University of Warsaw
Faculty of Economic Sciences

Introduction to Quantitative Advanced Macroeconomics

Master Thesis Seminar

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Intertemporal consumption choice

Utility Maximization Problem

The household maximizes utility from consumption in two periods

$$\max_{c_1,\, c_2,\, a} \quad U = \ln c_1 + \beta \ln c_2$$

$$\text{subject to} \quad c_1 + a = y_1$$

$$c_2 = y_2 + (1+r)\, a$$

Logarithmic utility for easy derivations, discount factor $\beta \in [0,1]$

Exogenous variables: incomes y_1, y_2 and the real interest rate r

Choice variables: consumption c_1, c_2 and assets at the end of period 1 a

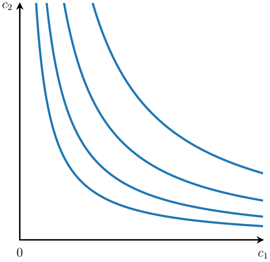
Lifetime budget constraint:

$$a = \frac{c_2 - y_2}{1+r} \quad \rightarrow \quad c_1 + \frac{c_2 - y_2}{1+r} = y_1 \quad \rightarrow \quad \boldsymbol{c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}}$$

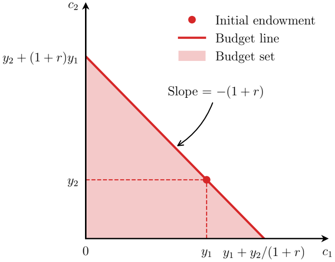
Utility Maximization Problem: graphical interpretation

We are looking for a specific **indifference curve** that is just tangent to the **budget line**. The point of tangency is the **optimal consumption** choice:

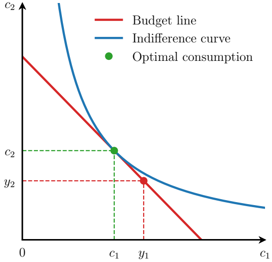
Indifference curve map



Lifetime budget constraint



Graphical solution



Method of Lagrange multipliers

Set up the Lagrangian

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

Derive the first order conditions (FOCs)

$$c_1 : \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda [-1] = 0 \quad \rightarrow \quad \lambda = \frac{1}{c_1}$$

$$c_2 : \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda \left[-\frac{1}{1+r} \right] = 0 \quad \rightarrow \quad \lambda = \beta (1+r) \frac{1}{c_2}$$

Obtain the optimality condition (**Euler equation**)

$$\frac{1}{c_1} = \beta (1+r) \frac{1}{c_2} \quad \rightarrow \quad \boldsymbol{c_2 = \beta (1+r) c_1}$$

Utility Maximization Problem: solution

Plug the Euler equation into the lifetime budget constraint

$$c_2 = \beta\left(1+r\right)c_1$$

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

$$c_1 + \beta c_1 = y_1 + \frac{y_2}{1+r}$$

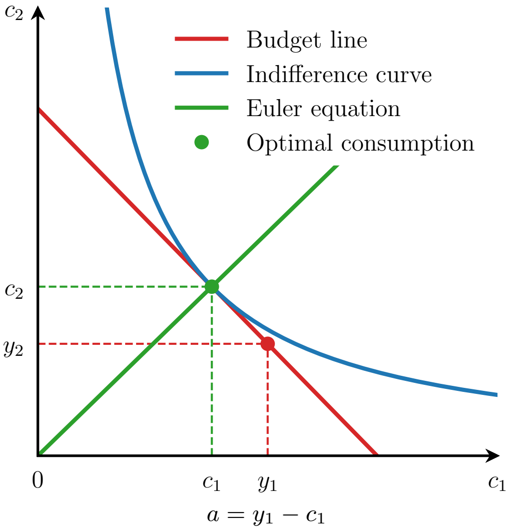
Optimal levels of consumption and assets

$$c_1 = \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right]$$

$$c_2 = \frac{\beta}{1+\beta}\left[\left(1+r\right)y_1 + y_2\right]$$

$$a = y_1 - c_1 = \frac{1}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right]$$

Utility Maximization Problem solution: graphical interpretation



Comparative Statics

Consumer is more patient (higher β)

$$\frac{\partial c_1}{\partial \beta} < 0, \quad \frac{\partial c_2}{\partial \beta} > 0, \quad \frac{\partial a}{\partial \beta} > 0$$

Higher income in the first period

$$\frac{\partial c_1}{\partial y_1} > 0, \quad \frac{\partial c_2}{\partial y_1} > 0, \quad \frac{\partial a}{\partial y_1} > 0$$

Higher (expected) income in the second period

$$\frac{\partial c_1}{\partial y_2} > 0, \quad \frac{\partial c_2}{\partial y_2} > 0, \quad \frac{\partial a}{\partial y_2} < 0$$

Comparative Statics: changes in real interest rate r

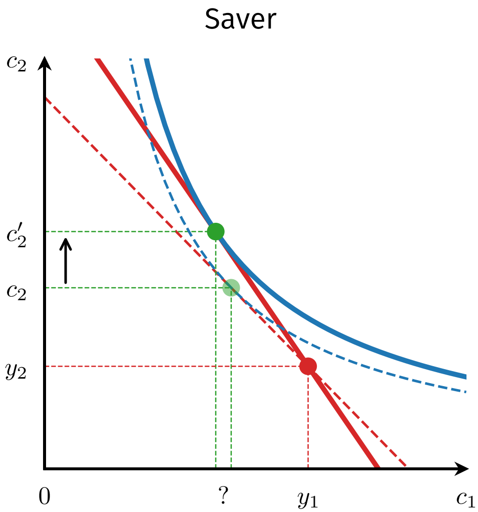
Substitution effect: as consumption in the future gets „cheaper", induces the agent to consume more in the second period and less in the first period

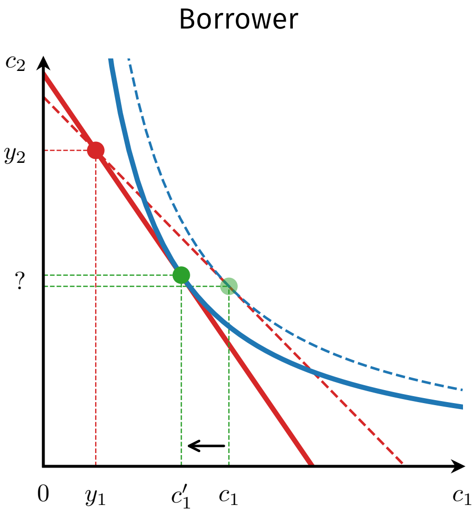
Income effect depends on the desired assets prior to interest rate change:

- Saver ($a > 0$): expansion of the budget set induces increases in consumption in both periods
- Borrower ($a < 0$): contraction of the budget set induces decreases in consumption in both periods

Effects of an increase in r	Saver			Borrower		
	c_1	c_2	a	c_1	c_2	a
Substitution	$-$	$+$	$+$	$-$	$+$	$+$
Income	$+$	$+$	$-$	$-$	$-$	$+$
Net	?	$+$	?	$-$	?	$+$

Comparative Statics: changes in real interest rate r





Effects of changes in interest rate in the data

Figure 5: Dynamic effects of a 25 basis point unanticipated interest rate cut on the expenditure of durable goods by housing tenure group. Grey areas are bootstrapped 90% confidence bands. Top row: UK (FES/LCFS data). Bottom row: US (CEX data).

Cloyne, Ferreira, Surico (2016) *Monetary policy when households have debt*

Additional constraints

Borrowing constraint

Now the agent cannot have negative assets

$$\begin{aligned} \max_{c_1, c_2, a} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r)\, a \\ & \boldsymbol{a \geq 0} \end{aligned}$$

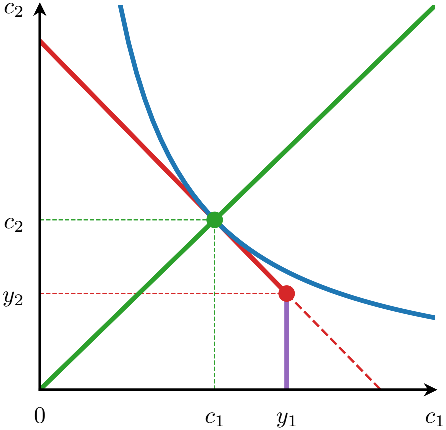
Either the agent would choose $a > 0$ and the constraint is not binding

Or they would like to choose $a < 0$ and the constraint is binding:

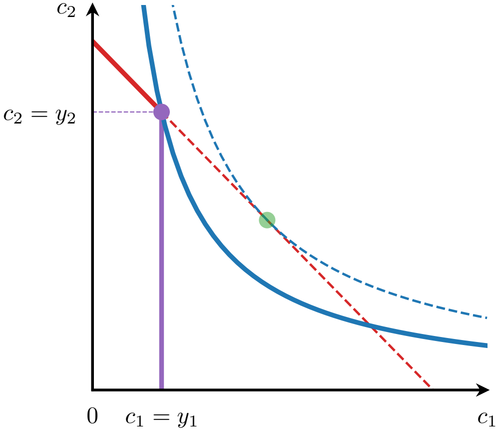
$$a = 0, \quad c_1 = y_1, \quad c_2 = y_2$$

Borrowing constraint: graphical interpretation

Case 1: constraint not binding



Case 2: constraint binding



In Case 2 the agent changes current consumption following any change in income

Two interest rates

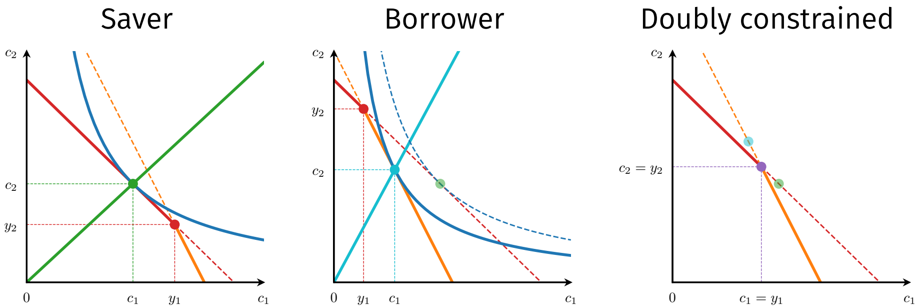
A similar, more realistic set-up is when the agent can freely borrow amount b, but at a higher interest rate $r^b > r$

$$\begin{aligned} \max_{c_1, c_2, a, b} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + a = y_1 + b \\ & c_2 + (1 + r^b) b = y_2 + (1 + r)\, a \\ & \boldsymbol{a \geq 0} \\ & \boldsymbol{b \geq 0} \end{aligned}$$

We now have three (sensible) cases:

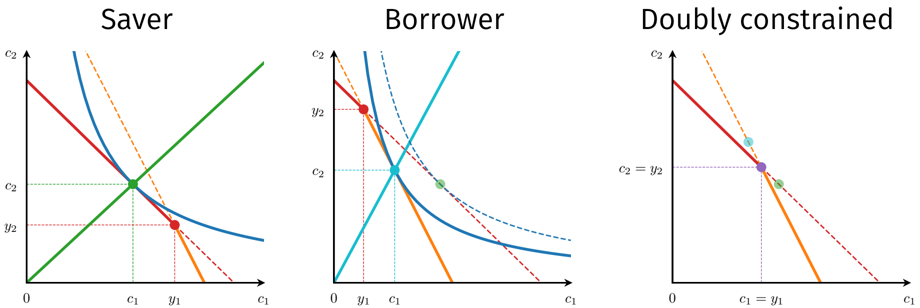
1. Saver: ($a > 0,\ b = 0$)
2. Borrower: ($a = 0$, $b > 0$)
3. Doubly constrained: ($a = 0$, $b = 0$)

Two interest rates: graphical interpretation



In the third case the agent behaves (locally) as if borrowing constrained

Sensitivity of c_1 to monetary policy (MP) & fiscal policy (FP) changes



MP: low

FP: zero

MP: high

FP: low

MP: zero

FP: high

Uncertainty

Uncertainty in income

Consider a two-period expected utility maximization problem

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = \ln c_1 + \beta \mathrm{E}\left[\ln c_2\right] \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1 + r)\, a \end{aligned}$$

First period income is certain and equals y

Second period income will be equal to either $y + e$ or $y - e$:

$$y_2 = \begin{cases} y + e & \text{with probability } 1/2 \\ y - e & \text{with probability } 1/2 \end{cases}$$

Uncertainty in income

Assume $\beta = 1$ and $r = 0$ for simplicity

Use budget constraints to express consumption levels

$$c_1 = y - a$$

$$c_2 = \begin{cases} y + e + a & \text{with probability } 1/2 \\ y - e + a & \text{with probability } 1/2 \end{cases}$$

Rewrite the problem as choosing the optimal a alone:

$$\max_a \quad U = \ln(y - a) + \frac{1}{2}\ln(y + e + a) + \frac{1}{2}\ln(y - e + a)$$

First order condition:

$$-\frac{1}{y - a} + \frac{1}{2}\frac{1}{y + e + a} + \frac{1}{2}\frac{1}{y - e + a} = 0$$

Precautionary saving

$$a = \frac{1}{2}\left(\sqrt{y^2 + 2e^2} - y\right)$$

When second period income is certain ($e = 0$) then (given $\beta = 1$ and $r = 0$) the household holds no assets in optimum and enjoys smooth consumption over time, since $c_1 = c_2 = y$

When there is uncertainty about second period income ($e > 0$), the household accumulates **precautionary savings** to self-insure against the scenario of low income in the second period.[1] The more uncertain second period income is, the higher is the stock of accumulated assets:

$$\frac{\partial a}{\partial e} = \frac{1}{2} \cdot \frac{1}{2\sqrt{y^2 + 2e^2}} \cdot 2 \cdot 2e = \frac{e}{\sqrt{y^2 + 2e^2}} > 0$$

[1] To get this result of "prudence", the utility function has to satisfy $u''' > 0$

Uncertainty in future income and ex-post rate of return

$$\max_{c_1,\, c_2,\, a} \quad U = \ln c_1 + \beta \mathrm{E}\left[\ln c_2\right]$$

$$\text{subject to} \quad c_1 + a = y_1$$

$$c_2 = y_2 + (1 + r_2)\, a$$

Set up the Lagrangian

$$\mathcal{L} = \ln \boldsymbol{c_1} + \beta \mathrm{E}\left[\ln \boldsymbol{c_2}\right] + \lambda_1 \left[y_1 - \boldsymbol{c_1} - \boldsymbol{a}\right] + \mathrm{E}\left[\lambda_2 \left[y_2 + (1 + r_2)\, \boldsymbol{a} - \boldsymbol{c_2}\right]\right]$$

First order conditions (FOCs)

$$\begin{aligned}
c_1 &: \quad \frac{1}{c_1} - \lambda_1 = 0 && \rightarrow \quad \lambda_1 = \frac{1}{c_1} \\
c_2 &: \quad \mathrm{E}\left[\beta \frac{1}{c_2}\right] - \mathrm{E}\left[\lambda_2\right] = 0 && \rightarrow \quad \lambda_2 = \beta \frac{1}{c_2} \\
a &: \quad -\lambda_1 + \mathrm{E}\left[\lambda_2 (1 + r_2)\right] = 0 && \rightarrow \quad \lambda_1 = \mathrm{E}\left[\lambda_2 (1 + r_2)\right]
\end{aligned}$$

Uncertainty in future income and ex-post rate of return

Resulting optimality condition

$$\frac{1}{c_1} = \mathrm{E}\left[\beta \frac{1}{c_2}\left(1 + r_2\right)\right]$$

We need to be extra careful not to break any expectation operators!

Rewrite the Euler equation in the following way

$$1 = \mathrm{E}\left[\beta \frac{c_1}{c_2}\left(1 + r_2\right)\right] \equiv \mathrm{E}\left[\beta \frac{u'\left(c_2\right)}{u'\left(c_1\right)} \cdot \left(1 + r_2\right)\right]$$

This is an asset pricing equation. Here the price of a unit of savings is one unit of first period consumption. The payoff from having an asset in the second period will be $\left(1 + r_2\right)$. The term $\beta \cdot c_1 / c_2$ (or $\beta \cdot u'\left(c_2\right) / u'\left(c_1\right)$ in the general case) is called the stochastic discount factor and measures the relative marginal utility of consumption across periods.

Asset pricing: general case

Investors can buy or sell as much of the payoff x_2 as they wish, at a price p_1

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = u\left(c_1\right) + \mathrm{E}\left[\beta u\left(c_2\right)\right] \\ \text{subject to} \quad & c_1 + p_1 \cdot a = y_1 \\ & c_2 = y_2 + x_2 \cdot a \end{aligned}$$

Set up the Lagrangian

$$\mathcal{L} = u\left(c_1\right) + \mathrm{E}\left[\beta u\left(c_2\right)\right] + \lambda_1\left[y_1 - \boldsymbol{c_1} - p_1 \cdot \boldsymbol{a}\right] + \mathrm{E}\left[\lambda\left[y_2 + x_2 \cdot \boldsymbol{a} - \boldsymbol{c_2}\right]\right]$$

Resulting optimality condition

$$p_1 \cdot u'\left(c_1\right) = \mathrm{E}\left[\beta u'\left(c_2\right) \cdot x_2\right] \quad \rightarrow \quad p_1 = \mathrm{E}\left[\beta \frac{u'\left(c_2\right)}{u'\left(c_1\right)} \cdot x_2\right] \equiv \mathrm{E}\left[m_2 \cdot x_2\right]$$

Pricing a bond: a simplified example

Utility function is logarithmic, $\beta = 0.95$ and $c_1 = 1$

Second period consumption can take two values: high $c_2^h = 1.1$ and low $c_2^l = 0.9$, with $q = 0.5$ being the probability of the low state

Use $p_1 = \mathrm{E}\left[m_2 \cdot x_2\right]$ to price bonds and stocks in this economy

Stochastic discount factor

$$\mathrm{E}\left[m_2\right] = \mathrm{E}\left[\beta \frac{u'\left(c_2\right)}{u'\left(c_1\right)}\right] = \beta \mathrm{E}\left[\frac{c_1}{c_2}\right] = \beta\left[q \cdot \frac{c_1}{c_2^l} + (1-q) \cdot \frac{c_1}{c_2^h}\right] \approx 0.9596$$

Price and return of a bond that pays off $x_2^b = 1$ with certainty

$$p_1^b = \mathrm{E}[m_2 \cdot x_2^b] = \mathrm{E}\left[m_2 \cdot 1\right] \approx 0.9596$$

$$1 + r_2^b = \frac{x_2^b}{p_1^b} = \frac{1}{0.9596} \approx 1.0421 \quad \rightarrow \quad r_2^b \approx 4.2\%$$

Pricing a stock: a simplified example

A stock pays dividend $d_2^h = 1.2$ in high state and $d_2^l = 0.8$ in low state, with a resale value of $p_2^s = 0$ for simplicity (so that $\mathrm{E}[x_2^s] = 1$)

$$p_1^s = \mathrm{E}[m_2 \cdot x_2^s] = \mathrm{E}[m_2 \cdot (d_2 + p_2^s)] = \mathrm{E}[m_2 \cdot d_2]$$

Important to remember that (unless SDF m_2 and d_2 are independent)

$$\mathrm{E}[m_2 \cdot d_2] \neq \mathrm{E}[m_2] \cdot \mathrm{E}_1[d_2]$$

The stock price and expected return are calculated as follows

$$p_1^s = \beta \left[q \frac{c_1}{c_2^l} d_2^l + (1-q) \frac{c_1}{c_2^h} d_2^h \right] \approx 0.9404$$

$$\mathrm{E}[1 + r_2^s] = \frac{\mathrm{E}[x_2^s]}{p_1^s} = \frac{1}{0.9404} \approx 1.0634 \quad \rightarrow \quad \mathrm{E}[r_2^s] \approx 6.3\%$$

Equity risk premium

The stock is cheaper than a bond, although their expected payoffs are identical

This is because stock dividends and the SDF exhibit negative covariance (while stock dividends and future consumption exhibit positive covariance)

Investors receive higher payoff in the state where consumption is high anyway, and a lower payoff when consumption is already low

The expected return on the stock needs then to be higher to motivate investors to hold the risky asset

$$\mathrm{E}[r_2^s - r_2^b] \approx 2.1\%$$

Current research suggests that the majority of equity risk premium arises due to the possibility of drawdowns in the 10-30% range, typical for recessions where income (consumption) risk increases significantly

Possible topics

- Income and wealth inequality
- Human capital accumulation
- Lifecycle profiles of income, consumption and assets
- Pension system design and analysis
- Economic growth
- Business cycles modeling
- Monetary policy design and analysis
- Housing and financial markets imperfections
- Sovereign debt default
- … and more