University of Warsaw
Faculty of Economic Sciences

Models of inequality

Advanced Macroeconomics IE: Lecture 13

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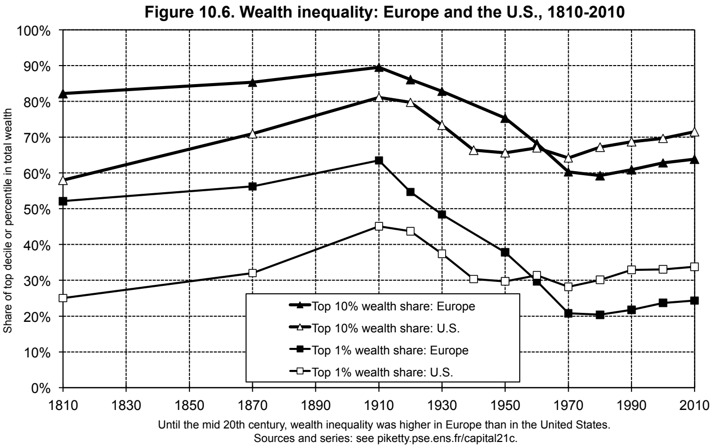
University of Warsaw

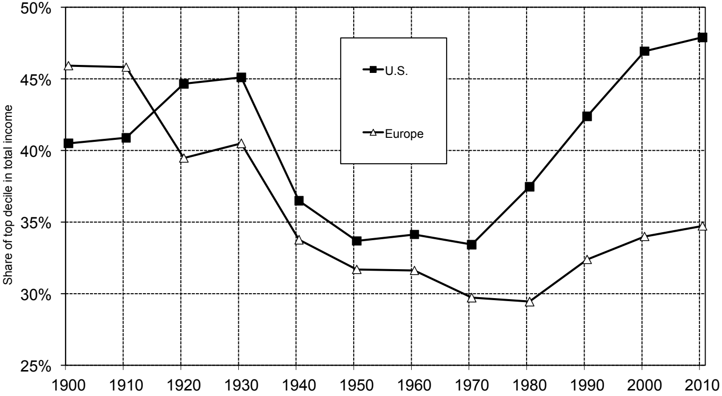
Evolution of top wealth



Piketty (2014) Capital in the Twenty-First Century

Evolution of top incomes

Figure 9.8. Income inequality: Europe vs. the United States, 1900-2010



The top decile income share was higher in Europe than in the U.S. in 1900-1910; it is a lot higher in the U.S. in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c.

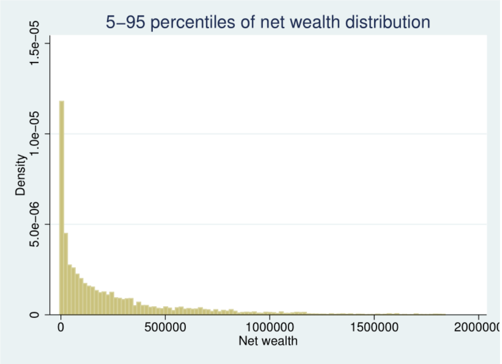
Piketty (2014) Capital in the Twenty-First Century

Lecture plan

1. Model of (top) wealth inequality based on Jones (2015)
2. Simple model of precautionary savings and the role of borrowing constraints
3. Quantitative models of income and wealth inequality based on De Nardi (2015)

(Top) wealth inequality – Jones (2015)

US wealth distribution

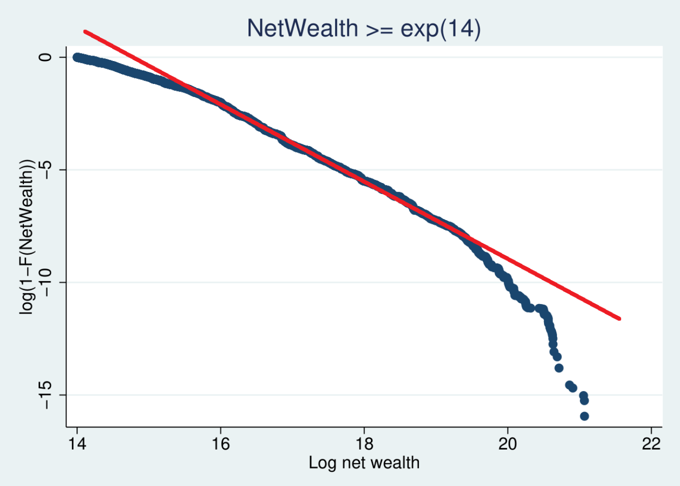


Ben Moll's lecture

Features of US Wealth Distribution:

- right skewness
- heavy upper tail, well approximated by a Pareto distribution

US wealth distribution



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Properties of Pareto distribution

When a variable (e.g. wealth) is Pareto distributed, it satisfies:

$$\text{Pr}\left[\text{wealth} > a\right] = \left(a/a_{min}\right)^{-1/\eta}$$

which means the fraction of people with wealth greater than some cutoff is proportional to the cutoff raised to some power

Under Pareto distribution, computation of "top shares" is easy. The fraction of wealth going to the top p percentiles is given by:

$$(100/p)^{\eta-1}$$

The higher η is, the more unequal the distribution:

- For $\eta = 0.5$ the top 1% wealth share is $100^{-0.5} = 10\%$
- For $\eta = 0.75$ the top 1% wealth share is $100^{-0.25} \approx 32\%$
- Piketty (2014): in the US the top 1% wealth share $\approx 33\%$, in UK and France between 25% and 30%

Pareto wealth distribution: core intuition

Assume (for now) that the size of population does not change

Suppose households face a constant probability of death d

Then the probability that an individual is of at least age x is:

$$\text{Pr}\left[\text{age} > x\right] = (1-d)^x \approx e^{-dx}$$

Assume (for now) that everyone receives the same initial wealth = 1

Let the wealth of households (dynasties) increase with age at rate μ:

$$a\left(x\right) = (1+\mu)^x \approx e^{\mu x} \quad \rightarrow \quad x\left(a\right) = (1/\mu) \cdot \ln a$$

Then we can easily map the probability of holding at least
some amount of wealth to the probability of being old enough:

$$\text{Pr}\left[\text{wealth} > a\right] = \text{Pr}\left[\text{age} > x\left(a\right)\right] = \exp\left(-\left(d/\mu\right) \cdot \ln a\right) = a^{-d/\mu}$$

Wealth is Pareto distributed with $\eta = \mu/d$

Demographics

Maintain the assumption of constant death probability

Allow population size to change over time

Define a (crude) birth rate $b_t \equiv B_t / N_t$ and assume it's constant

Population growth rate n is the difference between crude birth and death rates:

$$n = b - d \quad \rightarrow \quad b = n + d$$

Share of people aged x in the population is given by:

$$sh(x) = b\left(\frac{1-d}{1+n}\right)^x \approx b(1-d-n)^x = b(1-b)^x \approx be^{-bx}$$

Probability that a person is at least of age x:

$$\Pr[\text{age} > x] = \int_x^\infty be^{-bs}\, ds = e^{-bx}$$

Households' choice

Households solve the following utility maximization problem:

$$\max \quad U = \sum_{t=0}^{\infty} \left[\beta\left(1-d\right)\right]^{t} \ln c_t$$

$$\text{subject to} \quad a_{t+1} = \left(1+r-\tau\right) a_t - c_t$$

where households do not receive any labor income and τ is a tax on wealth

Euler equation:

$$c_{t+1} = \beta\left(1-d\right)\left(1+r-\tau\right) c_t$$

Guess-and-verify that households consume a fixed fraction α of their wealth:

$$\alpha a_{t+1} = \beta\left(1-d\right)\left(1+r-\tau\right)\alpha a_t$$

$$\alpha\left[\left(1+r-\tau\right) a_t - \alpha a_t\right] = \beta\left(1-d\right)\left(1+r-\tau\right)\alpha a_t$$

$$\left(1+r-\tau\right) - \alpha = \beta\left(1-d\right)\left(1+r-\tau\right)$$

Wealth dynamics

Budget constraint then determines the dynamics of wealth:

$$a_{t+1} = (1 + r - \tau - \alpha)\, a_t \equiv (1 + \mu)\, a_t \quad \rightarrow \quad a_t = (1 + \mu)^t\, a_0$$

Let $a_t\,(x)$ denote the wealth of a person aged x at time period t:

$$a_t\,(x) = (1 + \mu)^x\, a_{t-x}\,(0)$$

Assume that newly born agents inherit wealth of the deceased:

$$a_t\,(0) = \frac{dK_t}{B_t} = \frac{dK_t}{bN_t} = \frac{d}{b} k_t$$

Assume the BGP economy with exogenous technological progress:

$$k_t = (1 + g)^t\, k_0 \quad \rightarrow \quad k_t = (1 + g)^x\, k_{t-x}$$

Wealth inherited by newborns in period $t - x$:

$$a_{t-x}\,(0) = \frac{d}{b} k_{t-x} = \frac{d}{b}\,(1 + g)^{-x}\, k_t$$

Wealth distribution

Wealth of people aged x at time period t:

$$a_t(x) = (1+\mu)^x \cdot \frac{d}{b}(1+g)^{-x} k_t \approx \frac{d}{b} k_t \cdot e^{(\mu-g)x}$$

Age x needed to accumulate wealth a:

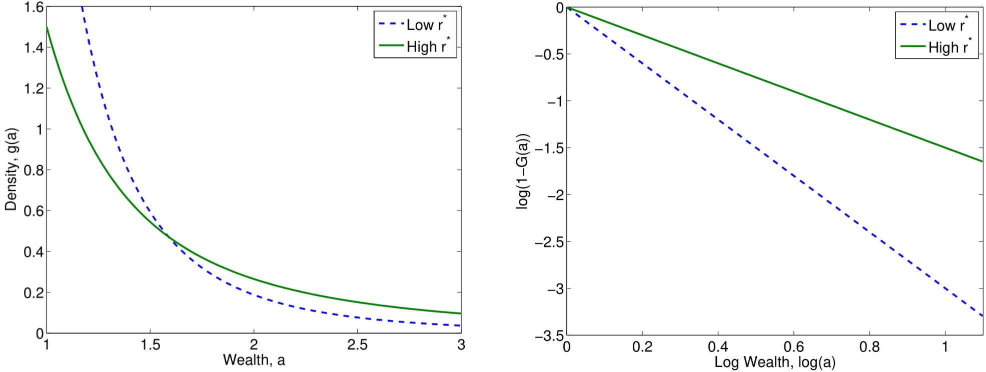
$$x(a_t) = \frac{1}{\mu-g} \cdot \ln\left[\frac{a_t}{(d/b)\,k_t}\right]$$

Probability of holding wealth of at least a is then given by:

$$\begin{aligned}\text{Pr}\left[\text{wealth} > a\right] &= \text{Pr}\left[\text{age} > x(a)\right] = e^{-bx(a)} \\ &= \exp\left[-\frac{b}{\mu-g} \cdot \ln\left[\frac{a_t}{(d/b)\,k_t}\right]\right] = \left[\frac{a_t}{(d/b)\,k_t}\right]^{-\frac{b}{\mu-g}}\end{aligned}$$

Wealth is Pareto distributed with $\eta = \dfrac{\mu-g}{b} = \dfrac{r-\tau-\alpha-g}{n+d}$

Wealth distribution (Partial Equilibrium)



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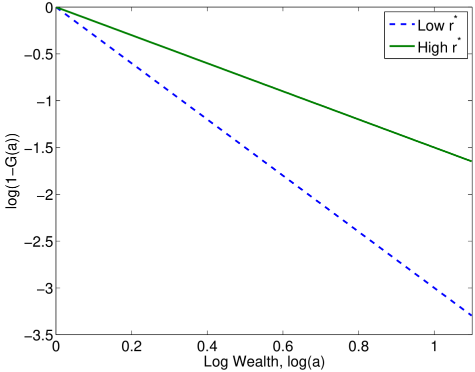
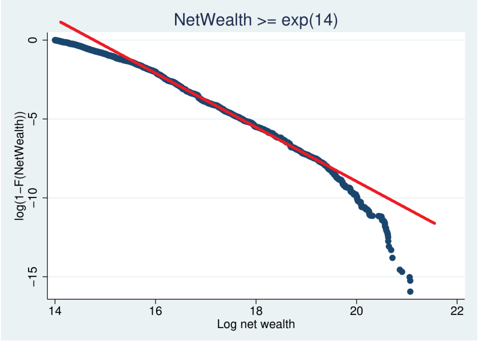
Wealth distribution (Partial Equilibrium)

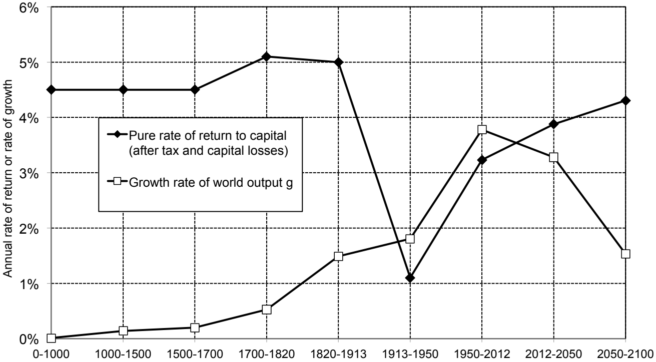




Ben Moll's lecture

Piketty (2014): importance of $r\,(-\tau) - g\,(-n)$

Figure 10.10. After tax rate of return vs. growth rate at the world level, from Antiquity until 2100



The rate of return to capital (after tax and capital losses) fell below the growth rate during the 20th century, and may again surpass it in the 21st century. Sources and series : see piketty.pse.ens.fr/capital21c

Piketty (2014) Capital in the Twenty-First Century

Wealth inequality (Partial Equilibrium)

Wealth is Pareto distributed with $\eta = \dfrac{r - \tau - \alpha - g}{n + d}$

Those lucky to live a long life (members of long-lived dynasties) will accumulate greater stocks of wealth

Recall that the higher η is, the more unequal the distribution

Piketty (2014): increase in $r - g\,(-n)$ increases wealth inequality

19th century: low g and low $n \rightarrow$ high inequality

Middle 20th century: high g and $n \rightarrow$ low inequality

21st century: declining g and $n \rightarrow$ back to 19th century (?)

Piketty's prescription: increase τ to counteract g and n

Wealth inequality (General Equilibrium)

Relationship between aggregate capital and individual wealth:

$$K_t = \sum_{x=0}^{\infty} sh(x)\, N_t \cdot a_t(x) = \sum_{x=0}^{\infty} b(1-b)^x\, N_t \cdot \frac{dk_t}{b}(1+\mu-g)^x$$

$$\approx dK_t \sum_{x=0}^{\infty} (1+\mu-g-b)^x = \frac{dK_t}{1-(1+\mu-g-b)}$$

Real interest rate under General Equilibrium is given by:

$$d = -(\mu-g-b) = -(r-\tau-\alpha-g-d-n)$$

$$r = n+g+\tau+\alpha$$

Wealth inequality coefficient under General Equilibrium:

$$\eta = \frac{r-\tau-\alpha-g}{n+d} = \frac{n+g+\tau+\alpha-g-\tau-\alpha}{n+d} = \frac{n}{n+d}$$

Wealth inequality is determined purely by demography!

Takeaway

$$\eta^{PE} = \frac{r - g - \tau - \alpha}{n + d} \quad \text{vs} \quad \eta^{GE} = \frac{n}{n + d}$$

If wealth tax is redistributed in lump-sum, then $\eta^{GE} = \frac{n - \tau}{n + d}$

Piketty is right to highlight the link between $r - g$, population growth, taxes and top wealth inequality (under PE)

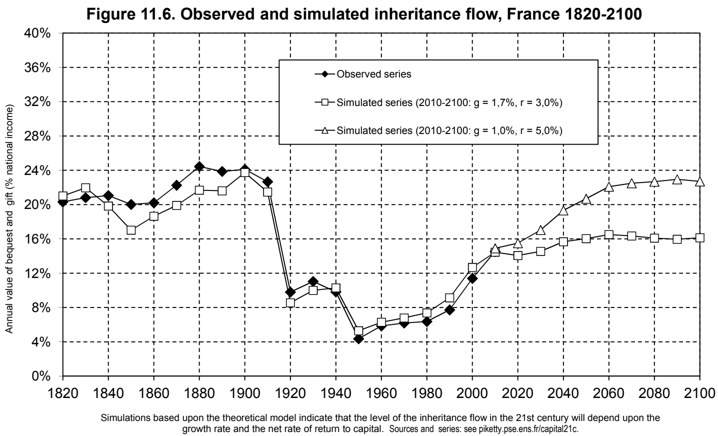
But these results are fragile and can disappear under GE

All above results hinge on the assumptions regarding inheritance

Need for richer framework, including bequests, social mobility, progressive taxation, micro- and macroeconomic shocks, and multiple risk-return asset portfolios

More research needed (empirics & theory)

Inheritance is key



Piketty (2014) Capital in the Twenty-First Century

Precautionary savings and borrowing constraints

Household's problem under uncertainty I

Consider a two-period expected utility maximization problem:

$$\max_{c_t,\, c_{t+1},\, a} \quad U = \ln(c_t) + \beta \mathrm{E}_t\left[\ln(c_{t+1})\right]$$

$$\text{subject to} \quad c_t + a = y_t$$

$$c_{t+1} = y_{t+1} + (1+r)\,a$$

First period income is certain and equals y

Second period income will be equal to either $y + e$ or $y - e$:

$$y_{t+1} = \begin{cases} y + e & \text{with probability } 1/2 \\ y - e & \text{with probability } 1/2 \end{cases}$$

Household's problem under uncertainty II

Assume $\beta = 1$ and $r = 0$ for simplicity

Use budget constraints to express consumption levels:

$$c_t = y - a$$

$$c_{t+1} = \begin{cases} y + e + a & \text{w. prob. } 1/2 \\ y - e + a & \text{w. prob. } 1/2 \end{cases}$$

Rewrite the problem as choosing the optimal a alone:

$$\max_a \quad U = \ln(y - a) + \frac{1}{2}\ln(y + e + a) + \frac{1}{2}\ln(y - e + a)$$

First order condition:

$$-\frac{1}{y-a} + \frac{1}{2}\frac{1}{y+e+a} + \frac{1}{2}\frac{1}{y-e+a} = 0$$

Solution: full solution

$$a = \frac{1}{2}\left(\sqrt{y^2 + 2e^2} - y\right)$$

Precautionary savings

$$a = \frac{1}{2}\left(\sqrt{y^2 + 2e^2} - y\right)$$

When second period income is certain, then the household holds no assets in optimum and enjoys smooth consumption over time, since $c_t = c_{t+1} = y$.

When there is uncertainty about second period income ($e > 0$), the household accumulates **precautionary savings** to self-insure against the scenario of low income in the second period.[1]

We can easily demonstrate that the more variable second period income is, the higher is the stock of accumulated assets:

$$\frac{\partial a}{\partial e} = \frac{1}{2} \cdot \frac{1}{2\sqrt{y^2 + 2e^2}} \cdot 2 \cdot 2e = \frac{e}{\sqrt{y^2 + 2e^2}} > 0$$

[1] To get this result of "prudence", the utility function has to satisfy: $u''' > 0$.

Incomplete markets and borrowing constraints

But what if they could purchase insurance against shocks?

- insurance would pay $+e$ under negative shock and $-e$ under positive
- consumption would always equal y, no matter the state of the world
- expected insurance payout = 0, should be available at low cost
- its absence serves as evidence for **market incompleteness**

Households could not borrow in period 2 to smooth out the shocks

If $\beta(1+r)=1$, infinitely lived households do not change consumption (much) under income shocks when borrowing constraints are absent

- individual asset holdings follow random walk & wealth distribution is indeterminate

Market incompleteness and borrowing constraints generate stationary wealth distribution under idiosyncratic shocks

- typically under GE $\beta(1+r)<1 \rightarrow$ "excess" savings
- capital taxation may increase welfare

Income and wealth inequality – De Nardi (2015)

Basic infinitely-lived Bewley model

Framework proposed by Bewley (1977)

Labor market status z_t (e.g. $z_t = \{0, 1\}$) evolves according to the transition matrix P (with stationary distribution $\bar{P}$)

Households want to maximize lifetime expected utility:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}} \quad & U = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right] \\ \text{subject to} \quad & c_t + a_{t+1} = z_t w + \left(1 + r\right) a_t \\ & a_{t+1} \geq \underline{a} \\ & z_{t+1} \sim P\left(z_t\right) \end{aligned}$$

Solution: infinite sequence of consumption plans $\{c_t\}_{t=0}^{\infty}$

Can rewrite this problem as: choosing today's consumption and tomorrow's assets only, conditional on today's assets and labor market status

Recursive formulation of household's problem

We can rewrite the utility function into the value function:

$$V\left(a_t, z_t\right) = \max_{c_t,\, a_{t+1}} \left\{u\left(c_t\right) + \beta \mathrm{E}_t\left[V\left(a_{t+1}, z_{t+1}\right) | z_t\right]\right\}$$

$$\text{subject to} \quad c_t + a_{t+1} = z_t w + (1+r)\, a_t$$

$$a_{t+1} \geq \underline{a}$$

Even more compactly:

$$V\left(a_t, z_t\right) = \max_{a_{t+1} \geq \underline{a}} \left\{u\left(z_t w + (1+r)\, a_t - a_{t+1}\right) + \beta \mathrm{E}_t\left[V\left(a_{t+1}, z_{t+1}\right) | z_t\right]\right\}$$

Solution is the policy function A which maps from $\left(a_t, z_t\right)$ to a_{t+1}:

$$a_{t+1} = A\left(a_t, z_t\right)$$

We can also use the budget constraint to obtain
the policy function C which maps from $\left(a_t, z_t\right)$ to c_t:

$$c_t = C\left(a_t, z_t\right) = z_t w + (1+r)\, a_t - A\left(a_t, z_t\right)$$

Simplified analytical example: setup

Based on section “No-trade equilibria” in Ragot (2018)

Agents either employed ($z_t = 1$) or unemployed ($z_t = 0$)

Probabilities of flows: employed to unemployed s, unemployed to employed p

Employed receive wage w

Unemployed generate “home production” b

Capital-less economy: only assets are borrowing contracts

Since $\underline{a} = 0$, unemployed can’t borrow and in equilibrium employed save 0

Unemployed are borrowing constrained, but employed are not

Euler equation of employed will determine the real interest rate

Full solution ▸ here

Simplified analytical example: real interest rate

Real interest rate is pinned down by the Euler equation of employed:

$$u'\left(c_t^E\right) = \beta\left(1+r\right)\left[\left(1-s\right)u'\left(c_{t+1}^E\right) + su'\left(c_{t+1}^U\right)\right]$$

There is no borrowing or saving, so that:

$$c^E = w \quad \text{and} \quad c^U = b$$

Assume CRRA utility function:

$$u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma} \quad \rightarrow \quad u'\left(c\right) = c^{-\sigma}$$

After some algebra:

$$1+r = \frac{1}{\beta}\left[1-s+s\left(\frac{w}{b}\right)^{\sigma}\right]^{-1} < \frac{1}{\beta}$$

Rethinking taxation

Consumption of employed and unemployed after applying: linear labor income tax τ^w, linear consumption tax τ^c, lump-sum tax τ and lump-sum transfer v

$$c^E = \frac{1}{1+\tau^c}\left[\left(1-\tau^w\right)w - \tau + v\right] \quad \text{and} \quad c^U = \frac{1}{1+\tau^c}\left[b - \tau + v\right]$$

Expected utility (veil of ignorance a'la Rawls):

$$U = \frac{p}{s+p} \cdot u\left(c^E\right) + \frac{s}{s+p} \cdot u\left(c^U\right)$$

Can be shown that (under balanced govt. budget with positive govt. expenditure):

- linear labor income tax is preferred to linear consumption tax, and both are preferred to lump-sum tax
- progressive taxes are preferred to linear taxes
- lump-sum transfer is welfare-improving (but directed transfer is even better)
- in environment with physical capital, positive tax rate on capital can be welfare-improving

A more realistic example

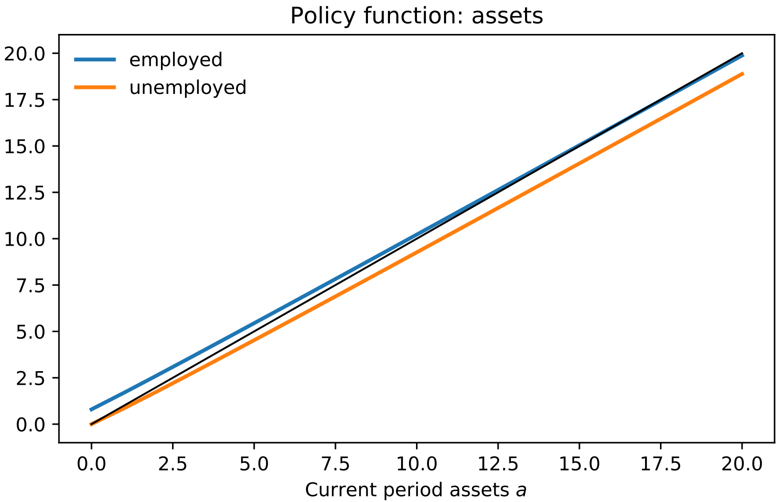
Analytical solutions do not exist

Solutions are obtained using a variety of computational methods

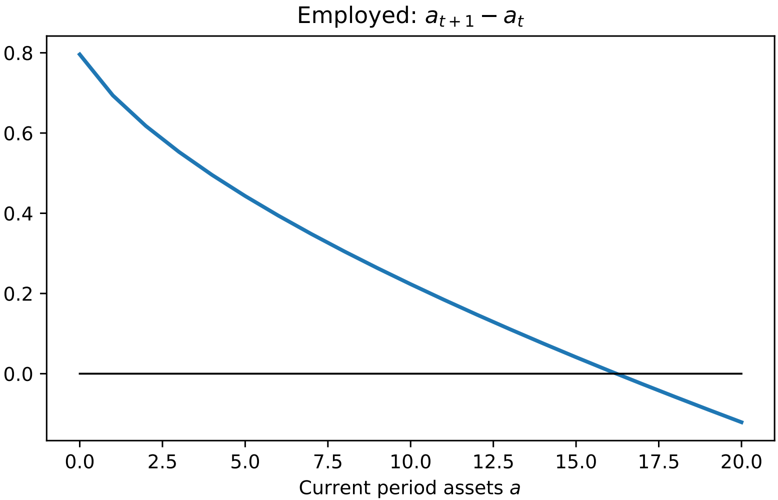
Example model:

- households have low ("unemployed") or high ("employed") labor productivity
- low productivity is 10% of high productivity
- $z = [0.1,\ 1]$, $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$, $\bar{P} = [0.5,\ 0.5]$
- borrowing constraint $\underline{a} = 0$
- $u\left(c\right) = \ln c$, $\beta = 0.96$
- $r = 2\%$ (Partial Equilibrium interest rate)

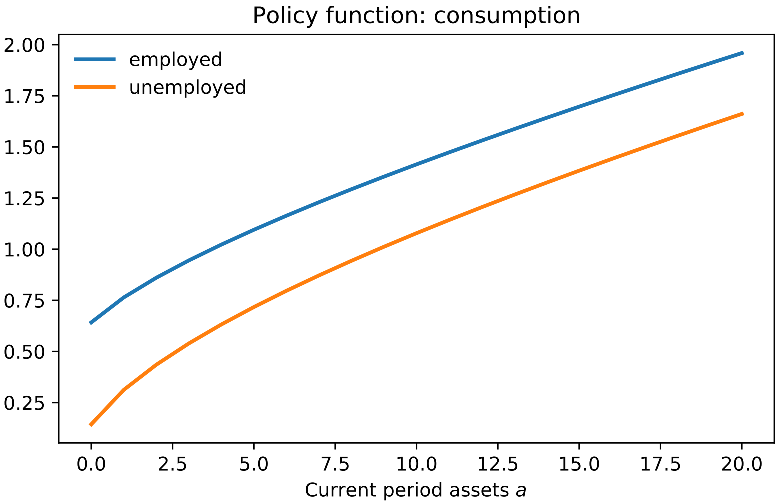
Policy functions (Partial Equilibrium)



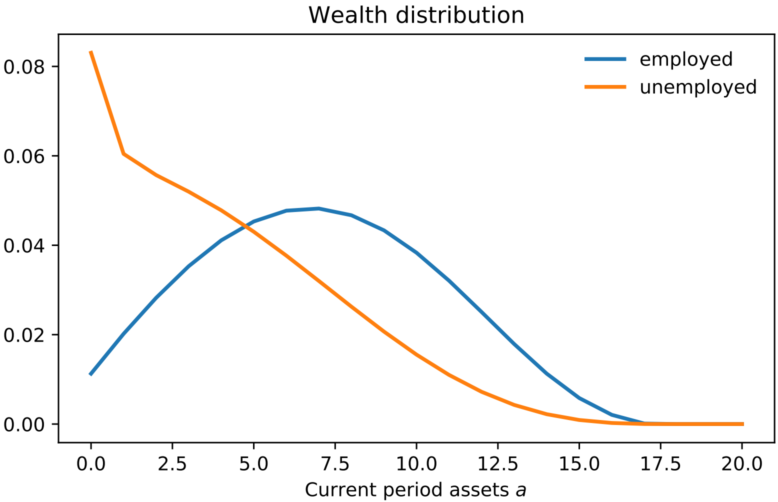
Policy functions (Partial Equilibrium)



Policy functions (Partial Equilibrium)



Wealth distribution (Partial Equilibrium)



General Equilibrium

Households and firms take prices w and r as given

Assume standard production function:

$$Y = K^{\alpha} L^{1-\alpha}$$

Prices depend on the supply of factors of production:

$$L = N \cdot z\bar{P}$$

$$K = \int_{\underline{a}}^{\infty} a \, dg(a)$$

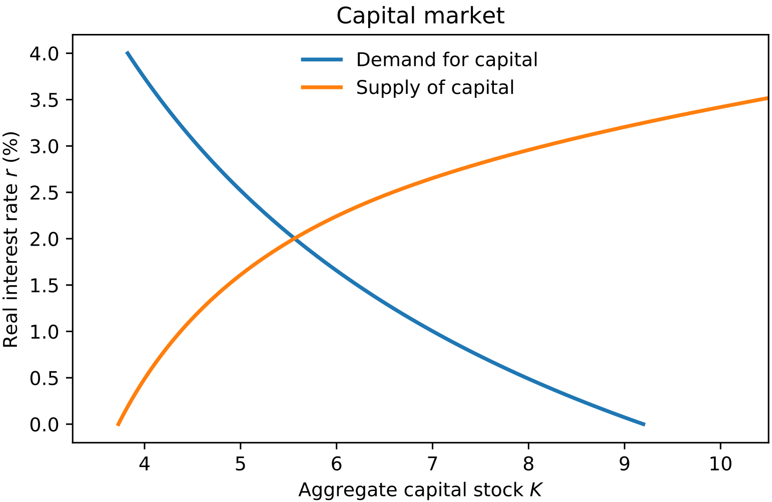
Expressions for prices:

$$w = \alpha K^{\alpha} L^{-\alpha}$$

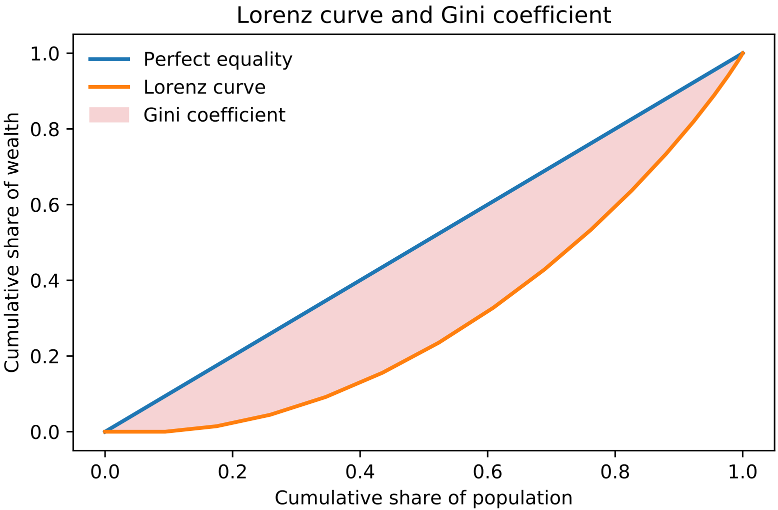
$$r = (1-\alpha) K^{\alpha-1} L^{1-\alpha} - \delta$$

Market clearing: equalize capital supply (households) with capital demand (firms)

Capital market equilibrium



Lorenz curve and Gini coefficient for wealth



Some issues

Since under General Equilibrium $\beta(1+r) < 1$, households do not want to save without bound (good for computational reasons)

Households are willing to hold positive assets because:

- there is a borrowing constraint
- they don't want to have low assets when unemployed $\rightarrow$ very low consumption
- no reason to increase assets if this possibility is small and in distant future

Hard to generate households with very high wealth

Inequality does not matter much for aggregate outcomes:

- policy functions close to linear
- households with low assets have low consumption
  $\rightarrow$ impact on aggregate consumption small
- not that many of borrowing-constrained households

Aiyagari (1994)

Aiyagari (1994) approximates the earnings of US workers by an AR(1) process:

$$\ln z_t = \rho \ln z_{t-1} + \varepsilon_t$$

Autocorrelation $\rho = 0.6$ and standard deviation $\sigma_\varepsilon = 0.2$

Earning levels discretized to 7 possible values

	% wealth in top		
Gini	1%	5%	20%
U.S. data, 1989 SCF			
.78	29	53	80
Aiyagari Baseline			
.38	3.2	12.2	41.0
Aiyagari higher variability			
.41	4.0	15.6	44.6

Huggett (1996)

Huggett (1996): overlapping generations variant of the Bewley model

Households can live for up to T periods and face age-dependent survival probability ω

Value function is age-dependent:

$$V_t\left(a_t, z_t\right) = \max_{c_t,\, a_{t+1}} \left\{u\left(c_t\right) + \beta \omega_{t+1} \mathrm{E}_t\left[V_{t+1}\left(a_{t+1}, z_{t+1}\right) | z_t\right]\right\}$$

$$\text{subject to} \quad c_t + a_{t+1} = e_t\left(z_t\right) w + \left(1 + r\right) a_t + b_t$$

$$a_{t+1} \geq \underline{a}$$

where b_t are bequest from the deceased (redistributed equally)
plus Social Security payments to retirees

Partial Equilibrium very easy to solve for since V_T is known

De Nardi (2004)

De Nardi (2004): Huggett model with intergenerational links

- voluntary bequests from parents to children (utility from giving)
- transmission of labor productivity from parents to children

Transfer wealth ratio	Wealth Gini	Percentage wealth in the top 1%	5%	20%	40%	60%	Percentage with negative or zero wealth
U.S. data, 1989 SCF							
.60	.78	29	53	80	93	98	5.8–15.0
Equal bequests to all (Huggett)							
.67	.67	7	27	69	90	98	17
Unequal bequests to children (unintentional)							
.38	.68	7	27	69	91	99	17
Parent's bequest motive							
.55	.74	14	37	76	95	100	19
Parent's bequest motive and productivity inheritance							
.60	.76	18	42	79	95	100	19

Lifetime wealth profiles

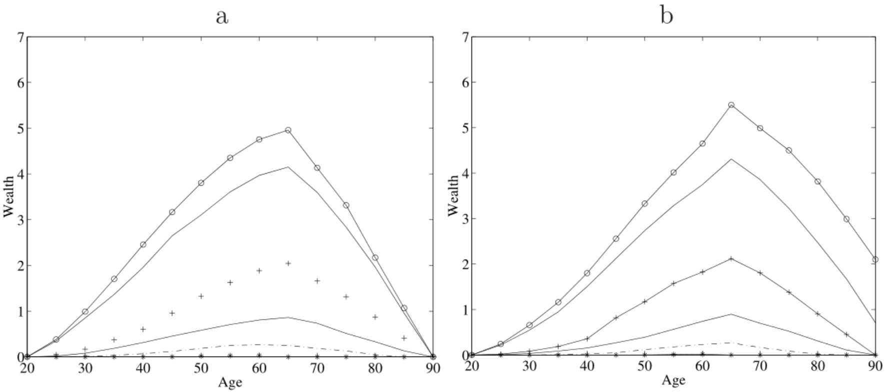


Figure 2: Wealth .1, .3, .5, .7, .9, .95 quantiles. No links, equal bequests to all, panel a, and Bequest motive, panel b .

De Nardi (2015)

Cagetti and De Nardi (2006)

Entrepreneurs: households who declare being self-employed, own a privately held business (or a share of one), and have an active management role in it

Small fraction of the population, but hold a large share of wealth

Top %	1	5	10	20
Whole population				
percentage of total net worth held	30	54	67	81
Entrepreneurs				
percentage of households in a given percentile	63	49	39	28
percentage of net worth held in a given percentile	68	58	53	47

Cagetti and De Nardi (2006)

Altruistic agents care about their children

Agents decide whether to run a business or work for a wage

Entrepreneurial production function depends
on entrepreneurial ability and working capital

Borrowing for working capital is constrained by agents' assets

Rationale for holding high levels of wealth

Wealth Gini	Fraction of entrepreneurs	Percentage wealth in the top 1%	5%	20%	40%
U.S. data					
0.78	7.55%	30	54	81	94
Baseline model with entrepreneurs					
0.78	7.50%	31	60	83	94

Takeaway

Bequests and entrepreneurship can account for the observed wealth inequality

Changes in these assumptions can yield vastly different welfare effects of policies!

Ahn et al. (2017): To get macroeconomic effects of inequality, two assets are needed:

- "wealthy hand-to-mouth" agents: e.g. low liquid assets and a (mortgaged) house
- consumption choices of these agents matter for aggregate consumption, as they consume a lot and are a significant fraction of the population

Solution to precautionary savings problem I

Rewrite the FOC:

$$
\begin{aligned}
\frac{1}{y-a} &= \frac{1}{2}\frac{1}{y+e+a} + \frac{1}{2}\frac{1}{y-e+a} \quad \Big| \quad \cdot 2 \\
\frac{2}{y-a} &= \frac{1}{y+e+a} + \frac{1}{y-e+a} \\
\frac{2}{y-a} &= \frac{y-e+a+y+e+a}{(y+e+a)(y-e+a)} \\
\frac{2}{y-a} &= \frac{2y+2a}{y^2-ye+ya+ey-e^2+ea+ay-ae+a^2} \\
\frac{2}{y-a} &= \frac{2(y+a)}{y^2+2ay-e^2+a^2} \\
\frac{1}{y-a} &= \frac{y+a}{y^2+2ay-e^2+a^2}
\end{aligned}
$$

Solution to precautionary savings problem II

Cross-multiply the above equation to get:

$$\begin{aligned} y^2 + 2ay - e^2 + a^2 &= (y+a)(y-a) \\ y^2 + 2ay - e^2 + a^2 &= y^2 - a^2 \\ 2ay + 2a^2 - e^2 &= 0 \end{aligned}$$

The result is the following quadratic equation for a:

$$a^2 + ay - \frac{e^2}{2} = 0$$

The above quadratic equation has two roots: back

$$a = \frac{-y + \sqrt{y^2 + 2e^2}}{2} \quad \text{or} \quad a = \frac{-y - \sqrt{y^2 + 2e^2}}{2}$$

Discard the second root – in this case for $e = 0$ we get $a = -y$ and $c_2 = 0$, which is clearly not the solution of the consumer's problem

Simplified analytical example: problem of employed

$$V^E(a_t) = \max_{c_t,\, a_{t+1}} \left\{ u(c_t) + \beta\left[(1-s)\,V^E(a_{t+1}) + sV^U(a_{t+1})\right]\right\}$$

$$\text{subject to} \quad c_t + a_{t+1} = w + (1+r)\,a_t$$

Incorporate constraint directly into the objective:

$$V^E(a_t) = \max_{a_{t+1}} \left\{ \begin{array}{c} u\left(w + (1+r)\,a_t - a_{t+1}\right) \\ +\beta\left[(1-s)\,V^E(a_{t+1}) + sV^U(a_{t+1})\right] \end{array} \right\}$$

First order condition (with respect to a_{t+1}):

$$0 = -u'\left(c_t^E\right) + \beta\left[(1-s)\,\frac{\partial V^E(a_{t+1})}{\partial a_{t+1}} + s\,\frac{\partial V^U(a_{t+1})}{\partial a_{t+1}}\right]$$

Envelope condition (with respect to a_t):

$$\frac{\partial V^E(a_t)}{\partial a_t} = (1+r)\,u'\left(c_t^E\right)$$

Simplified analytical example: problem of unemployed

$$V^U(a_t) = \max_{c_t,\, a_{t+1}} \left\{ u(c_t) + \beta \left[pV^E(a_{t+1}) + (1-p) V^U(a_{t+1}) \right] \right\}$$

$$\text{subject to} \quad c_t + a_{t+1} = b + (1+r) a_t$$

$$a_{t+1} \geq 0$$

Incorporate constraints directly into the objective:

$$V^U(a_t) = \max_{a_{t+1}} \left\{ \begin{array}{c} u(b + (1+r) a_t - a_{t+1}) \\ +\beta \left[pV^E(a_{t+1}) + (1-p) V^U(a_{t+1}) \right] \end{array} \right\} + \mu a_{t+1}$$

First order condition (with respect to a_{t+1}):

$$0 = -u'\left(c_t^U\right) + \beta \left[p \frac{\partial V^E(a_{t+1})}{\partial a_{t+1}} + (1-p) \frac{\partial V^U(a_{t+1})}{\partial a_{t+1}} \right] + \mu$$

Envelope condition (with respect to a_t):

$$\partial V^U(a_t) / \partial a_t = (1+r) u'\left(c_t^U\right)$$

Simplified analytical example: joint problem

Optimality conditions:

$$u'\left(c_t^E\right) = \beta\left[(1-s)\frac{\partial V^E\left(a_{t+1}\right)}{\partial a_{t+1}} + s\frac{\partial V^U\left(a_{t+1}\right)}{\partial a_{t+1}}\right]$$

$$u'\left(c_t^U\right) = \beta\left[p\frac{\partial V^E\left(a_{t+1}\right)}{\partial a_{t+1}} + (1-p)\frac{\partial V^U\left(a_{t+1}\right)}{\partial a_{t+1}}\right] + \mu$$

$$\frac{\partial V^E\left(a_{t+1}\right)}{\partial a_{t+1}} = (1+r)\,u'\left(c_{t+1}^E\right)$$

$$\frac{\partial V^U\left(a_{t+1}\right)}{\partial a_{t+1}} = (1+r)\,u'\left(c_{t+1}^U\right)$$

Resulting in (recall that $\mu > 0$):

$$u'\left(c_t^E\right) = \beta(1+r)\left[(1-s)\,u'\left(c_{t+1}^E\right) + su'\left(c_{t+1}^U\right)\right]$$

$$u'\left(c_t^U\right) > \beta(1+r)\left[pu'\left(c_{t+1}^E\right) + (1-p)\,u'\left(c_{t+1}^U\right)\right]$$

◂ back