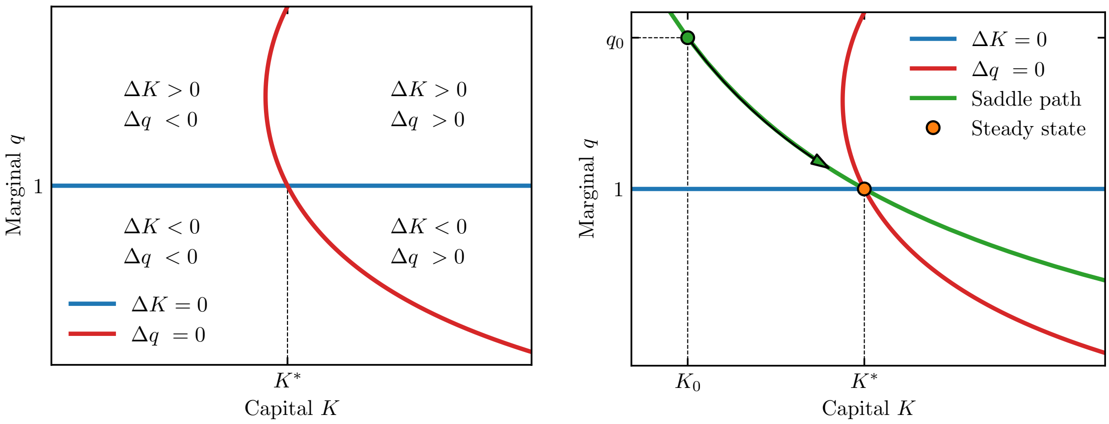
[7] The model also admits a "trivial" steady state of $K^* = 0$, which we ignore as in that case the firm does not exist.

[8] By dropping the time subscripts we obtain equations that are exact (only) in continuous time.

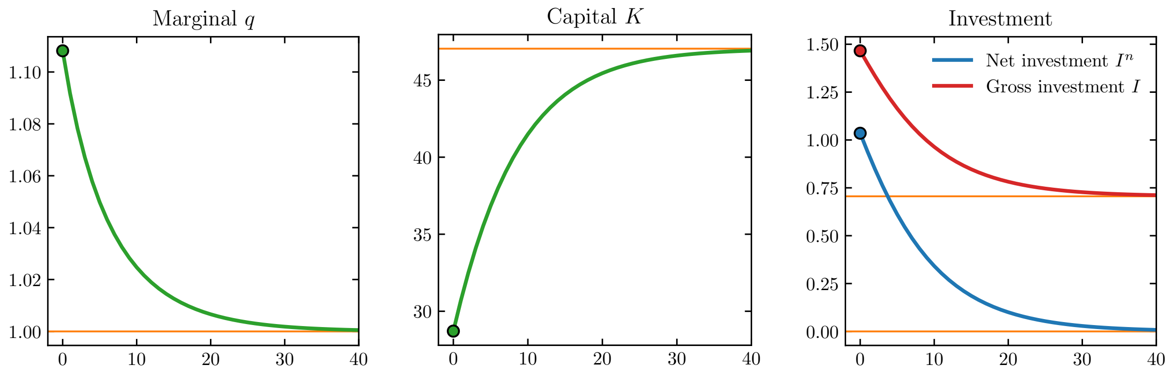
Assuming a certain functional form of $F$ we can easily plot the $\Delta K = 0$ and $\Delta q = 0$ curves.[9]

Now let us figure out how the system behaves outside of the steady state. We can easily see from the $\Delta K$ equation that if $q > 1$ then $\Delta K > 0$ and if $q < 1$ then $\Delta K < 0$.

How does $\Delta q$ behave? Consider a point along the $\Delta q = 0$ curve. Without changing $q$, let us increase $K$. Since $F_K$ is a decreasing function, we now subtract a smaller value than before, and $\Delta q$ changes from 0 to some positive value. That means that in the region to the "right" of the $\Delta q = 0$ curve (higher values of $K$) we have $\Delta q > 0$, and analogously to the "left" of the $\Delta q = 0$ curve we have $\Delta q < 0$:




Our phase diagram is now divided into four regions. In two of them (southwest and northeast) the variables are "pushed" away from the steady state. In the other two regions (northwest and southeast) the variables are "pulled" toward the steady state. This system exhibits saddle path stability,[10] where a single optimal path (called the "saddle path" or "transition path") takes the system from any initial level of $K$ to the desired level of $K^*$, with $q$ dictating the optimal pace of investment along the way. It does not mean that we could not get to the steady state by using some different path, but that all alternative paths are associated with lower PDV stream of dividends and lower firm value. The graphs below depict the dynamic behavior of variables over time for the case where the initial level of capital is lower than desired ($K_0 < K^*$):




In the neoclassical case capital would immediately jump from $K_0$ to $K^*$. Here this adjustment takes time, as due to the presence of adjustment costs the firm would incur unnecessary losses trying to immediately close the gap, as it is now optimal to spread the investment process over multiple time periods.

The speed of convergence to the desired capital stock level depends on the value of $\chi$: the larger it is, the longer the transition period.

[9] I use here the Cobb-Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$ with $A > 0$, $L = N = 1$ and $\alpha \in (0, 1)$.
[10] See the Appendix on how to establish the stability properties of dynamic systems.

## 3.3 Tobin's (1969) $q$ and the stock market value

From the point of view of the firm's manager, since $q_t$ is the shadow price of capital installed at the end of period $t$ in the firm, a natural approach to value the firm is $V_t = q_t K_{t+1}$. From the outsiders' standpoint $q$ is unobservable. But agents in the economy form some expectations about future dividend flows, reflected in the stock-market valuation of the firm. It turns out that we can get useful information from this valuation. Let us start with the expression for $q_t$:

$$q_t = \frac{1}{1+r}\left[F_{K,t+1} - \delta + \frac{\chi}{2}\left(\frac{I^n_{t+1}}{K_{t+1}}\right)^2 + q_{t+1}\right] \quad | \quad \cdot K_{t+1} = K_{t+2} - I^n_{t+1}$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_{K,t+1}\cdot K_{t+1} - \delta K_{t+1} + \frac{\chi}{2}\frac{\left(I^n_{t+1}\right)^2}{K_{t+1}} + q_{t+1}\left(K_{t+2} - I^n_{t+1}\right)\right]$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_{K,t+1}\cdot K_{t+1} - \delta K_{t+1} + \frac{\chi}{2}\frac{\left(I^n_{t+1}\right)^2}{K_{t+1}} + q_{t+1}K_{t+2} - \left(1+\chi\frac{I^n_{t+1}}{K_{t+1}}\right) I^n_{t+1}\right]$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_{K,t+1}\cdot K_{t+1} - \delta K_{t+1} - I^n_{t+1}\left(1+\frac{\chi}{2}\frac{I^n_{t+1}}{K_{t+1}}\right) + q_{t+1}K_{t+2}\right]$$

$$q_t K_{t+1} = \sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left[F_{K,t+j}\cdot K_{t+j} - \delta K_{t+j} - I^n_{t+j}\left(1+\frac{\chi}{2}\frac{I^n_{t+j}}{K_{t+j}}\right)\right]$$

Recall that:

$$V_t = \sum_{j=1}^{\infty}\frac{D_{t+j}}{(1+r)^j}$$

$$= \sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left[F\left(K_{t+j}, L_{t+j}\right) - w_{t+j}L_{t+j} - \delta K_{t+j} - I^n_{t+j}\left(1+\frac{\chi}{2}\frac{I^n_{t+j}}{K_{t+j}}\right) + B_{t+j+1} - (1+r)B_{t+j}\right]$$

We can without the loss of generality ignore $B$ and focus on the case where the firm never issues bonds (recall the Modigliani-Miller theorem). We will also use one of the properties of the neoclassical production function and the FOC for employment:

$$F\left(K_t, L_t\right) = F_{K,t}\cdot K_t + F_{L,t}\cdot L_t = F_{K,t}\cdot K_t + w_t L_t \quad \rightarrow \quad F_{K,t}\cdot K_t = F\left(K_t, L_t\right) - w_t L_t$$

Then we can write:

$$q_t K_{t+1} = \sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left[F\left(K_{t+j}, L_{t+j}\right) - w_{t+j}L_{t+j} - \delta K_{t+j} - I^n_{t+i}\left(1+\frac{\chi}{2}\frac{I^n_{t+j}}{K_{t+j}}\right)\right] = V_t$$

So it would appear that we could extract $q_t$ using the formula: $q_t = V_t/K_{t+1}$. Is that true?

## 3.4 Hayashi's (1982) theorem

In general, $V/K$ is the average $Q$, not marginal $q$:

$$Q \equiv \frac{\text{firm market value}}{\text{firm book value}}$$

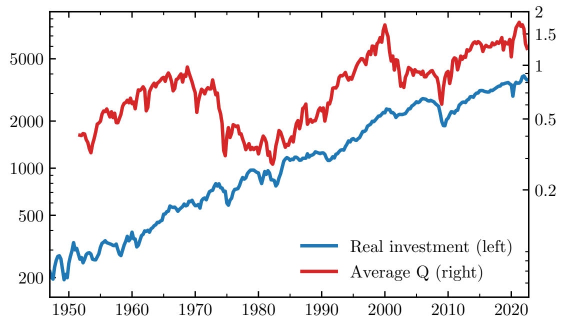
However, under certain (restrictive) assumptions these two concepts coincide:

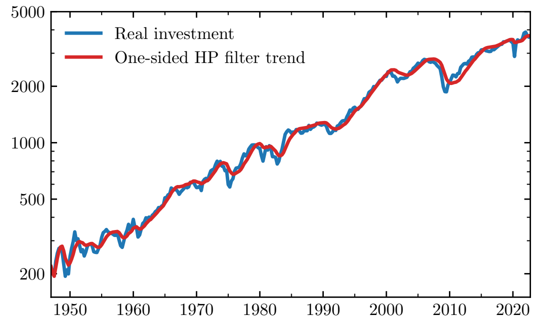
1. Production function and total adjustment cost function exhibit constant returns to scale.
2. Capital goods are homogeneous.
3. Stock market is efficient (uses fundamental pricing).

The firm should then invest whenever $V/K > 1$ and disinvest when $V/K < 1$.

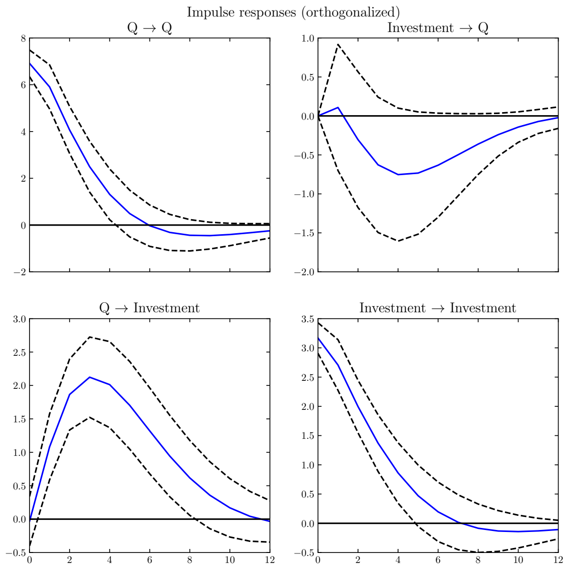
## 3.5 Using average $Q$ to forecast real aggregate investment

Even if the assumptions of the Hayashi's theorem are not satisfied in reality, average $Q$ does provide information that can be used in forecasting investment. The following graph shows the relevant time series for the US at quarterly frequency.[11] As can be easily seen, real investment is increasing over time and average $Q$ is usually far from 1. However, one can notice that significant drops in $Q$ usually translate to drops in investment a few quarters later. To perform formal analysis, I first detrend (the logarithms of) both time series using one-sided Hodrick-Prescott (1997) filter. This filter can be thought of as a special moving average filter that isolates deviations from trend at business cycle frequency:

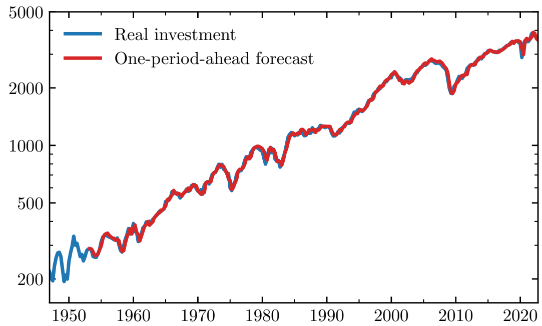


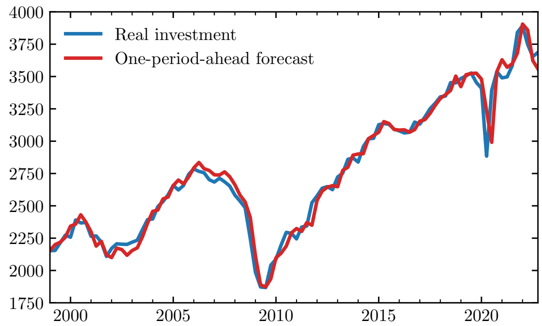




Next I set up a simple VAR model on HP-deviations from trend of investment and $Q$. Using a variety of lag selection criteria I decide to include two lags. Then I estimate the VAR model and produce the Impulse Response Functions plot, seen on the right. The HP-deviations of investment and average $Q$ exhibit significant autocorrelation. While the shocks to investment do not translate statistically significantly to changes in $Q$, shocks to $Q$ indeed translate to changes in investment, influencing it over the horizon of up to 8 quarters, and having the strongest impact for 2-4 quarters after the initial shock to $Q$.




Finally, I use my VAR model to produce one quarter ahead forecasts for cyclical deviations of investment, which I combine with the previously isolated HP trend. Thanks to information provided by $Q$, the model performs particularly well in forecasting the peaks and throughs of the investment time series:




[11] US quarterly real investment data can be downloaded from https://fred.stlouisfed.org/series/GPDIC1. Average $Q$ is approximated by the ratio between nonfinancial corporate business equities and net worth.