Marcin Bielecki, Advanced Macroeconomics IE, Spring 2022

1 Discrete vs continuous time optimization

1.1 Consumer's problem in discrete time: Euler equation

The consumer solves the following utility maximization problem:

$$\begin{aligned}
\max_{\{c_t,\, a_{t+1}\}_{t=0}^{\infty}} \quad & U = \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\
\text{subject to} \quad & c_t + a_{t+1} = y_t + (1+r)\, a_t \quad \text{for all } t = 0, 1, 2, \ldots, \infty \\
\text{(TVC)} \quad & \lim_{t \to \infty} \lambda_t a_{t+1} = 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

Note that the budget constraint can be also rewritten as (we shall make use of this later):

$$\Delta a_{t+1} \equiv a_{t+1} - a_t = y_t + r a_t - c_t$$

Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[y_t + (1+r)\, a_t - c_t - a_{t+1} \right] \\
&= \ldots + \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[y_t + (1+r)\, a_t - c_t - a_{t+1} \right] + \lambda_{t+1} \left[y_{t+1} + (1+r)\, a_{t+1} - c_{t+1} - a_{t+2} \right] + \ldots
\end{aligned}$$

First order conditions:

$$\begin{aligned}
c_t &: \quad \beta^t c_t^{-\sigma} - \lambda_t = 0 & \rightarrow \quad & \lambda_t = \beta^t c_t^{-\sigma} \\
a_{t+1} &: \quad -\lambda_t + \lambda_{t+1} (1+r) = 0 & \rightarrow \quad & \lambda_t = (1+r)\, \lambda_{t+1}
\end{aligned}$$

Euler equation:

$$\beta^t c_t^{-\sigma} = (1+r)\, \beta^{t+1} c_{t+1}^{-\sigma} \quad \rightarrow \quad c_{t+1}^{\sigma} = \beta\, (1+r)\, c_t^{\sigma}$$

Discount factor β can be also related to the discount rate ρ (psychological interest rate):

$$\beta \equiv \frac{1}{1+\rho}$$

The Euler equation can be accordingly rewritten as:

$$\left(\frac{c_{t+1}}{c_t} \right)^{\sigma} = \frac{1+r}{1+\rho} = \frac{(1+r)(1-\rho)}{(1+\rho)(1-\rho)} = \frac{1 + r - \rho - r\rho}{1-\rho^2} \approx 1 + r - \rho$$

Where in the approximation we have discarded the second order terms.

Finally, we can use the first-order Taylor expansion of the power function:

$$\frac{c_{t+1}}{c_t} = (1 + r - \rho)^{\frac{1}{\sigma}} \approx 1 + \frac{r-\rho}{\sigma} \quad \rightarrow \quad \frac{\Delta c_{t+1}}{c_t} \approx \frac{r-\rho}{\sigma}$$

The obtained expression is an approximation of the "true" Euler equation in discrete time.

It turns out that in continuous time the analogue of the above expression will be exact.

1.2 Consumer's problem in continuous time: Euler equation

$$\begin{aligned}
\max \quad & U = \int_0^\infty e^{-\rho t} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \\
\text{subject to} \quad & \dot{a}_t = r a_t + y_t - c_t \quad \text{for all } t \in (0, \infty) \\
\text{(TVC)} \quad & \lim_{t \to \infty} \lambda_t a_t = 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

where we have used the following notational convention:

$$\dot{a}_t \equiv \frac{\mathrm{d}a_t}{\mathrm{d}t}$$

How would we solve this "normally"? We would set up a Lagrangian:

$$\mathcal{L} = \int_0^\infty e^{-\rho t} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t + \int_0^\infty \lambda_t \left[y_t + r a_t - c_t - \dot{a}_t \right] \mathrm{d}t$$

But immediately we run into a problem: we don't know how should we take the derivative of $\mathrm{d}a_t/\mathrm{d}t$ w.r.t. a_t. We need to somehow get rid of this expression. Let us split the constraint in the Lagrangian:

$$\mathcal{L} = \int_0^\infty e^{-\rho t} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t + \int_0^\infty \lambda_t \left[y_t + r a_t - c_t \right] \mathrm{d}t - \int_0^\infty \lambda_t \dot{a}_t \, \mathrm{d}t$$

and integrate the last expression **by parts**[1]:

$$\int_0^\infty \lambda_t \dot{a}_t \, \mathrm{d}t = \lim_{t \to \infty} \lambda_t a_t - \lambda_0 a_0 - \int_0^\infty a_t \dot{\lambda}_t \, \mathrm{d}t$$

Now we can rewrite our Lagrangian, using the transversality condition (TVC) to simplify the expression:

$$\mathcal{L} = \int_0^\infty e^{-\rho t} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t + \int_0^\infty \lambda_t \left[y_t + r a_t - c_t \right] \, \mathrm{d}t - \lambda_0 a_0 + \int_0^\infty a_t \dot{\lambda}_t \, \mathrm{d}t$$

Let us derive the first order conditions (note that we take the derivative w.r.t. a_t and not a_{t+1}):

$$\begin{aligned}
c_t \quad &: \quad e^{-\rho t} \cdot c_t^{-\sigma} - \lambda_t = 0 \quad \rightarrow \quad \lambda_t = e^{-\rho t} \cdot c_t^{-\sigma} \\
a_t \quad &: \quad \lambda_t r + \dot{\lambda}_t = 0
\end{aligned}$$

Euler equation

We need to obtain an expression for $\mathrm{d}\lambda_t/\mathrm{d}t$. Differentiate the FOC for consumption w.r.t. time:

$$\dot{\lambda}_t = \frac{\mathrm{d}\left(e^{-\rho t} \cdot c_t^{-\sigma}\right)}{\mathrm{d}t} = \left(-\rho e^{-\rho t}\right) \cdot c_t^{-\sigma} + e^{-\rho t} \cdot \left(-\sigma c_t^{-\sigma-1}\right) \dot{c}_t = -\rho e^{-\rho t} c_t^{-\sigma} - \sigma e^{-\rho t} c_t^{-\sigma-1} \dot{c}_t$$

Combine the above result with the FOC for assets:

$$\begin{aligned}
e^{-\rho t} \cdot c_t^{-\sigma} \cdot r - \rho e^{-\rho t} c_t^{-\sigma} - \sigma e^{-\rho t} c_t^{-\sigma-1} \dot{c}_t = 0 \quad & | \quad : e^{-\rho t} \cdot c_t^{-\sigma} \\
r - \rho - \sigma \frac{\dot{c}_t}{c_t} = 0 &
\end{aligned}$$

From the above we can obtain the Euler equation for consumption in continuous time:

$$\frac{\dot{c}_t}{c_t} = \frac{r - \rho}{\sigma}$$

[1] For a very clear derivation of the formula, see e.g. **here**.

1.3 Consumer's problem in continuous time: Hamiltonian

The procedure for finding the first order conditions in continuous time can be streamlined and expressed as a "recipe" for a Hamiltonian function. The recipe is as follows:

1. Take the utility function and "disregard" the integral and dt symbols
2. Add in a multiplier, e.g. λ_t and multiply it with the constraint, **without** the "dotted" variable
3. When taking the first order condition w.r.t. the "dotted" variable, set it not to 0, **but** to $-\dot{\lambda}_t$

Apply the recipe to our example. Hamiltonian:

$$\mathcal{H} = e^{-\rho t} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[y_t + r a_t - c_t \right]$$

First order conditions:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial c_t} = 0 \quad &: \quad e^{-\rho t} \cdot c_t^{-\sigma} - \lambda_t = 0 \\ \frac{\partial \mathcal{H}}{\partial a_t} = -\dot{\lambda}_t \quad &: \quad \lambda_t r = -\dot{\lambda}_t \end{aligned}$$

You can easily verify that the resulting first order conditions are identical to ones obtained before.

Euler equation

The Euler equation can be also obtained via a different approach, called the log-differentiation:

$$\begin{aligned} \lambda_t &= e^{-\rho t} \cdot c_t^{-\sigma} \quad | \quad \ln(\cdot) \\ \ln \lambda_t &= -\rho t - \sigma \ln c_t \quad | \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ \frac{\mathrm{d}(\ln \lambda_t)}{\mathrm{d}t} &= -\rho - \sigma \frac{\mathrm{d}(\ln c_t)}{\mathrm{d}t} \\ \frac{1}{\lambda_t} \frac{\mathrm{d}\lambda_t}{\mathrm{d}t} &= -\rho - \sigma \frac{1}{c_t} \frac{\mathrm{d}c_t}{\mathrm{d}t} \\ \frac{\dot{\lambda}_t}{\lambda_t} + \rho &= -\sigma \frac{\dot{c}_t}{c_t} \\ \frac{\dot{c}_t}{c_t} &= \frac{-\dot{\lambda}_t / \lambda_t - \rho}{\sigma} \end{aligned}$$

Now use the FOC for assets:

$$-\frac{\dot{\lambda}_t}{\lambda_t} = r \quad \rightarrow \quad \frac{\dot{c}_t}{c_t} = \frac{r - \rho}{\sigma}$$

Obviously, the result is the same as before. You can use whichever approach you prefer.

1.4 Consumer's problem in continuous time: optimal consumption

Now we can write down the lifetime budget constraint and obtain the optimal levels of consumption.

Recall the formulas for the lifetime budget constraint and optimal consumption in the discrete time case:

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = (1+r)\, a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \equiv \tilde{a}_0 + \tilde{y}_0$$

$$c_t = [\beta\,(1+r)]^t\, c_0$$

$$c_0 = (1-\beta)\,(\tilde{a}_0 + \tilde{y}_0)$$

We want to obtain similar formulas for the continuous time case. Start with the budget constraint:

$$\dot{a}_t = y_t + r a_t - c_t$$

This is a first-order differential equation in a_t with time-varying coefficient $(y_t - c_t)$ and can be solved using standard techniques, which I do not cover here. However, the result is very intuitive:

$$\int_0^{\infty} e^{-rt} \cdot c_t \,\mathrm{d}t = a_0 + \int_0^{\infty} e^{-rt} \cdot y_t \,\mathrm{d}t \equiv a_0 + \tilde{y}_0$$

The Euler equation dictates the optimal growth of consumption:

$$\frac{\dot{c}_t}{c_t} = \frac{r-\rho}{\sigma} \qquad \rightarrow \qquad c_t = c_0 \cdot e^{(r-\rho)/\sigma \cdot t}$$

After substituting the Euler equation into the lifetime budget constraint, we get:

$$\int_0^{\infty} e^{-rt} \cdot c_0 \cdot e^{(r-\rho)/\sigma \cdot t} \,\mathrm{d}t = a_0 + \tilde{y}_0$$

And so the optimal initial consumption can be expressed as:

$$c_0 = \mu\,(a_0 + \tilde{y}_0)$$

where the marginal propensity to consume out of lifetime wealth μ is given by:

$$\frac{1}{\mu} = \int_0^{\infty} e^{(r(1-\sigma)/\sigma - \rho/\sigma)\cdot t} \,\mathrm{d}t$$

For the logarithmic utility ($\sigma = 1$) the expression simplifies to:

$$\frac{1}{\mu} = \int_0^{\infty} e^{-\rho t} \,\mathrm{d}t = -\frac{1}{\rho} \cdot \left[e^{-\rho t}\right]_0^{\infty} = -\frac{1}{\rho}\left(e^{-\rho \cdot \infty} - e^{-\rho \cdot 0}\right) = -\frac{1}{\rho}\,(0-1) = \frac{1}{\rho}$$

$$c_0 = \rho\,(a_0 + \tilde{y}_0)$$

Which is analogous to the discrete time case:

$$1-\beta = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = \frac{\rho\,(1-\rho)}{(1+\rho)\,(1-\rho)} = \frac{\rho-\rho^2}{1-\rho^2} \approx \rho$$

$$c_0 \approx \rho\,(\tilde{a}_0 + \tilde{y}_0)$$