Marcin Bielecki, Advanced Macroeconomics IE, Spring 2024
Homework 1, deadline: February 27, 4:45 PM

Problem 1

Consider the following two-period utility maximization problem. This utility function belongs to the Constant Relative Risk Aversion (CRRA) class of functions that will be often used throughout our course.[1] An agent lives for two periods and in both receives some positive income. Solve for the optimal consumption values.

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = \frac{c_1^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r)\, a \end{aligned}$$

where $\sigma \geq 0$, $\beta \in [0,1]$ and $r \geq -1$.

(a) Rewrite the budget constraints into a single lifetime budget constraint and set up the Lagrangian.

(b) Obtain the first order conditions for c_1 and c_2. Express c_2 as a function of c_1.

(c) Using the lifetime budget constraint obtain the formulas for optimal c_1 and c_2.

(d) Using your results from (c), set $\sigma = 1$ and verify that the formulas for optimal c_1 and c_2 are identical to the ones we obtained in class for the utility function $U = \ln c_1 + \beta \ln c_2$.

(e) Return to expressions obtained in (c). Assume now that $y_2 = 0$. How does c_1 react when interest rate r increases? How does this reaction depend on σ?

Problem 2

Solve the following household problem that faces future income uncertainty:

$$\max_{c_1,\, c_2,\, a} \quad U = \ln c_1 + \mathrm{E}\left[\ln c_2\right] \qquad \text{subject to} \quad c_1 + a = y_1 \quad \text{and} \quad c_2 = y_2 + a$$

where for simplicity it was already assumed that $\beta = 1$ and $r = 0$. Moreover, assume that first period income is certain and equals y, while second period income will be equal to either $y + e$ or $y - e$, with 50-50% probability (where $0 \leq e < y$):

$$y_2 = \begin{cases} y + e & \text{with probability } 1/2 \\ y - e & \text{with probability } 1/2 \end{cases}$$

(a) Using budget constraints express c_1 and possible levels of c_2 as functions of y, e and a.

(b) Using the properties of the expected value, rewrite the household's utility in terms of possible realizations of current and future consumption, using expressions prepared in (a).

(c) Your problem should at this stage look like this: $\max \quad U(y, e, a)$ where a is the only choice variable. Find optimal a (no need for Lagrangian, simply calculate $\partial U / \partial a = 0$ and solve the resulting quadratic equation for a).

(d) What are the levels of optimal c_1, c_2 and a when $e = 0$? Provide economic interpretation of these results.

(e) What are the levels of optimal c_1, c_2 and a when $e > 0$? Show also that $\partial a / \partial e > 0$. Provide economic interpretation of these results.

[1] The CRRA function can be thought of as a generalized logarithmic function. For $\sigma = 1$ the CRRA function becomes logarithmic, which can be easily proven by using the L'Hôpital's rule to compute the following limit: $\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma}$

Problem 3

Consider the following two-period model with two assets (bonds and physical capital):

$$\begin{aligned} \max_{c_1, c_2, b, k} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + b + k = y_1 \\ & c_2 = y_2 + (1+r)\, b + (1-\delta)\, k \\ & y_2 = Ak^{\alpha} \end{aligned}$$

where b denotes bonds, $\delta \in (0,1)$ stands for capital depreciation rate, $A > 0$ is the level of technology and $\alpha \in (0,1)$ is the elasticity of production to capital.

(a) Write down the problem in the form of a Lagrangian.

(b) Find the optimal values of c_1, c_2, b and k. *Hint: once you find the optimal k, you can treat it as a parameter.*

(c) Calculate the derivative of optimal k with respect to r. Provide intuition for this result. *Hint: you can assume $\alpha = 1/2$ to simplify the calculations.*

(d) What is the effect of an increase in A on c_1, c_2, b and k? Provide intuition for this result.

Problem 4

In this economy households enter period 1 with a unit share of firm stock each ($\tilde{s} = 1$) and receive endowment $y_1 = 1$. They can then use these resources to either consume or use to purchase stocks s and bonds b, at their respective prices p^s and p^b. Each unit of a bond will pay a unit of consumption in period 2 with certainty, while both labor income and stock payoff are subject to uncertainty. The firms are going to generate revenue $y_2 = \{y^l, y^h\}$ where $y^l < y^h$, with the probability of the low state denoted by q. The firm (stock) owners will receive a fraction $\alpha \in (0,1)$ of firm's revenue, while workers will receive a fraction $1 - \alpha$ of firm's revenue as their labor income. The problem of the household is then:

$$\begin{aligned} \max_{c_1,\, c_2,\, s,\, b} \quad & U = \frac{c_1^{1-\sigma}}{1-\sigma} + \beta \mathrm{E}\left[\frac{c_2^{1-\sigma}}{1-\sigma}\right] \\ \text{subject to} \quad & c_1 + p^s s + p^b b = y_1 + p^s \tilde{s} \\ & c_2 = (1-\alpha)\, y_2 + d_2 \cdot s + b \\ & d_2 = \alpha y_2 \end{aligned}$$

(a) Using the Lagrangian approach, derive the first order conditions of the households.

(b) Combine the FOCs w.r.t. c_1, c_2 and s to obtain the "stock" Euler equation.
Combine the FOCs w.r.t. c_1, c_2 and b to obtain the "bond" Euler equation.

(c) Examine the equilibrium where $s = 1$ (households are satisfied with current holdings of stocks and there are no splits or mergers) and $b = 0$ (nobody issues or buys bonds). Using the budget constraints and properties of the expected value find the expressions for asset prices.

(d) Assume that: $\sigma = 1$, $q = 1/2$, $y^l = 1 - e$, $y^h = 1 + e$, with $e \in [0,1)$. Find the asset prices and their expected returns. Calculate the equity risk premium. When would it be equal to 0? Why? *Hint: the resulting stock price will be independent of e, which is an artifact due to our simplifying assumptions.*

(e) Suppose now that during the low state both labor and asset income are subject to additional, symmetric idiosyncratic risk, and are given respectively by $(1 \pm z)\,(1-\alpha)\, y_2^l$ and $(1 \pm z)\, \alpha y_2^l$. Each household randomly draws either positive or negative $z \in [0,1)$ with 50-50% probability. Assume the same numerical values as in (d). Calculate asset prices and expected returns. How does the equity risk premium depend on z^2? Why? *Hint: the resulting stock price will be independent of z, which is an artifact due to our simplifying assumptions.*