Marcin Bielecki, Applied Macroeconomics, Fall 2018

1 Dynamic consumption choice

1.1 Two-period optimization

An agent solves the following problem:

$$\begin{aligned} \max_{c_t, c_{t+1}, a} \quad & U = \log c_t + \beta \log c_{t+1} \\ \text{subject to} \quad & c_t + a = y_t \\ & c_{t+1} = y_{t+1} + (1+r)\, a \\ & \beta \in [0,1]\,, \text{ given} \\ & y_t, y_{t+1} \geq 0, \text{ given} \\ & r \geq -1, \text{ given} \\ & c_t, c_{t+1} > 0 \end{aligned}$$

where c_t and c_{t+1} denote agent's consumption in the first and second period, y_t and y_{t+1} denote agent's income in the first and second period, β is the **discount factor**, and a denotes wealth / net assets (possibly negative) held by the agent at the end of period 1 (at the very beginning of period 2), which yield real rate of interest r.

First construct the lifetime budget constraint:

$$a = \frac{c_{t+1} - y_{t+1}}{1+r}$$

$$\begin{aligned} c_t + \frac{c_{t+1} - y_{t+1}}{1+r} &= y_t \\ c_t + \frac{c_{t+1}}{1+r} &= y_t + \frac{y_{t+1}}{1+r} \end{aligned}$$

The lifetime budget constraint says that the present discounted value (PDV) of income (plus initial wealth which is here assumed to be 0) has to equal the PDV of consumption. Rewrite the budget constraint so that one side of the equation equals 0:

$$y_t + \frac{y_{t+1}}{1+r} - c_t - \frac{c_{t+1}}{1+r} = 0$$

Set up the Lagrangian:

$$\mathcal{L} = \log c_t + \beta \log c_{t+1} + \lambda \left[y_t + \frac{y_{t+1}}{1+r} - c_t - \frac{c_{t+1}}{1+r} \right]$$

where λ is the Lagrange multiplier with the following economic interpretation: by how much would the agent's utility increase if the budget constraint was marginally relaxed, i.e. the agent had marginally higher PDV of income ($\partial U / \partial y_t$).

The Lagrangian has two choice variables: c_t and c_{t+1}. Therefore, we will need to calculate two first order conditions.

First order conditions (FOCs):

$$\begin{aligned} c_t \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} + \lambda [-1] = 0 \quad &\rightarrow \quad \lambda = \frac{1}{c_t} \\ c_{t+1} \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta \cdot \frac{1}{c_{t+1}} + \lambda \left[-\frac{1}{1+r} \right] = 0 \quad &\rightarrow \quad \lambda = \beta (1+r) \frac{1}{c_{t+1}} \end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta(1+r)c_t$$

An equation linking in optimum consumption in different time periods is called the intertemporal (from Latin: “between time (periods)”) condition or the **Euler equation**.

The Euler equation can be also obtained in the following way. Suppose that the agent has found the optimal solution to the problem. She now considers consuming x units less in the first period so that she can consume $(1+r)x$ units more in the second period. Her utility level is now given by:

$$U = \log(c_t - x) + \beta\log(c_{t+1} + (1+r)x)$$

If the solution was truly optimal, the derivative of the utility function with respect to x, evaluated at $x=0$, should be equal to 0:

$$\frac{\partial U}{\partial x} = \frac{1}{c_t - x}\cdot(-1) + \beta\frac{1}{c_{t+1} + (1+r)x}\cdot(1+r)$$
$$\frac{\partial U}{\partial x}|_{x=0} = -\frac{1}{c_t} + \beta(1+r)\frac{1}{c_{t+1}} = 0 \quad \rightarrow \quad c_{t+1} = \beta(1+r)c_t$$

Plug the optimality condition into the budget constraint:

$$c_t + \frac{c_{t+1}}{1+r} = y_t + \frac{y_{t+1}}{1+r}$$
$$c_t + \frac{\beta(1+r)c_t}{1+r} = y_t + \frac{y_{t+1}}{1+r}$$
$$c_t + \beta c_t = y_t + \frac{y_{t+1}}{1+r}$$

Finally, we can obtain the optimal levels of consumption and assets at the end of the first period:

$$c_t = \frac{1}{1+\beta}\left[y_t + \frac{y_{t+1}}{1+r}\right]$$
$$c_{t+1} = \frac{\beta}{1+\beta}\left[(1+r)y_t + y_{t+1}\right]$$
$$a = y_t - \frac{1}{1+\beta}\left[y_t + \frac{y_{t+1}}{1+r}\right] = \frac{\beta}{1+\beta}y_t - \frac{1}{1+\beta}\frac{y_{t+1}}{1+r}$$

1.1.1 Comparative statics

Let us take a look at how changes in the parameters of our problem affect the agent’s choices.

Changes in β
The higher is β, the more patient is our agent. Intuitively, since the agent with higher β cares more about the future, she would save more in the first period (and thus consume less in the first period) in order to consume more in the second period:

$$\frac{\partial c_t}{\partial \beta} = -\frac{1}{(1+\beta)^2}\left[y_t + \frac{y_{t+1}}{1+r}\right] < 0$$
$$\frac{\partial c_{t+1}}{\partial \beta} = \frac{1+\beta-\beta}{(1+\beta)^2}\left[(1+r)y_t + y_{t+1}\right] > 0$$
$$\frac{\partial a}{\partial \beta} = \frac{1+\beta-\beta}{(1+\beta)^2}y_t - \left[-\frac{1}{(1+\beta)^2}\right]\frac{y_{t+1}}{1+r} > 0$$

Changes in y_t
An increase in first period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional first period income to second period consumption, savings will also increase:

$$\frac{\partial c_t}{\partial y_t} = \frac{1}{1+\beta} > 0$$
$$\frac{\partial c_{t+1}}{\partial y_t} = \frac{\beta}{1+\beta}(1+r) > 0$$
$$\frac{\partial a}{\partial y_t} = \frac{\beta}{1+\beta} > 0$$

Changes in y_{t+1}
An increase in second period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional second period income to first period consumption, savings will decrease:

$$\frac{\partial c_t}{\partial y_{t+1}} = \frac{1}{1+\beta}\frac{1}{1+r} > 0$$
$$\frac{\partial c_{t+1}}{\partial y_{t+1}} = \frac{\beta}{1+\beta} > 0$$
$$\frac{\partial a}{\partial y_{t+1}} = -\frac{1}{1+\beta}\frac{1}{1+r} < 0$$

Changes in r
An increase in the interest rate generates two effects: substitution effect and income effect. Substitution effect is generated by the change in relative prices of consumption in the first period versus consumption in the second period. It induces the agent to consume more in the second period and consume less in the first period. Income effect is generated by the fact that the PDV of income changes due to the changes in the discounting rate. The sign of the effect depends on whether an agent is a saver or a borrower. An increase in interest rates creates a positive income effect for the saver, increasing her consumption in both periods. Conversely, an increase in interest rates creates a negative income effect for the borrower, decreasing her consumption in both periods.

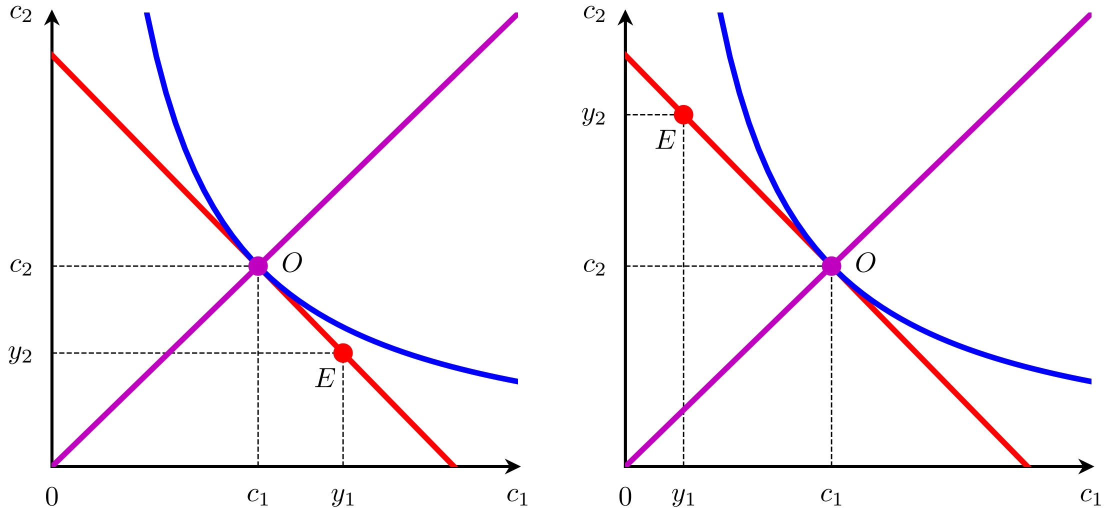
These effects can be summarized by the following table, where pluses and minuses describe the direction of change in a variable for which we can be certain of the direction, regardless of the assumed utility function:

Effects of an	Saver			Borrower		
increase in r	c_t	c_{t+1}	a	c_t	c_{t+1}	a
Income	+	+	-	-	-	+
Substitution	-	+	+	-	+	+
Net	?	+	?	-	?	+

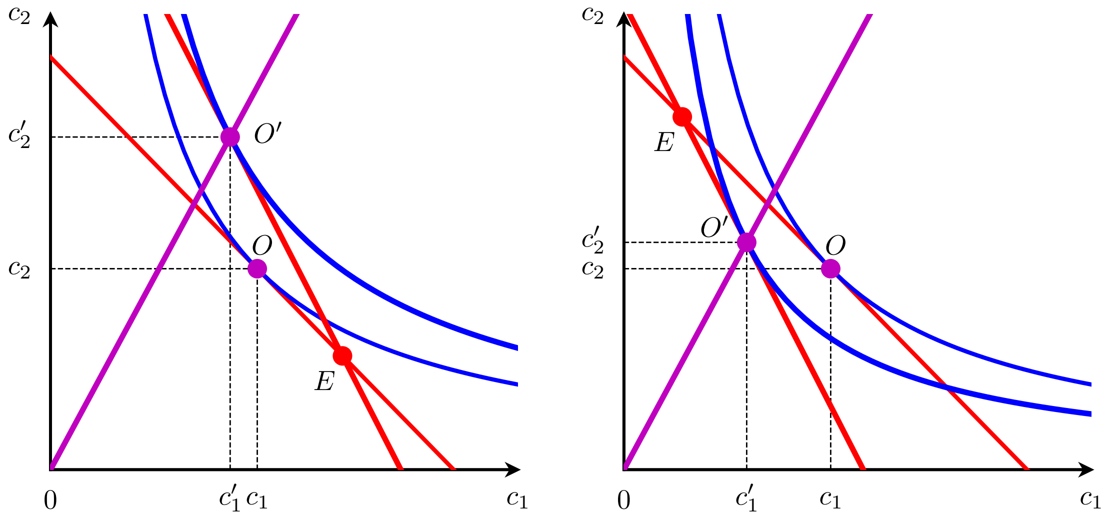
There are however also changes in variables marked with question marks, where the direction of change in a variable depends on the relative strength of substitution and income effects. In the case of a saver, if the income effect is stronger (weaker) than the substitution effect, first period consumption increases (decreases) and savings decrease (increase). In the case of a borrower, if the income effect is stronger (weaker) than the substitution effect, second period consumption decreases (increases).

Saver **Borrower**

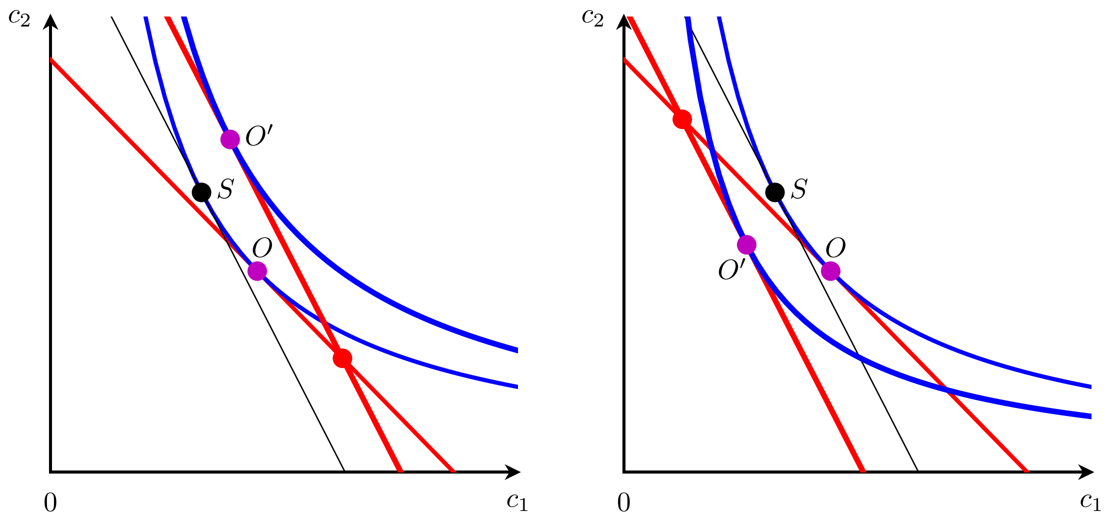
Optimal choice is labeled O. E denotes the initial endowment point.



Reaction to an increase in r. New optimal point is labeled O'.



Decomposition of effects. $O \to S$ is pure substitution effect while $S \to O'$ is pure income effect

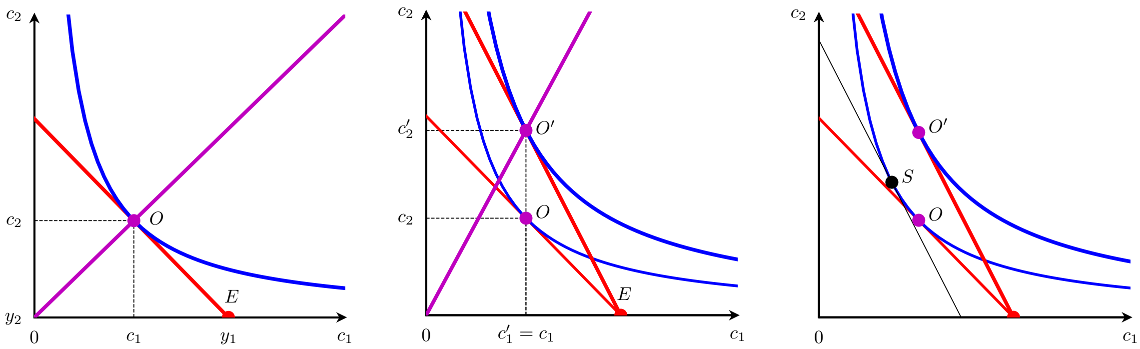


The relative strength of the two effects depends on the shape of the indifference curves. For our agent:

$$\frac{\partial c_t}{\partial r} = \frac{1}{1+\beta}\left[-\frac{y_{t+1}}{(1+r)^2}\right] < 0$$
$$\frac{\partial c_{t+1}}{\partial r} = \frac{\beta}{1+\beta} y_t > 0$$
$$\frac{\partial a}{\partial r} = -\frac{1}{1+\beta}\left[-\frac{y_{t+1}}{(1+r)^2}\right] > 0$$

In response to the increase in the interest rate, first period consumption decreases (if only $y_{t+1} > 0$), second period consumption increases (if only $y_t > 0$) and savings increase (if only $y_{t+1} > 0$). The negative relationship between the level of interest rates and consumption is a usual assumption in macroeconomics, and is present in the undergraduate macroeconomic models such as IS-LM and AS-AD.

The case of logarithmic utility and $y_{t+1} = 0$ is a very special case where both first period consumption and savings stay constant while second period consumption increases in response to the increase in the interest rate. The income and substitution effects have exactly equal strength and cancel each other out.



2 Overlapping Generations model

The overlaping generations (OLG) model was first developed by **Allais (1947)**, **Samuelson (1958)** and **Diamond (1965)**. Here I present the simple, classic version of the model where agents live for two periods: in the first they are "young" and work, and in the second they are "old" (retired) and have to finance their consumption from previously accumulated savings. I also later discuss social security (pensions) issues.

2.1 The Diamond (1965) model

Households

Each household lives for two periods. In period t there are N_t^y young households born and each of them supplies one unit of labor, so that the total labor supply L_t is equal to the number of young households N_t^y. The rate of growth of young agents is assumed to be constant for simplicity and denoted with n:

$$\frac{L_{t+1}}{L_t} = \frac{N_{t+1}^y}{N_t^y} = \frac{(1+n)\, N_t^y}{N_t^y} = 1+n$$

The number of old agents in period t is denoted with N_t^o. All young survive into the old age, but all old die with certainty. Therefore, the number of old agents in period $t+1$ is equal to the number of young agents in period t. The rate of growth of the entire population (denoted with N_t) is also equal to n:

$$N_t = N_t^y + N_t^o = N_t^y + N_{t-1}^y = (1+n)\, N_{t-1}^y + N_{t-1}^y = (2+n)\, N_{t-1}^y$$

$$\frac{N_{t+1}}{N_t} = \frac{N_{t+1}^y + N_{t+1}^o}{N_t^y + N_t^o} = \frac{(1+n)^2\, N_{t-1}^y + (1+n)\, N_{t-1}^y}{(2+n)\, N_{t-1}^y} = \frac{(1+n)\,(2+n)}{2+n} = 1+n$$

A household born in period t faces the following utility maximization problem:

$$\begin{aligned} \max_{c_t^y,\, c_{t+1}^o,\, a_{t+1}} \quad & U = \log c_t^y + \beta \log c_{t+1}^o \\ \text{subject to} \quad & c_t^y + a_{t+1} = w_t \\ & c_{t+1}^o = (1+r_{t+1})\, a_{t+1} \end{aligned}$$

where c_t^y denotes consumption of household born in period t when young, and c_{t+1}^o denotes consumption of household born in period t when old (consumption takes place in period $t+1$). It is assumed that young households receive wage income w_t and the old households sell their assets to consume. Note that both wage w_t and interest rate r_{t+1} are time-dependent and will be determined in the market equilibrium.

The lifetime budget constraint of a household born in period t is:

$$c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} = w_t$$

The present discounted value (PDV) of consumption has to equal to the PDV of income, the latter being equal to the wage income earned while young.

Set up the Lagrangian:

$$\mathcal{L} = \log c_t^y + \beta \log c_{t+1}^o + \lambda \left[w_t - c_t^y - \frac{c_{t+1}^o}{1+r_{t+1}} \right]$$

First order conditions:

$$\begin{aligned} c_t^y \quad &: \quad \frac{1}{c_t^y} - \lambda = 0 & \rightarrow \quad & \lambda = \frac{1}{c_t^y} \\ c_{t+1}^o \quad &: \quad \frac{\beta}{c_{t+1}^o} - \frac{\lambda}{1+r_{t+1}} = 0 & \rightarrow \quad & \lambda = \beta\,(1+r_{t+1})\, \frac{1}{c_{t+1}^o} \end{aligned}$$

We obtain the familiar Euler equation:

$$\frac{1}{c_t^y} = \beta\left(1+r_{t+1}\right)\frac{1}{c_{t+1}^o} \quad \rightarrow \quad c_{t+1}^o = \beta\left(1+r_{t+1}\right)c_t^y \tag{1}$$

Plug the optimality condition into the lifetime budget constraint:

$$\begin{aligned} c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} &= w_t \\ c_t^y + \frac{\beta\left(1+r_{t+1}\right)c_t^y}{1+r_{t+1}} &= w_t \\ \left(1+\beta\right)c_t^y &= w_t \end{aligned}$$

$$\begin{aligned} c_t^y &= \frac{1}{1+\beta}w_t \\ c_{t+1}^o &= \frac{\beta}{1+\beta}\left(1+r_{t+1}\right)w_t \\ a_{t+1} &= \frac{\beta}{1+\beta}w_t \end{aligned}$$

Note that the consumption of young and their savings are independent of the interest rate, which greatly simplifies the following analysis. This is a consequence of the substitution and income effects canceling out due to the assumption of logarithmic utility and zero retirement income.

Firms

For simplicity let's assume that the behavior of the entire firms sector is summarized by a single representative firm. This firm hires capital K and labor L and produces goods Y according to the following Cobb-Douglas production function:

$$Y_t = K_t^\alpha\left(A_tL_t\right)^{1-\alpha}$$

where A represents the productivity of the economy, growing at rate g, while $\alpha \in (0,1)$ represents the elasticity of output with respect to capital.

The representative firm aims to maximize its profits. The price of the good is normalized to 1 (so that all other prices are expressed in units of the final good) and the profit maximization problem is given by:

$$\begin{aligned} \max_{K_t,\,L_t} \quad & \Pi_t = Y_t - \left(r_t+\delta\right)K_t - w_tL_t \\ \text{subject to} \quad & Y_t = K_t^\alpha\left(A_tL_t\right)^{1-\alpha} \end{aligned}$$

where $\delta \in (0,1)$ is the capital depreciation rate. Here it is convenient to directly include the constraint in the objective function:

$$\max_{K_t,\,L_t} \quad \Pi_t = K_t^\alpha\left(A_tL_t\right)^{1-\alpha} - \left(r_t+\delta\right)K_t - w_tL_t$$

FOCs:

$$\begin{aligned} K_t \quad &: \quad \alpha K_t^{\alpha-1}\left(A_tL_t\right)^{1-\alpha} - \left(r_t+\delta\right) = 0 \quad \rightarrow \quad r_t = \alpha\left(\frac{K_t}{A_tL_t}\right)^{\alpha-1} - \delta \\ L_t \quad &: \quad \left(1-\alpha\right)K_t^\alpha A_t^{1-\alpha}L_t^{-\alpha} - w_t = 0 \qquad \rightarrow \quad w_t = \left(1-\alpha\right)A_t\left(\frac{K_t}{A_tL_t}\right)^\alpha \end{aligned}$$

The prices of factors of production depend on the level of capital per effective labor, defined as:

$$\hat{k}_t \equiv \frac{K_t}{A_tL_t}$$

The factor prices can then be rewritten as:

$$\begin{aligned} r_t &= \alpha\hat{k}_t^{\alpha-1} - \delta \\ w_t &= \left(1-\alpha\right)A_t\hat{k}_t^\alpha \end{aligned}$$

General equilibrium

We now know how the households and firms behave in isolation. However, they are obviously interconnected: how much the households save will matter for how much capital gets accumulated in the economy, while the prices of factors of production matter for the households' choices. The way to combine this information is to impose that the economy is in general equilibrium and all markets clear.

The capital that will be available for production in period $t+1$ is equal to the end-of-period savings of time period t young:

$$K_{t+1} = N_t^y a_{t+1}$$

We can express the above relationship in per effective labor terms:

$$\frac{K_{t+1}}{A_t L_t} = \frac{N_t^y}{L_t} \frac{a_{t+1}}{A_t}$$

$$\frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1}}{A_t} \frac{L_{t+1}}{L_t} = \frac{a_{t+1}}{A_t}$$

$$\hat{k}_{t+1} (1+g)(1+n) = \frac{a_{t+1}}{A_t}$$

From the solution of the households problem we obtained already the expression for assets of the young:

$$a_{t+1} = \frac{\beta}{1+\beta} w_t$$

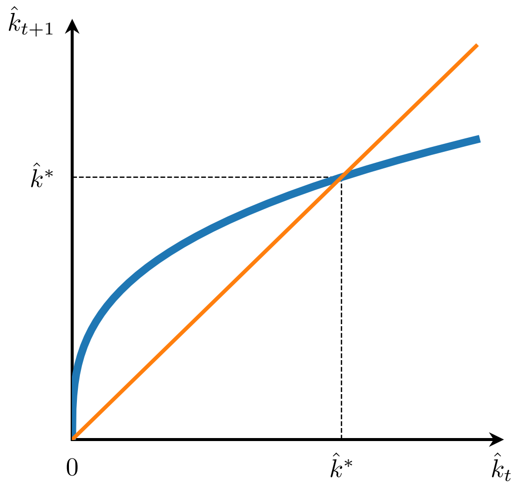
From the problem of the firms we also obtained the expression for the real wage:

$$w_t = (1-\alpha) A_t \hat{k}_t^{\alpha}$$

Combine all pieces of information:

$$\hat{k}_{t+1} = \frac{\beta(1-\alpha)}{1+\beta} \frac{1}{(1+g)(1+n)} \hat{k}_t^{\alpha}$$

The above is a dynamic equation that describes the evolution of capital per effective labor over time:



with the steady state value of capital per effective labor equal to:

$$\hat{k}^* = \left[\frac{\beta(1-\alpha)}{1+\beta} \frac{1}{(1+g)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

Note that under our assumptions the behavior of the model closely resembles that of the Solow-Swan model. In fact, we can show that the expression $\frac{\beta(1-\alpha)}{1+\beta}$ corresponds to the constant saving rate.

Saving rate of the simple OLG economy

We can derive the saving rate of this economy, and show that it is constant, just like in the Solow-Swan case. This model however will give us the link between the saving rate and households' preferences.

The total supply of savings in this economy is determined by the asset accumulation of the young agents:

$$S_t = N_t^y a_{t+1} = L_t \cdot \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta} (1-\alpha) K_t^\alpha A_t^{1-\alpha} L_t^{1-\alpha}$$

By definition, the economy's saving rate is the ratio between total savings and output:

$$s \equiv \frac{S_t}{Y_t} = \frac{\frac{\beta(1-\alpha)}{1+\beta} K_t^\alpha A_t^{1-\alpha} L_t^{1-\alpha}}{K_t^\alpha (A_t L_t)^{1-\alpha}} = \frac{\beta(1-\alpha)}{1+\beta}$$

As expected, the saving rate s depends positively on households' discount factor β:

$$\frac{\partial s}{\partial \beta} = (1-\alpha) \frac{(1+\beta)-\beta}{(1+\beta)^2} = \frac{1-\alpha}{(1+\beta)^2} > 0$$

The more patient the households are (the higher β is), the higher the aggregate saving rate in an economy.

The forward equation in capital per effective labor $\hat{k}$ which we have derived previously is identical to the Solow-Swan case, once we assume that $\delta = 1$. This is justified by the fact that each period of time represents decades in real world. Just think about how few machines and other pieces of equipment that were used in 1990 are still in use today. This time I will not approximate $ng \approx 0$, since over the timespan of decades the product of population and technological growth is not trivial:

$$\hat{k}_{t+1} = \frac{s}{1+n+g+ng} \hat{k}_t^\alpha$$

$$\hat{k}^* = \left(\frac{s}{1+n+g+ng} \right)^{\frac{1}{1-\alpha}}$$

Back to Solow-Swan: Golden Rule saving rate

Let us consider again the Solow-Swan economy. We can ask the following question: which saving rate maximizes consumption in the steady state? It turns out that it is easier to first find the answer to: which level of capital per effective labor maximizes consumption in the steady state?

$$\hat{c}^* = \hat{y}^* - \hat{i}^* = (\hat{k}^*)^\alpha - (\delta+n+g+ng) \hat{k}^*$$

$$\frac{\partial \hat{c}^*}{\partial \hat{k}^*} = \alpha (\hat{k}^*)^{\alpha-1} - (\delta+n+g+ng) = 0$$

$$\hat{k}^*_{GR} = \left(\frac{\alpha}{\delta+n+g+ng} \right)^{\frac{1}{1-\alpha}}$$

The symbol $\hat{k}^*_{GR}$ denotes the level of capital per effective labor in the steady state consistent with the golden rule saving rate. By comparing this expression to the one we have derived for any steady state, we can immediately notice that the golden rule saving rate has to satisfy:

$$s_{GR} = \alpha$$

If the economy's saving rate is higher than s_{GR}, the economy is dynamically inefficient, as we could increase the consumption of both current and future generations. For the OLG economy to be dynamically inefficient, the following has to hold:

$$\frac{\beta(1-\alpha)}{1+\beta} > \alpha \quad \rightarrow \quad \beta > \frac{\alpha}{1-2\alpha}$$

If we assume that $\alpha = 0.3$, then the condition is equivalent to $\beta > 0.75$, which is quite possible for "patient" societies.

2.2 Pensions

The considerations from the previous sections are very relevant for the construction of the retirement systems. We will analyze two types: "fully funded" and "pay-as-you-go" systems. In both cases the government will collect a social security contribution τ_t from young agents and pay pensions p_t to the retired. The modified budget constraints are:

$$c_t^y + a_{t+1} = w_t - \tau_t$$
$$c_{t+1}^o = (1 + r_{t+1})\, a_{t+1} + p_{t+1}$$

The difference between the systems stems from the different relationships between τ and p.

Fully funded

In the fully funded system the government collects the social security contributions and invests them in financial markets, just as the households do. The rate of return on those "mandatory" savings is assumed to be equal to the rate of return on households' savings. Thus the pensions are determined by:

$$p_{t+1} = (1 + r_{t+1})\, \tau_t$$

Include this information in the households' budget constraints:

$$c_t^y + a_{t+1} = w_t - \tau_t$$
$$c_{t+1}^o = (1 + r_{t+1})\, a_{t+1} + (1 + r_{t+1})\, \tau_t$$

And produce the lifetime budget constraint:

$$a_{t+1} = \frac{c_{t+1}^o}{1 + r_{t+1}} - \tau_t$$
$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} - \tau_t = w_t - \tau_t$$
$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t$$

The lifetime budget constraint is identical to the case of no retirement system. The optimality condition is still given by the Euler equation (1). Thus, the chosen consumption level will be unchanged:

$$c_t^y = \frac{1}{1 + \beta} w_t$$

However, since the young have less disposable income, their private savings will be equal to:

$$a_{t+1} = w_t - c_t^y - \tau_t = \frac{\beta}{1 + \beta} w_t - \tau_t$$

The capital in the next period will be the sum of voluntary (private) and mandatory (public) savings:

$$K_{t+1} = N_t^y\, (a_{t+1} + \tau_t) = N_t^y \left(\frac{\beta}{1 + \beta} w_t \right)$$

and will be independent of the pension system.

Pay-as-you-go (PAYG)

In the pay-as-you-go system the government collects the social security contributions and immediately spends them on the pensions of the currently old:

$$N_t^o p_t = N_t^y \tau_t$$
$$p_t = (1+n)\,\tau_t$$

Include this information in the households' budget constraints:

$$c_t^y + a_{t+1} = w_t - \tau_t$$
$$c_{t+1}^o = (1+r_{t+1})\,a_{t+1} + (1+n)\,\tau_{t+1}$$

To analyze the PAYG system easily, assume that $\tau_t = \tau$ and $\tau_{t+1} = (1+g)\,\tau$ (i.e. social security contributions grow together with technological improvements). The lifetime budget constraint becomes:

$$a_{t+1} = \frac{c_{t+1}^o - (1+n)\,(1+g)\,\tau}{1+r_{t+1}}$$
$$c_t^y + \frac{c_{t+1}^o - (1+n)\,(1+g)\,\tau}{1+r_{t+1}} = w_t - \tau$$
$$c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} = w_t + \frac{(n+g+ng-r_{t+1})\,\tau}{1+r_{t+1}}$$

The optimality condition is still given by the Euler equation 1 and the chosen level of consumption when young and savings will be given by:

$$c_t^y = \frac{1}{1+\beta}\left[w_t + \frac{(n+g+ng-r_{t+1})\,\tau}{1+r_{t+1}}\right]$$
$$a_{t+1} = w_t - \tau - \frac{1}{1+\beta}\left[w_t + \frac{(n+g+ng-r_{t+1})\,\tau}{1+r_{t+1}}\right]$$
$$a_{t+1} = \frac{\beta}{1+\beta}\,(w_t - \tau) - \frac{1}{1+\beta}\,\frac{1+n+g+ng}{1+r_{t+1}}\tau$$

Accordingly, the level of capital in the next period will be equal to:

$$K_{t+1} = N_t^y a_{t+1} = N_t^y \left[\frac{\beta}{1+\beta}\,(w_t - \tau) - \frac{1}{1+\beta}\,\frac{(1+n)\,(1+g)}{1+r_{t+1}}\tau\right]$$

Clearly, the right hand side depends negatively on τ. That is, compared to the no retirement system case, the economy will accumulate less capital. We have seen the situation when that would be beneficial: when the households save "too much", leading to the dynamically inefficient situation. In terms of interest rate, the dynamic inefficiency occurs whenever:

$$(1+r) < (1+n)(1+g)$$

Thus, if the market interest rate is below the "biological" interest rate, the PAYG system improves welfare. The opposite is true when:

$$(1+r) \geq (1+n)(1+g)$$

As many countries experience low fertility rates, and as a consequence low (or even negative) n, it would be optimal to switch from the PAYG to the fully funded system. However such a switch involves redirecting the social security contributions away from financing the pensions of currently old to the financial markets. This creates the need to find some other source of financing those pensions, which might involve welfare losses outweighing the benefits of switching systems. If you want to read more on these topics, check out some of **my papers**.