University of Warsaw
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Ramsey Model (Neoclassical Growth Model)

Advanced Macroeconomics IE

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Ramsey Model

Ramsey Model (Neoclassical Growth Model)

Authors: Ramsey (1928), Cass (1965), Koopmans (1965), Malinvaud

Other common names: Ramsey-Cass-Koompans model, RCK model

Extends the Solow model with optimal household decisions
regarding consumption and asset accumulation

Allows to evaluate welfare effects of economic policies

Core model of “modern” macroeconomics

Extensions of Ramsey model are used for both growth and business cycle analysis

Usually the model is presented in continuous time,
here time will be discrete like in the business cycle models we’ll learn later

Simplifications and assumptions

Closed economy

No government (for now)

Population grows at rate n (possibly negative)

Perfect competition in markets for goods and factors of production

Homogeneous final good with price normalized to 1 in every period, produced according to a neoclassical production function

All variables and prices expressed in real terms

Two groups of representative agents

- Households
- Firms

Households own factors of production directly and rent them to firms

Digression on the representative agent assumption

Pop-econ understanding of representative agent

- Households have identical preferences
- Households have identical endowments (streams of income)
- Everyone is identical and there is no trade in equilibrium

“Reality” of representative agent assumption

- There is lots of trade in financial assets under the hood!
- Financial markets are assumed to be complete
- Agents trade with each other to insure away all idiosyncratic (i.e. individual) risk: see Arrow-Debreu securities, Arrow securities, Radner equilibrium
- This works even if agents have heterogeneous endowments, preferences and subjective beliefs (although then “aggregate” welfare is elusive)

Digression on the representative agent assumption

Sufficient conditions for emergence of a representative firm

- Firms take prices of inputs and outputs as given
- Firms don't face binding borrowing constraints

Sufficient conditions for emergence of a **normative** representative household

- Households take prices as given
- Households don't face binding borrowing constraints
- Households' preferences admit a Gorman form with equal wealth coefficients
  $\hookrightarrow$ differences in endowments don't matter for aggregate demand (very restrictive)

Constantinides (1982): any complete-markets price-taking outcome looks as if it were the solution to a single, optimizing, **positive** representative household

But such representative agent might be unstable and her preferences may not reflect the preferences of any actual household (though we don't seem to have any alternatives)

Back to Ramsey: construction of the “dynastic” welfare function

Current family members care equally for all family members, present and future

$$U_t = u(c_t) + \beta U_{t+1}$$

“Dynastic” welfare function’s planning horizon becomes effectively infinite

$$U_0 = u(c_0) + \beta U_1 = u(c_0) + \beta\left[u(c_1) + \beta U_2\right] = u(c_0) + \beta u(c_1) + \beta^2 U_2$$
$$U_0 = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \beta^3 u(c_3) + \ldots$$

Using summation notation

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Whenever convenient we will use households’ **discount rate** ρ

$$\beta \equiv \frac{1}{1+\rho}, \quad \rho \equiv \frac{1}{\beta} - 1$$

Households' budget constraint

Labor, asset and dividend income is pooled together at the family level
and split between consumption (equal for all family members) and assets

$$C_t + Assets_{t+1} = w_t L_t + (1 + r_t)\, Assets_t + D_t \quad \Big| \quad : L_t$$
$$\frac{C_t}{L_t} + \frac{Assets_{t+1}}{L_t} = w_t + (1 + r_t)\, \frac{Assets_t}{L_t} + \frac{D_t}{L_t}$$

Since the number of workers L is proportional to population N we can safely ignore the distinction between consumption per worker (C/L) and per person (C/N) in U

As in the Solow model, small letter variables denote quantities per worker

$$c_t + \frac{L_{t+1}}{L_t} \frac{Assets_{t+1}}{L_{t+1}} = w_t + (1 + r_t)\, a_t + d_t$$
$$c_t + (1 + n)\, a_{t+1} = w_t + (1 + r_t)\, a_t + d_t$$

Households' problem

Households' Utility Maximization Problem (UMP)

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to $\quad c_t + (1+n)\, a_{t+1} = w_t + (1+r_t)\, a_t + d_t \quad$ for all $t = 0, 1, \ldots, \infty$

$a_0 > 0 \quad$ given

No Ponzi Game $\quad \lim_{t\to\infty} \beta^t \lambda_t a_{t+1} \geq 0$

Construct the Lagrangian and expand it around the choice variables in t: c_t and a_{t+1}

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - (1+n)\, a_{t+1}\right] \\
&= \ldots + \beta^t u\left(\boldsymbol{c_t}\right) + \ldots + \beta^t \lambda_t \left[w_t + (1+r_t)\, a_t - \boldsymbol{c_t} - (1+n)\, \boldsymbol{a_{t+1}}\right] \\
&\qquad + \beta^{t+1} \left\{\lambda_{t+1} \left[w_{t+1} + (1+r_{t+1})\, \boldsymbol{a_{t+1}} - c_{t+1} - (1+n)\, a_{t+2}\right]\right\} + \ldots
\end{aligned}$$

Households' problem

Expanded Lagranian

$$\mathcal{L} = \ldots + \beta^t u\left(\boldsymbol{c_t}\right) + \ldots + \beta^t \lambda_t \left[w_t + \left(1 + r_t\right) a_t - \boldsymbol{c_t} - \left(1 + n\right) \boldsymbol{a_{t+1}}\right] \\ + \beta^{t+1} \left\{\lambda_{t+1} \left[w_{t+1} + \left(1 + r_{t+1}\right) \boldsymbol{a_{t+1}} - c_{t+1} - \left(1 + n\right) a_{t+2}\right]\right\} + \ldots$$

First Order Conditions (FOCs)

$$\boldsymbol{c_t} : \quad \beta^t u'\left(c_t\right) + \beta^t \left[-\lambda_t\right] = 0 \quad \rightarrow \quad \lambda_t = u'\left(c_t\right)$$

$$\boldsymbol{a_{t+1}} : \quad \beta^t \lambda_t \left[-\left(1 + n\right)\right] + \beta^{t+1} \lambda_{t+1} \left(1 + r_{t+1}\right) = 0 \quad \rightarrow \quad \lambda_t = \frac{\beta\left(1 + r_{t+1}\right)}{1 + n} \lambda_{t+1}$$

Resulting Euler equation

$$u'\left(c_t\right) = \frac{\beta\left(1 + r_{t+1}\right)}{1 + n} u'\left(c_{t+1}\right)$$

Constant Relative Risk Aversion utility function

We will use the Constant Relative Risk Aversion (CRRA) utility function with $\sigma > 0$

$$u\left(c_t\right) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \quad \rightarrow \quad u'\left(c_t\right) = c_t^{-\sigma}$$

For $\sigma = 1$ the CRRA function collapses to the logarithmic function

Parameter σ is also the inverse of Intertemporal Elasticity of Substitution (IES)

The higher σ is, the "smoother" is the desired consumption path
and consumption reacts less to changes in real interest rate

Euler equation for the CRRA function (recall that $\beta = 1/\left(1+\rho\right)$)

$$c_t^{-\sigma} = \frac{\beta\left(1 + r_{t+1}\right)}{1+n} c_{t+1}^{-\sigma} \quad \rightarrow \quad c_{t+1} = \left[\frac{1 + r_{t+1}}{\left(1+\rho\right)\left(1+n\right)}\right]^{1/\sigma} c_t$$

Consumption increases over time if only $1 + r_{t+1} > \left(1+\rho\right)\left(1+n\right)$

Firms' problem

For now assume that technology level is constant

Perfectly competitive, representative firms maximize profits / dividends in every period

$$\begin{aligned}
\max_{K_t,\, L_t} \quad & D_t = 1 \cdot Y_t - w_t L_t - r_t^K K_t \\
\text{subject to} \quad & Y_t = F\left(K_t, L_t\right) \\
& r_t^K = r_t + \delta
\end{aligned}$$

Production function in intensive (per worker) form

$$y_t = \frac{Y_t}{L_t} = \frac{F\left(K_t, L_t\right)}{L_t} = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) = F\left(k_t, 1\right) \equiv f\left(k_t\right)$$

Rewritten profit maximization problem

$$\max_{k_t,\, L_t} \quad D_t = L_t \left[f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t\right]$$

Firms' problem

Firms' Profit Maximization Problem (PMP)

$$\max_{k_t,\, L_t} \quad D_t = L_t \left[f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t \right]$$

First Order Conditions (FOCs)

$$\begin{aligned} k_t &: \quad L_t \left[f'\left(k_t\right) - \left(r_t + \delta\right) \right] = 0 & \rightarrow \quad r_t &= f'\left(k_t\right) - \delta \\ L_t &: \quad f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t = 0 & \rightarrow \quad w_t &= f\left(k_t\right) - k_t f'\left(k_t\right) \end{aligned}$$

Factor prices in equilibrium depend on the level of capital per worker k

Economic profits are equal to 0

$$D_t = L_t \left[f\left(k_t\right) - \left(f\left(k_t\right) - k_t f'\left(k_t\right) \right) - \left(f'\left(k_t\right) - \delta + \delta \right) k_t \right] = 0$$

General equilibrium

Equilibrium in the asset market requires that $a = k$ in every period

$$c_t + (1+n)\,a_{t+1} = w_t + (1+r_t)\,a_t + d_t$$
$$c_t + (1+n)\,k_{t+1} = w_t + (1+r_t)\,k_t + d_t$$

We now plug in firm profits per worker (alternatively we could plug in factor prices)

$$c_t + (1+n)\,k_{t+1} = w_t + (1+r_t)\,k_t + [f\,(k_t) - w_t - (r_t + \delta)\,k_t]$$
$$c_t + (1+n)\,k_{t+1} = f\,(k_t) + (1-\delta)\,k_t$$

Equation analogous to the fundamental equation of the Solow model

$$(1+n)\,k_{t+1} = (f\,(k_t) - c_t) + (1-\delta)\,k_t$$
$$(1+n)\,k_{t+1} = \boldsymbol{s_t} f\,(k_t) + (1-\delta)\,k_t$$

but the saving rate $\boldsymbol{s}$ is now endogenous

Ramsey model dynamic equations

We already have the **resource constraint** ($Y_t = C_t + I_t$)

$$(1+n)\,k_{t+1} = f\,(k_t) + (1-\delta)\,k_t - c_t$$

Plug in the interest rate to get the final form of the **Euler equation**

$$c_{t+1} = \left[\frac{1 + f'\,(k_{t+1}) - \delta}{(1+\rho)\,(1+n)}\right]^{1/\sigma} c_t$$

We now have a system of two dynamic equations in capital and consumption per worker

This time we don't have global stability (as in Solow) but **saddle path stability**

We have a **unique** solution: there exists precisely one sequence of optimal (preference-consistent) saving rates $\{s_t\}_{t=0}^{\infty}$ leading the system towards the steady state

Other paths can also lead to steady state, but involve welfare losses (EE not satisfied)

The decentralized solution is **efficient** and the government cannot improve upon it **[HW]**

Steady state

The steady state satisfies $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$

$$c^* = \left[\frac{1 + f'(k^*) - \delta}{(1+\rho)(1+n)}\right]^{1/\sigma} c^*$$

$$(1+n) k^* = f(k^*) + (1-\delta) k^* - c^*$$

Start with the Euler equation

$$(1+\rho)(1+n) = 1 + f'(k^*) - \delta$$

$$f'(k^*) = (1+\rho)(1+n) - (1-\delta)$$

$$f'(k^*) \simeq \rho + \delta + n$$

Knowing k^* we find c^* using the resource constraint

$$c^* = f(k^*) + (1-\delta) k^* - (1+n) k^*$$

$$c^* = f(k^*) - (\delta + n) k^*$$

Steady state

For the Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} \quad \rightarrow \quad y_t = f(k_t) = k_t^{\alpha} \quad \rightarrow \quad f'(k_t) = \alpha k_t^{\alpha-1}$$

Steady state level of capital per worker k^*

$$\alpha (k^*)^{\alpha-1} = \rho + \delta + n$$

$$k^* = \left(\frac{\alpha}{\rho + \delta + n} \right)^{1/(1-\alpha)}$$

Steady state level of capital per worker in the Solow model

$$k^* = \left(\frac{s}{\delta + n} \right)^{1/(1-\alpha)}$$

The (steady state) saving rate in the Ramsey model equals $s_{GR} = \alpha$ if only $\rho = 0$

Since $\rho \geq 0$, Ramsey economy is always **dynamically efficient**

Ramsey model dynamics

Ramsey model dynamics: analytical solution

Assume Cobb-Douglas production function ($Y = K^\alpha L^{1-\alpha}$), logarithmic utility ($\sigma = 1$) and total capital depreciation ($\delta = 1$)

$$\begin{aligned} \text{Euler equation} &: \quad \frac{c_{t+1}}{c_t} = \frac{\alpha k_{t+1}^{\alpha-1}}{(1+\rho)(1+n)} \\ \text{Resource constraint} &: \quad (1+n)\, k_{t+1} = k_t^\alpha - c_t \end{aligned}$$

"Guess-and-verify" that the saving rate is constant (just like in the Solow model)

$$c_t = (1-s)\, y_t = (1-s)\, k_t^\alpha$$

Euler equation

$$\frac{(1-s)\, k_{t+1}^\alpha}{(1-s)\, k_t^\alpha} = \frac{\alpha k_{t+1}^{\alpha-1}}{(1+\rho)(1+n)} \quad \rightarrow \quad (1+n)\, k_{t+1} = \frac{\alpha}{1+\rho} k_t^\alpha = \alpha\beta k_t^\alpha$$

Resource constraint

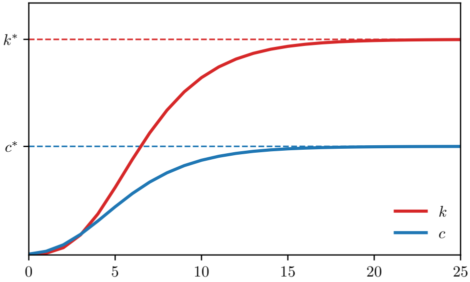
$$\alpha\beta k_t^\alpha = k_t^\alpha - c_t \quad \rightarrow \quad c_t = (1-\alpha\beta)\, k_t^\alpha \quad \rightarrow \quad s = \alpha\beta \leq \alpha$$

Ramsey model dynamics: analytical solution

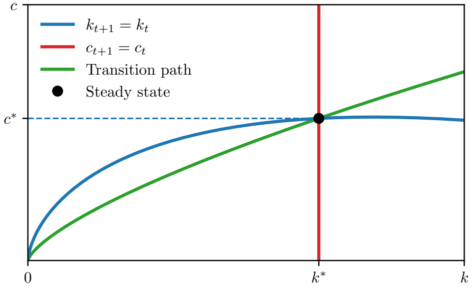
Since $\{s_t\}_{t=0}^{\infty} = \{\alpha\beta\}$ the **transition path** is given by

$$k_{t+1} = \frac{\alpha\beta}{1+n} k_t^{\alpha}$$
$$c_t = (1-\alpha\beta)\, k_t^{\alpha}$$

Dynamics of c and k



Phase diagram



Ramsey model dynamics: numerical solution

Ramsey model has no analytical solutions except in a few special cases

Solutions can be easily found using quasi-analytical or numerical methods

- Linear approximation of dynamic equations around the steady state
- Newton methods for solving systems of nonlinear equations
- Numerical methods for solving systems of differential equations
- Dynamic programming methods
- Shooting algorithm

Ramsey model dynamics: numerical solution

Shooting algorithm description

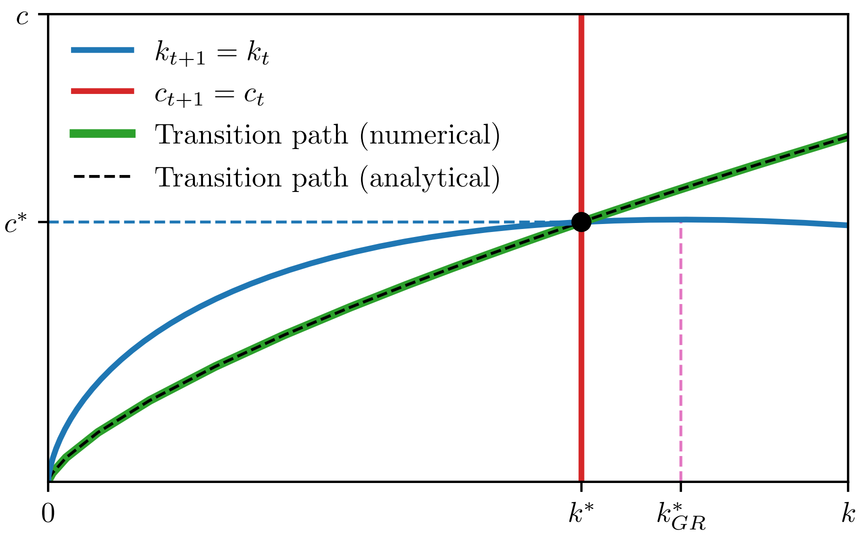
1. For some initial k_0 propose intial consumption c_0
2. Calculate resulting path using dynamic equations

$$k_{t+1} = \frac{k_t^{\alpha} + (1-\delta)\, k_t - c_t}{1+n}$$

$$c_{t+1} = \left[\frac{1 + \alpha k_{t+1}^{\alpha-1} - \delta}{(1+\rho)\,(1+n)}\right]^{1/\sigma} c_t$$

3. Calculate convergence criterion (what was the “miss” relative to the steady state)
4. Find c_0 minimizing the “miss” for the given k_0
5. The resulting sequences $\{k_t,\, c_t\}_{t=0}^{\infty}$ lie on the transition path

Ramsey model dynamics: numerical solution



Phase diagram construction

From the Euler equation get condition for $c_{t+1} = c_t$

$$c_{t+1} = \left[\frac{1 + f'(k_{t+1}) - \delta}{(1+\rho)(1+n)}\right]^{1/\sigma} c_t \quad \rightarrow \quad f'(k_{t+1}) \simeq \rho + \delta + n$$

If $k_t < k^*$ then $k_{t+1} < k^*$, $f'(k_{t+1}) > f'(k^*)$, $r_{t+1} > r^*$ and $c_{t+1} > c_t$

From the resource constraint get condition for $k_{t+1} = k_t$

$$(1+n)\,k_{t+1} = f(k_t) + (1-\delta)\,k_t - c_t \quad \rightarrow \quad c_t = f(k_t) - (\delta + n)\,k_t$$

If $c_t < f(k_t) - (\delta + n)\,k_t$ then $k_{t+1} > k_t$

The transition path lies in those areas of the graph where
at the same time $c_{t+1} > c_t$ and $k_{t+1} > k_t$ **or** at the same time $c_{t+1} < c_t$ and $k_{t+1} < k_t$

Transition path shape

In a special case where $\sigma = \alpha$, optimal consumption is linear in k

$$c_t \simeq \left[\frac{\rho + \delta + n}{\alpha} - (\delta + n)\right] k_t$$

What if $\sigma \neq \alpha$?

- If $\sigma < \alpha$ then the transition path is convex and convergence is quicker
- If $\sigma > \alpha$ then the transition path is concave and convergence is slower

Empirically relevant is the last case ($\alpha \approx 1/3$ and $\sigma \approx 2$)

Whenever technology improves, consumption increases immediately:
this will be a crucial mechanism in the Real Business Cycles model

Saving rate along the transition

Saving rate in the Ramsey model

$$s \equiv \frac{y-c}{y} \quad \rightarrow \quad s^* = \frac{y^* - \left[y^* - (\delta+n)\, k^*\right]}{y^*} = \frac{(\delta+n)\, k^*}{y^*}$$

For the Cobb-Douglas production function

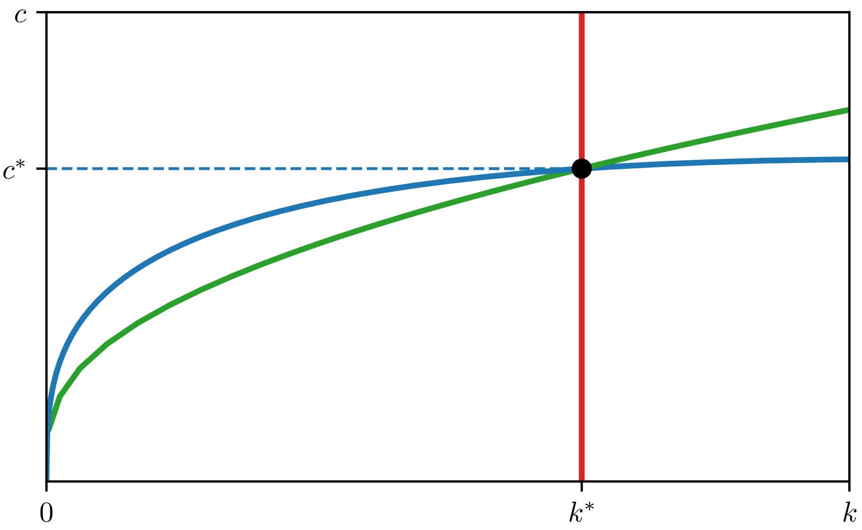
$$y = k^\alpha \quad \text{and} \quad k^* = \left(\frac{\alpha}{\rho+\delta+n}\right)^{1/(1-\alpha)} \quad \rightarrow \quad s^* = (\delta+n)\,(k^*)^{1-\alpha} = \frac{\delta+n}{\rho+\delta+n}\alpha \leq \alpha$$

If $s^* = 1/\sigma$ then $\{s_t\}_{t=0}^{\infty} = s^*$. What if $s^* \neq 1/\sigma$?

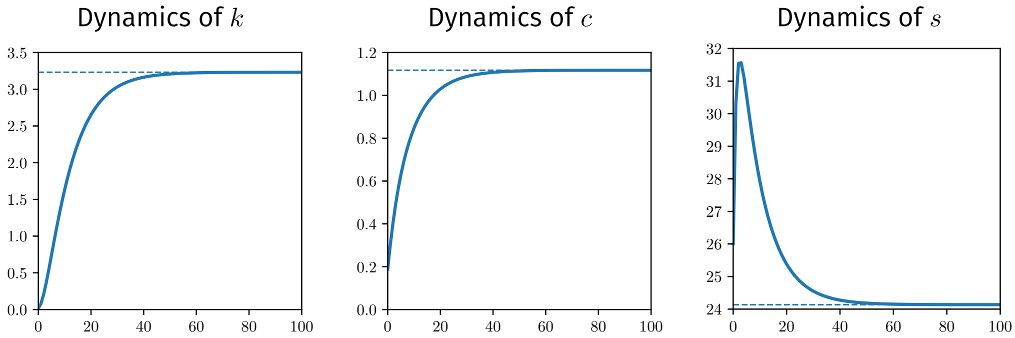
- If $s^* > 1/\sigma$ then the saving rate is initially lower than s^* and rises over time
- If $s^* < 1/\sigma$ then the saving rate is initially higher than s^* and falls over time

It seems the last case is empirically relevant ($s^* \approx 0.2$ and $\sigma \approx 2$)

Ramsey model dynamics: "realistic" parameter values



Ramsey model dynamics: “realistic” parameter values



Since $s_t \geq s^*$ in the “realistic” Ramsey model, convergence speed is even higher than in Solow

Saving rate along the transition: Stone-Geary utility function

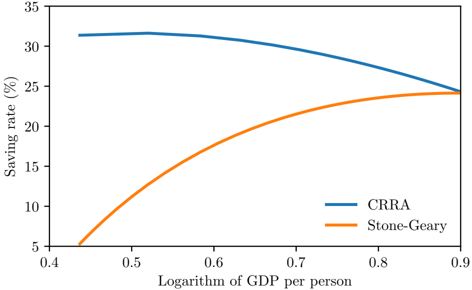
Convergence speed is much lower under the Stone-Geary utility

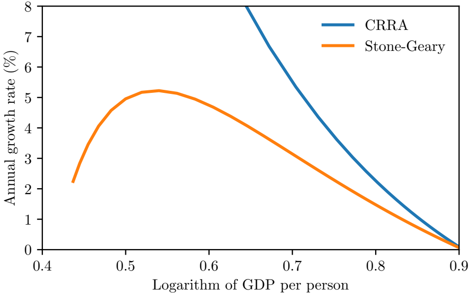
$$u(c_t) = \frac{(c_t - \bar{c})^{1-\sigma} - 1}{1-\sigma}$$

In economies producing barely above $\bar{c}$ per person the saving rate is almost 0

Saving rate increases with GDP per person even when $s^* < 1/\sigma$

Economies with “middle” levels of GDP per person grow the fastest





Technological progress

Firms' problem

Assume that technology A grows at rate g

$$A_{t+1} = (1+g)\,A_t$$

Consistent with Kaldor stylized facts, the production function can be expressed as

$$Y_t = F\left(K_t, A_t L_t\right)$$

Production function in intensive (per effective labor) form

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = \frac{F\left(K_t, A_t L_t\right)}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, \frac{A_t L_t}{A_t L_t}\right) = F(\hat{k}_t, 1) = f(\hat{k}_t)$$

Firms maximize their profits in every period

$$\max_{K_t,\, L_t} \quad D_t = 1 \cdot Y_t - w_t L_t - (r_t + \delta)\, K_t$$

$$\text{subject to} \quad Y_t = F\left(K_t, A_t L_t\right)$$

Firms' problem

Firms' Profit Maximization Problem (where $\hat{w}_t \equiv w_t / A_t$)

$$\max_{\hat{k}_t,\, L_t} \quad D_t = A_t L_t \left[f(\hat{k}_t) - \hat{w}_t - (r_t + \delta)\, \hat{k}_t \right]$$

First Order Conditions (FOCs)

$$\begin{aligned}
\hat{k}_t &: \quad A_t L_t \left[f'(\hat{k}_t) - (r_t + \delta) \right] = 0 && \rightarrow \quad r_t = f'(\hat{k}_t) - \delta \\
L_t &: \quad A_t \left[f(\hat{k}_t) - \hat{w}_t - (r_t + \delta)\, \hat{k}_t \right] = 0 && \rightarrow \quad \hat{w}_t = f(\hat{k}_t) - \hat{k}_t f'(\hat{k}_t)
\end{aligned}$$

Factor prices in equilibrium depend now on capital per effective labor $\hat{k}$

Households' problem

No need to solve again! Just use the definition of consumption per effective labor

$$\hat{c}_t \equiv \frac{c_t}{A_t} \quad \rightarrow \quad c_t = \hat{c}_t A_t$$

And rewrite the Euler equation

$$c_{t+1} = \left[\frac{1 + r_{t+1}}{(1+\rho)(1+n)}\right]^{1/\sigma} c_t$$

$$\hat{c}_{t+1} A_{t+1} = \left[\frac{1 + f'(\hat{k}_{t+1}) - \delta}{(1+\rho)(1+n)}\right]^{1/\sigma} \hat{c}_t A_t$$

$$\hat{c}_{t+1} = \left[\frac{1 + f'(\hat{k}_{t+1}) - \delta}{(1+\rho)(1+n)}\right]^{1/\sigma} \frac{\hat{c}_t}{1+g}$$

General equilibrium

Equilibrium in the asset market requires that $a = k$ in every period

$$
\begin{aligned}
c_t + (1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t + d_t \\
c_t + (1+n)\, k_{t+1} &= w_t + (1+r_t)\, k_t + d_t \quad \Big| \quad : A_t \\
\frac{c_t}{A_t} + (1+n)\, \frac{k_{t+1}}{A_t} &= \frac{w_t}{A_t} + (1+r_t)\, \frac{k_t}{A_t} + \frac{d_t}{A_t} \\
\hat{c}_t + (1+n)\, \frac{A_{t+1}}{A_t} \frac{k_{t+1}}{A_{t+1}} &= \hat{w}_t + (1+r_t)\, \hat{k}_t + \hat{d}_t
\end{aligned}
$$

Plug in firm profits per effective labor (alternatively we could plug in factor prices)

$$
\begin{aligned}
\hat{c}_t + (1+n)\,(1+g)\, \hat{k}_{t+1} &= \hat{w}_t + (1+r_t)\, \hat{k}_t + \left[f(\hat{k}_t) - \hat{w}_t - (r_t + \delta)\, \hat{k}_t \right] \\
(1+n)\,(1+g)\, \hat{k}_{t+1} &= f(\hat{k}_t) + (1-\delta)\, \hat{k}_t - \hat{c}_t
\end{aligned}
$$

We again get the resource constraint ($Y_t = C_t + I_t$)

Ramsey model dynamics

The system of two dynamic equations in consumption and capital per effective labor

$$\text{Euler equation} \quad : \quad \hat{c}_{t+1} = \left[\frac{1 + f'(\hat{k}_{t+1}) - \delta}{(1+\rho)(1+n)}\right]^{1/\sigma} \frac{\hat{c}_t}{1+g}$$

$$\text{Resource constraint} \quad : \quad (1+n)(1+g)\,\hat{k}_{t+1} = f(\hat{k}_t) + (1-\delta)\,\hat{k}_t - \hat{c}_t$$

Balanced Growth Path satisfies $\hat{c}_{t+1} = \hat{c}_t = \hat{c}^*$ and $\hat{k}_{t+1} = \hat{k}_t = \hat{k}^*$

$$\begin{aligned}(1+g)^\sigma &= \frac{1 + f'(\hat{k}^*) - \delta}{(1+\rho)(1+n)} \\ f'(\hat{k}^*) &= (1+g)^\sigma (1+\rho)(1+n) - (1-\delta) \\ f'(\hat{k}^*) &\simeq \rho + \delta + n + \sigma g\end{aligned}$$

$$\begin{aligned}(1+n)(1+g)\,\hat{k}^* &= f(\hat{k}^*) + (1-\delta)\,\hat{k}^* - \hat{c}^* \\ \hat{c}^* &= f(\hat{k}^*) - (\delta + n + g + ng)\,\hat{k}^* \\ \hat{c}^* &\simeq f(\hat{k}^*) - (\delta + n + g)\,\hat{k}^*\end{aligned}$$

Balanced Growth Path (BGP)

For the Cobb-Douglas production function

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} \quad \rightarrow \quad \hat{y}_t = f(\hat{k}_t) = \hat{k}_t^{\alpha} \quad \rightarrow \quad f'(\hat{k}_t) = \alpha \hat{k}_t^{\alpha-1}$$

BGP levels of variables per effective labor

$$\hat{k}^* = \left(\frac{\alpha}{\rho+\delta+n+\sigma g}\right)^{1/(1-\alpha)} \quad \text{and} \quad \hat{y}^* = \left(\frac{\alpha}{\rho+\delta+n+\sigma g}\right)^{\alpha/(1-\alpha)}$$

$$\hat{c}^* = \left(\frac{\alpha}{\rho+\delta+n+\sigma g}\right)^{\alpha/(1-\alpha)} - (\delta+n+g)\left(\frac{\alpha}{\rho+\delta+n+\sigma g}\right)^{1/(1-\alpha)}$$

If $\rho = 0$ and $\sigma = 1$ then we get the same level of capital per effective labor as in the Solow model, provided that $s = s_{GR} = \alpha$

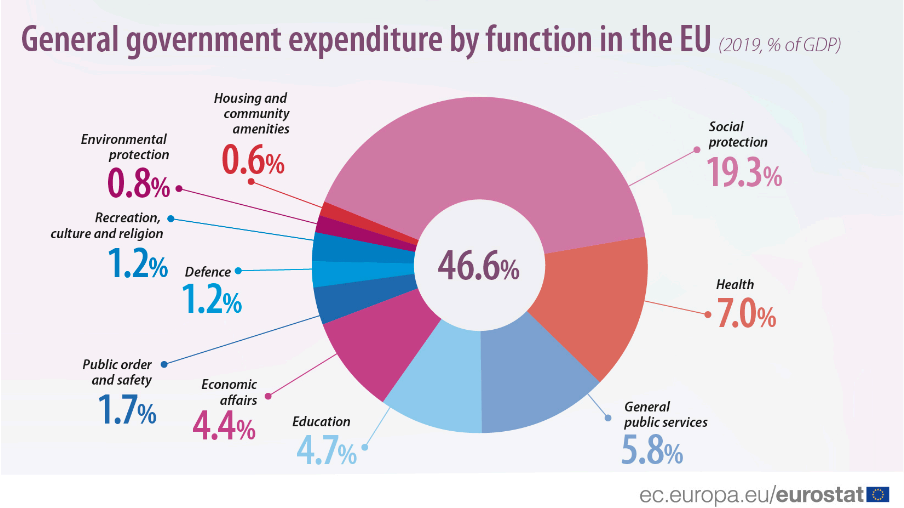
$$\hat{k}^* = \left(\frac{\alpha}{\delta+n+g}\right)^{1/(1-\alpha)}$$

The Ramsey model is still dynamically efficient, for any value of $\sigma > 0$

The higher σ is, the lower are $\hat{k}^*$, $\hat{y}^*$ and $\hat{c}^*$ (as if households were less patient)

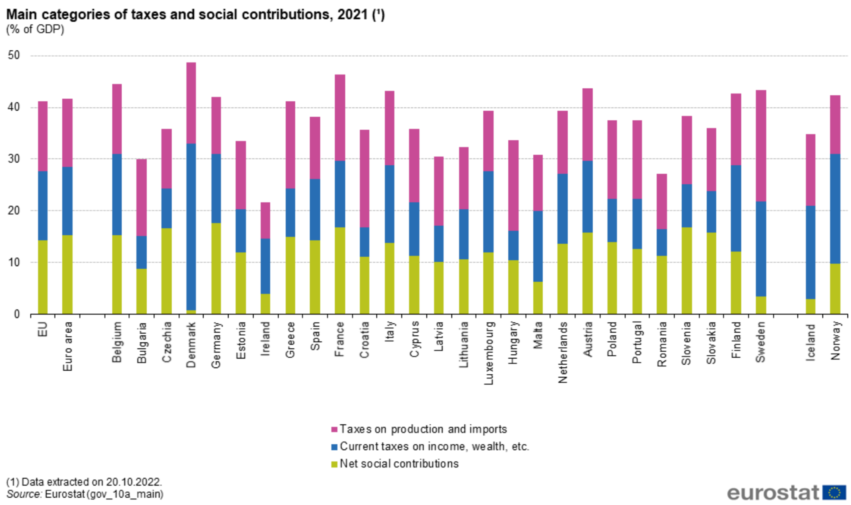
Government sector

General government expenditure in EU



Eurostat, interactive data

General government revenue in EU



Taxes and outlays

Assume constant population and technology for simplicity, $N = A = 1$

Consider a broad array of (linear) taxes

- Consumption tax τ^c
- Labor income tax τ^w
- Asset income tax τ^r
- Firm accounting profits tax τ^f
- Lump-sum tax τ

Divide public outlays into two broad groups

- Government spending per worker γ (schools, hospitals, roads, law enforcement, etc.)
- Transfers per worker v (for now equal to everybody)

Assume balanced budget in every period (Ricardian equivalence holds in Ramsey)

Assume that firms are capital owners and households' assets are claims on firm profits

Households' Utility Maximization Problem (UMP)

$$\max_{\{c_t,\, a_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$\text{subject to} \quad \left(1+\boldsymbol{\tau_t^c}\right) c_t + a_{t+1} = \left(1-\boldsymbol{\tau_t^w}\right) w_t + \left(1+\left(1-\boldsymbol{\tau_t^r}\right) r_t\right) a_t - \boldsymbol{\tau_t} + \boldsymbol{v_t}$$

Construct the Lagrangian and expand it around the choice variables in t, c_t and a_{t+1}

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\left(1-\tau_t^w\right) w_t + \left(1+\left(1-\tau_t^r\right) r_t\right) a_t - \tau_t + v_t - \left(1+\tau_t^c\right) c_t - a_{t+1}\right]$$

$$= \ldots + \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \ldots + \beta^t \lambda_t \left[\left(1-\tau_t^w\right) w_t + \left(1+\left(1-\tau_t^r\right) r_t\right) a_t - \tau_t + v_t - \left(1+\tau_t^c\right) c_t - a_{t+1}\right]$$

$$+ \beta^{t+1} \lambda_{t+1} \left[\left(1-\tau_{t+1}^w\right) w_{t+1} + \left(1+\left(1-\tau_{t+1}^r\right) r_{t+1}\right) a_{t+1} - \tau_{t+1} + v_{t+1} - \left(1+\tau_t^c\right) c_{t+1} - a_{t+2}\right]$$

$$+ \ldots$$

Households' problem

Expanded Lagranian

$$\mathcal{L} = \beta^t \frac{\boldsymbol{c_t}^{1-\sigma}}{1-\sigma} + \beta^t \lambda_t \left[\left(1-\tau_t^w\right) w_t + \left(1 + \left(1-\tau_t^r\right) r_t\right) a_t - \tau_t + v_t - \left(1+\tau_t^c\right) \boldsymbol{c_t} - \boldsymbol{a_{t+1}} \right] + \dots$$
$$+ \beta^{t+1} \lambda_{t+1} \left[\left(1-\tau_{t+1}^w\right) w_{t+1} + \left(1 + \left(1-\tau_{t+1}^r\right) r_{t+1}\right) \boldsymbol{a_{t+1}} - \tau_{t+1} + v_{t+1} - \left(1+\tau_t^c\right) c_{t+1} - a_{t+2} \right]$$

First Order Conditions (FOCs)

$$\boldsymbol{c_t} \ : \quad \beta^t c_t^{-\sigma} + \beta^t \lambda_t \left[-\left(1+\tau_t^c\right) \right] = 0 \qquad \rightarrow \quad \lambda_t = \frac{c_t^{-\sigma}}{1+\tau_t^c}$$

$$\boldsymbol{a_{t+1}} \ : \quad \beta^t \lambda_t [-1] + \beta^{t+1} \lambda_{t+1} \left[1 + \left(1-\tau_{t+1}^r\right) r_{t+1} \right] = 0 \quad \rightarrow \quad \lambda_t = \beta \left(1 + \left(1-\tau_{t+1}^r\right) r_{t+1}\right) \lambda_{t+1}$$

Resulting Euler equation

$$\frac{c_t^{-\sigma}}{1+\tau_t^c} = \beta \left(1 + \left(1-\tau_{t+1}^r\right) r_{t+1}\right) \frac{c_{t+1}^{-\sigma}}{1+\tau_{t+1}^c} \quad \rightarrow \quad c_{t+1} = \left[\frac{1 + \left(1-\tau_{t+1}^r\right) r_{t+1}}{1+\rho} \frac{1+\tau_t^c}{1+\tau_{t+1}^c} \right]^{1/\sigma} c_t$$

Firms' problem

Recall that we now assume that firms are direct capital owners

Tax code allows to treat capital depreciation δK as tax-deductible costs, but not the opportunity cost of holding capital rK

Firms' Profit Maximization Problem (PMP)

$$\max_{K_t,\, L_t} \quad D_t = (1-\tau_t^f)\left(1\cdot F\left(K_t, L_t\right) - w_t L_t - \delta K_t\right) - r_t K_t$$

$$\max_{k_t,\, L_t} \quad D_t = (1-\tau_t^f) L_t \left[f\left(k_t\right) - w_t - \delta k_t\right] - r_t L_t k_t$$

First Order Conditions (FOCs)

$$\begin{aligned}
k_t &: \quad (1-\tau_t^f) L_t \left[f'\left(k_t\right) - \delta\right] - r_t L_t = 0 &\rightarrow\quad r_t &= (1-\tau_t^f)\left(f'\left(k_t\right) - \delta\right) \\
L_t &: \quad (1-\tau_t^f)\left[f\left(k_t\right) - w_t - \delta k_t\right] - r_t k_t = 0 &\rightarrow\quad w_t &= f\left(k_t\right) - \delta k_t - \frac{r_t k_t}{1-\tau_t^f} \\
& & w_t &= f\left(k_t\right) - k_t f'\left(k_t\right)
\end{aligned}$$

Firms' problem

Economic profits are still 0

$$\begin{aligned}\frac{D_t}{L_t} &= \left(1-\tau_t^f\right)\left[f\left(k_t\right)-\delta k_t-\left(f\left(k_t\right)-k_t f'\left(k_t\right)\right)\right]-\left(1-\tau_t^f\right)\left[f'\left(k_t\right)-\delta\right]k_t \\ &= \left(1-\tau_t^f\right)\left[-\delta k_t+k_t f'\left(k_t\right)\right]-\left(1-\tau_t^f\right)\left[f'\left(k_t\right)-\delta\right]k_t=0\end{aligned}$$

There are however positive accounting profits D^f

$$\frac{D_t^f}{L_t}=f\left(k_t\right)-\delta k_t-\left(f\left(k_t\right)-k_t f'\left(k_t\right)\right)=\left(f'\left(k_t\right)-\delta\right)k_t>0$$

The tax distorts firms' decisions, disincentivizing them from holding capital

The tax does not affect wages directly, but lowers them indirectly via lower k

$$\frac{\partial w_t}{\partial k_t}=f'\left(k_t\right)-\left(f'\left(k_t\right)+k_t f''\left(k_t\right)\right)=-k_t f''\left(k_t\right)>0$$

Government sector

Government budget constraint assuming balanced budget (in per worker terms)

$$\gamma_t + v_t = \tau_t^f \left[f\left(k_t\right) - \delta k_t - w_t \right] + \tau_t^w w_t + \tau_t^r r_t a_t + \tau_t^c c_t + \tau_t$$

$$\gamma_t + v_t = \tau_t^f \left[f\left(k_t\right) - \delta k_t - \left(f\left(k_t\right) - k_t f'\left(k_t\right) \right) \right] + \tau_t^w w_t + \tau_t^r r_t a_t + \tau_t^c c_t + \tau_t$$

$$v_t = \tau_t^f \left[-\delta k_t + k_t f'\left(k_t\right) \right] + \tau_t^w w_t + \tau_t^r r_t a_t + \tau_t^c c_t + \tau_t - \gamma_t$$

$$\boldsymbol{v_t} = \tau_t^f \left[f'\left(k_t\right) - \delta \right] k_t + \tau_t^w w_t + \tau_t^r r_t a_t + \tau_t^c c_t + \tau_t - \gamma_t$$

General equilibrium

Since firms' "market" value equals "book" value ($q = 1$), $a = k$ in every period

$$
\begin{aligned}
(1+\tau_t^c)\, c_t + a_{t+1} &= (1-\tau_t^w)\, w_t + (1 + (1-\tau_t^r)\, r_t)\, a_t - \tau_t + v_t \\
k_{t+1} &= (1-\tau_t^w)\, w_t + (1 + (1-\tau_t^r)\, r_t)\, k_t - (1+\tau_t^c)\, c_t - \tau_t + v_t \\
k_{t+1} &= w_t + (1 + r_t)\, k_t - c_t - \tau_t^w w_t - \tau_t^r r_t k_t - \tau_t^c c_t - \tau_t \\
&\qquad + \tau_t^f \left[f'(k_t) - \delta\right] k_t + \tau_t^w w_t + \tau_t^r r_t k_t + \tau_t^c c_t + \tau_t - \gamma_t \\
k_{t+1} &= w_t + (1 + r_t)\, k_t - c_t + \tau_t^f \left(f'(k_t) - \delta\right) k_t - \gamma_t \\
k_{t+1} &= w_t + \left[1 + (1-\tau_t^f)\left(f'(k_t) - \delta\right)\right] k_t - c_t + \tau_t^f \left[f'(k_t) - \delta\right] k_t - \gamma_t \\
k_{t+1} &= f(k_t) - k_t f'(k_t) + (1 + f'(k_t) - \delta)\, k_t - c_t - \gamma_t \\
k_{t+1} &= f(k_t) + (1-\delta)\, k_t - c_t - \gamma_t
\end{aligned}
$$

We get the resource constraint ($Y_t = C_t + I_t + G_t$)

General equilibrium

Plug in the interest rate into the Euler equation

$$c_{t+1} = \left[\frac{1 + (1 - \tau^r_{t+1})(1 - \tau^f_{t+1})\left(f'\left(k_{t+1}\right) - \delta\right)}{1 + \rho} \frac{1 + \tau^c_t}{1 + \tau^c_{t+1}}\right]^{1/\sigma} c_t$$

Resource constraint

$$k_{t+1} = f\left(k_t\right) + \left(1 - \delta\right) k_t - c_t - \gamma_t$$

The equilibrium is only modified by

- Government spending γ
- Asset income tax τ^r
- Firm accounting profits tax τ^f
- Consumption tax τ^c, but only if it varies over time

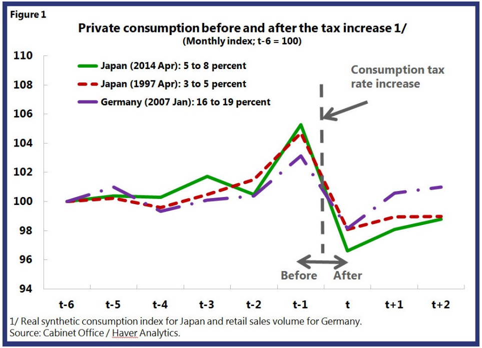
If the labor supply is inelastic, labor income tax τ^w, lump-sum tax τ and time-invariant consumption tax τ^c do not affect the equilibrium!

Effects of time-varying consumption tax

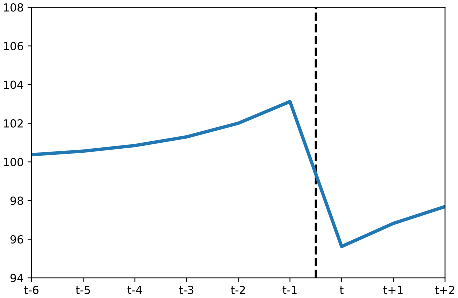
Whenever consumption tax is increased, consumption drops in the period of increase

If households are aware of the upcoming hike in advance, they consume more prior to tax change when consumption is still “cheaper”

Ramsey model can easily replicate consumption patterns observed in the data



Danninger (2014)



Effects of taxes in the long run

Assume from now on that all taxes are time-invariant

$$\text{Euler equation} \quad : \quad c_{t+1} = \left[\frac{1 + (1 - \tau^r)(1 - \tau^f)(f'(k_{t+1}) - \delta)}{1 + \rho}\right]^{1/\sigma} c_t$$

$$\text{Resource constraint} \quad : \quad k_{t+1} = f(k_t) + (1 - \delta) k_t - c_t - \gamma_t$$

Steady state level of capital per worker

$$1 + \rho = 1 + (1 - \tau^r)(1 - \tau^f)(f'(k^*) - \delta)$$

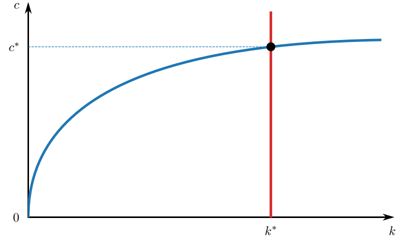
$$f'(k^*) = \delta + \frac{\rho}{(1 - \tau^r)(1 - \tau^f)}$$

Asset income and firm profit taxes decrease capital per worker in the long run

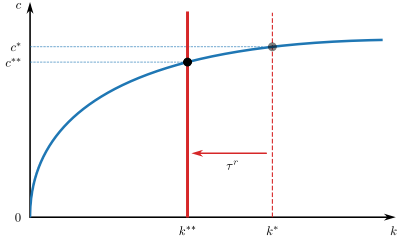
$$c^* = f(k^*) - \delta k^* - \gamma^*$$

Private consumption per worker is crowded out by government expenditure γ^*

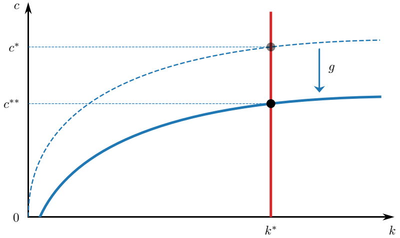
$$\gamma = 0, v > 0; \quad \tau^r = \tau^f = 0; \quad \tau, \tau^c, \tau^w \geq 0$$



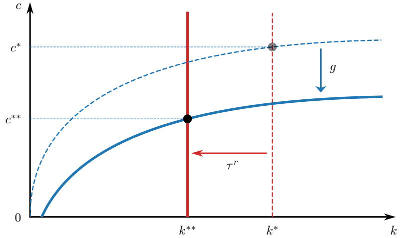
$$\gamma = 0, v > 0; \quad \tau^r, \tau^f \geq 0; \quad \tau = \tau^c = \tau^w = 0$$



$$\gamma > 0, v = 0; \quad \tau^r = \tau^f = 0; \quad \tau, \tau^c, \tau^w \geq 0$$



$$\gamma > 0, v = 0; \quad \tau^r, \tau^f \geq 0; \quad \tau = \tau^c = \tau^w = 0$$



Redistribution impossibility result

Demonstrated by Judd (1985) and Chamley (1986)

It is impossible to increase household welfare by taxing capital and transferring tax revenue equally to all households

Can we improve welfare if we target transfers to workers alone?

Introduce two household types

- Worker households (of count N^w) work and don't save (hand-to-mouth)

$$c_t^w = w_t + v_t$$

- Capitalist households (of count N^c) don't work and have only asset income, they solve the usual UMP

Capitalists' and firms' problems

UMP of capitalists

$$\max_{\{c_t,\, a_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \frac{(c_t^c)^{1-\sigma}}{1-\sigma}$$

$$\text{subject to} \quad c_t^c + a_{t+1}^c = (1 + (1 - \boldsymbol{\tau^r})\, r_t)\, a_t^c$$

Solution

$$c_{t+1}^c = \left[\frac{1 + (1 - \tau^r)\,(f'\,(k_{t+1}) - \delta)}{1+\rho}\right]^{1/\sigma} c_t^c$$

This time firms aren't taxed

$$r_t = f'\,(k_t) - \delta$$

$$w_t = f\,(k_t) - k_t f'\,(k_t)$$

General equilibrium

Aggregate capital is equal to total assets of capitalists

$$K_t = Assets_t$$
$$k_t \cdot N^w = a_t^c \cdot N^c$$
$$k_t = a_t^c \frac{N^c}{N^w}$$

Transfer per worker

$$v_t = \frac{\tau^r r_t a_t^c N^c}{N^w} = \tau^r r_t k_t$$

Steady state level of capital per worker

$$f'(k^*) = \delta + \frac{\rho}{1-\tau^r}$$

Steady state consumption of worker households

$$c^{w*} = w^* + v^* = f(k^*) - k^* f'(k^*) + \tau^r \left(f'(k^*) - \delta\right) k^*$$

Redistribution impossibility result

It suffices to show that steady state worker consumption depends negatively on τ^r

Assuming $f(k) = k^\alpha$ and $\delta = 0$

$$c^{w*} = (k^*)^\alpha - k^* \cdot \alpha (k^*)^{\alpha-1} + \tau^r \left(\alpha (k^*)^{\alpha-1} - 0\right) k^* = (1 - \alpha - \alpha\tau^r) \left[\frac{\alpha (1 - \tau^r)}{\rho}\right]^{\alpha/(1-\alpha)}$$

$$\ln c^{w*} = \ln (1 - \alpha - \alpha\tau^r) + \frac{\alpha}{1-\alpha} (\ln \alpha + \ln (1 - \tau^r) - \ln \rho)$$

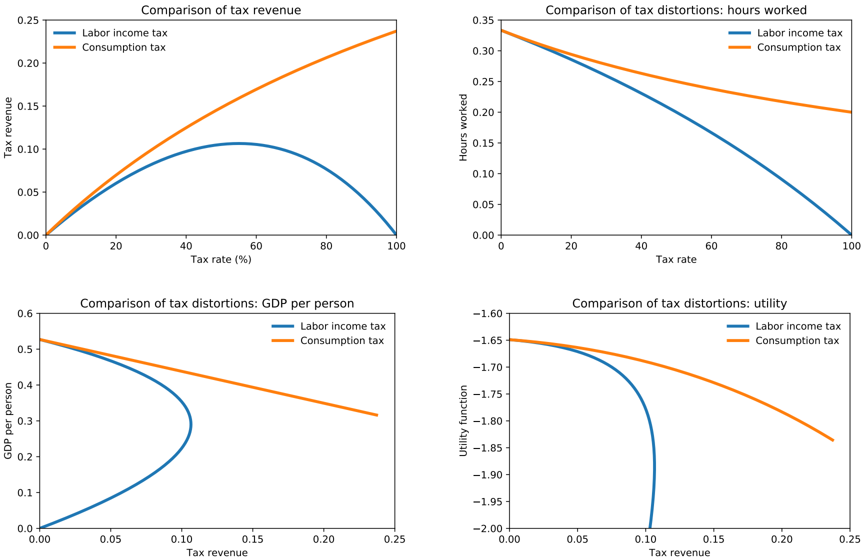
$$\frac{\partial \ln c^{w*}}{\partial \tau^r} = \frac{\alpha}{1 - \alpha + \alpha\tau^r} + \frac{\alpha}{1-\alpha}\left(-\frac{1}{1-\tau^r}\right) = \frac{\alpha}{1 - \alpha + \alpha\tau^r} - \frac{\alpha}{1 - \alpha + \alpha\tau^r - \tau^r} < 0$$

While workers' cosumption can be higher in the short run, taxing capitalists lowers capital per worker in the long run, decreasing wages

Since the tax introduces a deadweight social loss, transfers do not offset lower wages

How to "break" the result? Aiyagari (1995), Conesa et al. (2007), Straub and Werning (2014)

Taxes with endogenous labor supply



Taxes with endogenous labor supply

Why do Americans work more than Europeans? See Prescott (2004)

		OECD data average 2000-2018			Model	
		United States	France	Germany	United States	"Europe"
GDP per hour (PPP $)	y/h	59	56	56	59	56
Average labor tax wedge	τ^w	26%	44%	44%	26%	44%
Average hours worked	h	1790	1530	1400	1790	1430
GDP per worker (PPP $)	y	102 500	86 200	78 200	102 500	79 700

With endogenous labor supply only the lump-sum tax τ is non-distortionary

Consumption tax τ^c is preferred to labor income tax τ^w

Since lump-sum tax is unfairly regressive, the "best" tax
in the Ramsey framework would be a progressive consumption tax

Taxes with unemployment risk

In Ramsey model all workers are always employed

How the welfare ranking of taxes is affected by unemployment risk?

Based on section “No-trade equilibria” in Ragot (2018)

A worker can be employed or unemployed

Probabilities of flows: employed to unemployed s, unemployed to employed p

Employed receive wage w, unemployed generate “home production” b

Capital-less economy: only assets are borrowing contracts

Add borrowing constraint: unemployed can’t borrow

Since unemployed can’t borrow, employed can’t save and everyone’s assets are 0

Employed are unconstrained, real interest rate is pinned down by their Euler equation

Taxes with unemployment risk

Real interest rate is determined by the Euler equation of the employed

$$u'(c_t^E) = \beta\,(1+r)\left[(1-s)\,u'(c_{t+1}^E) + su'(c_{t+1}^U)\right]$$

Since there is no saving nor borrowing

$$c^E = w \quad > \quad c^U = b$$

Equilibrium real interest rate is lower than households' discount rate

$$\frac{1}{1+r} = \beta\left[(1-s) + s\frac{u'(b)}{u'(w)}\right] > \frac{1}{1+\rho} \quad \rightarrow \quad r < \rho$$

Households try to self-insure against unemployment risk via precautionary saving

As a result the model economy "saves" too much! Similar to dynamic inefficiency

Government can improve welfare by providing (partial) unemployment insurance

This is an example of an incomplete markets model

Taxes with unemployment risk

Consumption of employed and unemployed after applying linear labor income tax τ^w, linear consumption tax τ^c, lump-sum tax τ and lump-sum transfer v

$$c^E = \frac{(1-\tau^w)\,w - \tau + v}{1+\tau^c} \quad \text{and} \quad c^U = \frac{b - \tau + v}{1+\tau^c}$$

Expected utility (behind the veil of ignorance a'la Rawls):

$$\mathrm{E}\left[U\right] = \frac{p}{s+p} u(c^E) + \frac{s}{s+p} u(c^U)$$

Labor income tax is preferred to consumption tax, both are preferred to lump-sum tax

Progressive taxes are preferred to linear taxes (need to ensure incentive compatibility)

Lump-sum transfers improve welfare (directed transfers even better, but be aware of IC)

Taxing asset income and firm profits can be welfare improving

Reality is complicated! For in-depth discussion on optimal taxation see Mirrlees Review