Marcin Bielecki, Applied Macroeconomics, Fall 2019
Example exam problems, with solutions

Problem 1

Let's examine the role of taxes in the Solow-Swan model. In the following model economy, income is taxed with rate τ and (for simplicity) the tax revenues are used for government consumption which is useless from the point of view of households. The behavior of the economy may be summarized by the following three equations:

$$K_{t+1} = (1-\delta) K_t + I_t$$
$$I_t = s \left[(1-\tau) Y_t \right]$$
$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

Assume that population (and also number of employees) grows at rate n and technology at rate g, so that $L_{t+1} = (1+n) L_t$ and $A_{t+1} = (1+g) A_t$, respectively.

(a) Transform the three equations into per effective labor form, i.e. divide them by $A_t L_t$. Make use of notational convention $\hat{x}_t \equiv X_t / (A_t L_t)$.

$$K_{t+1} = (1-\delta) K_t + I_t \quad | \quad : A_t L_t$$
$$\frac{K_{t+1}}{A_t L_t} = (1-\delta) \frac{K_t}{A_t L_t} + \frac{I_t}{A_t L_t}$$
$$\frac{L_{t+1}}{L_t} \frac{A_{t+1}}{A_t} \frac{K_{t+1}}{A_{t+1} L_{t+1}} = (1-\delta) \hat{k}_t + \hat{i}_t$$
$$(1+n)(1+g) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \hat{i}_t$$

$$I_t = s \left[(1-\tau) Y_t \right] \quad | \quad : A_t L_t$$
$$\frac{I_t}{A_t L_t} = s \left[(1-\tau) \frac{Y_t}{A_t L_t} \right]$$
$$\hat{i}_t = s \left[(1-\tau) \hat{y}_t \right]$$

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \quad | \quad : A_t L_t$$
$$\frac{Y_t}{A_t L_t} = K_t^\alpha (A_t L_t)^{-\alpha}$$
$$\hat{y}_t = \left(\frac{K_t}{A_t L_t} \right)^\alpha$$
$$\hat{y}_t = \hat{k}_t^\alpha$$

(b) Find the balanced growth path level of capital per effective labor $\hat{k}^*$ in this economy.

$$(1+n)(1+g) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \hat{i}_t$$
$$\hat{i}_t = s \left[(1-\tau) \hat{y}_t \right]$$
$$\hat{y}_t = \hat{k}_t^\alpha$$

$$(1+n+g+ng) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + s \left[(1-\tau) \hat{k}_t^\alpha \right]$$

For $\hat{k}_t = \hat{k}_{t+1} = \hat{k}^*$:

$$\begin{aligned}
(1+n+g+ng)\,\hat{k}^* &= (1-\delta)\,\hat{k}^* + s\,(1-\tau)\left(\hat{k}^*\right)^{\alpha} \quad | \quad -(1-\delta)\,\hat{k}^* \\
(\delta+n+g+ng)\,\hat{k}^* &= s\,(1-\tau)\left(\hat{k}^*\right)^{\alpha} \quad | \quad :(\delta+n+g+ng)\left(\hat{k}^*\right)^{\alpha} \\
\left(\hat{k}^*\right)^{1-\alpha} &= \frac{s\,(1-\tau)}{\delta+n+g+ng} \quad | \quad ^{1/(1-\alpha)} \\
\hat{k}^* &= \left[\frac{s\,(1-\tau)}{\delta+n+g+ng}\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

(c) Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's balanced growth path level of capital per effective labor $\hat{k}^*$.
From the expression obtained in (b) it can be easily seen that balanced growth path level of capital per effective labor $\hat{k}^*$ depends negatively on δ, n, g, τ, and positively on s.

(d) Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's balanced growth path level of consumption per effective labor $\hat{c}^*$.

$$\hat{c}^* = (1-s)\,[(1-\tau)\,\hat{y}^*] = (1-s)\,(1-\tau)\left(\hat{k}^*\right)^{\alpha} = (1-s)\,(1-\tau)\left[\frac{s\,(1-\tau)}{\delta+n+g+ng}\right]^{\frac{\alpha}{1-\alpha}}$$

Balanced growth path level of consumption per effective labor $\hat{c}^*$ depends negatively on δ, n, g, τ. For levels of s below the golden rule level, balanced growth path level of consumption per effective labor $\hat{c}^*$ depends positively on s, and negatively for levels of s above the golden rule level.

(e) Households care about the level of consumption per capita, i.e. c_t. This variable grows at rate g once the economy reaches its balanced growth path. Discuss whether low g or high g is better from the point of view of households.
By definition, the level of consumption per capita along the balanced growth path is equal to $c_t^* = \hat{c}^* A_t$. We have shown that $\hat{c}^*$ depends negatively on g, but this represents only the level effect. By higher g the level of technology A_t grows faster (growth effect) and thus c_t^* is actually higher when g is higher. Therefore, households "like" to have fast technology growth as it increases their consumption per capita.

Problem 2

Consider the following two-period utility maximization problem. This utility function belongs to the CRRA (Constant Relative Risk Aversion) class of functions which can be thought of as generalized logarithmic functions. An agent lives for two periods and in both receives some positive income y.

$$\begin{aligned}
\max \quad & U = \frac{c_t^{1-\sigma}-1}{1-\sigma} + \beta\frac{c_{t+1}^{1-\sigma}-1}{1-\sigma} \\
\text{subject to} \quad & c_t + a_{t+1} = y_t \\
& c_{t+1} = y_{t+1} + (1+r)\,a_{t+1}
\end{aligned}$$

where $\sigma \geq 0$,[1] $\beta \in [0,1]$ and $r \geq 0$.

[1] For $\sigma = 1$ the CRRA function becomes logarithmic: $U = \ln c_t + \beta \ln c_{t+1}$. This can be proven by using L'Hôpital's rule to compute the following limit: $\lim_{\sigma\to 1}\left(c^{1-\sigma}-1\right)/(1-\sigma)$

(a) Rewrite the budget constraints into a single lifetime budget constraint and set up the Lagrangian.

$$\begin{aligned} c_{t+1} &= y_{t+1} + (1+r)\, a_{t+1} \\ a_{t+1} &= \frac{c_{t+1} - y_{t+1}}{1+r} \end{aligned}$$

$$\begin{aligned} c_t + a_{t+1} &= y_t \\ c_t + \frac{c_{t+1} - y_{t+1}}{1+r} &= y_t \\ c_t + \frac{c_{t+1}}{1+r} &= y_t + \frac{y_{t+1}}{1+r} \end{aligned}$$

Lagrangian:

$$\mathcal{L} = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma} - 1}{1-\sigma} + \lambda \left[y_t + \frac{y_{t+1}}{1+r} - c_t - \frac{c_{t+1}}{1+r} \right]$$

(b) Obtain the first order conditions for c_t and c_{t+1}. Express c_{t+1} as a function of c_t.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= c_t^{-\sigma} + \lambda\, [-1] = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{t+1}} &= \beta c_{t+1}^{-\sigma} + \lambda \left[-\frac{1}{1+r} \right] = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda &= c_t^{-\sigma} \\ \lambda &= \beta\, (1+r)\, c_{t+1}^{-\sigma} \end{aligned}$$

Join conditions to get the Euler equation:

$$\begin{aligned} c_t^{-\sigma} &= \beta\, (1+r)\, c_{t+1}^{-\sigma} \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \beta\, (1+r) \\ c_{t+1} &= [\beta\, (1+r)]^{1/\sigma}\, c_t \end{aligned}$$

(c) Using the lifetime budget constraint obtain the formulas for optimal c_t and c_{t+1}.

$$\begin{aligned} c_t + \frac{c_{t+1}}{1+r} &= y_t + \frac{y_{t+1}}{1+r} \\ c_t + \frac{[\beta\, (1+r)]^{1/\sigma}\, c_t}{1+r} &= y_t + \frac{y_{t+1}}{1+r} \\ c_t \left[1 + \beta^{\frac{1}{\sigma}}\, (1+r)^{\frac{1}{\sigma}-1} \right] &= y_t + \frac{y_{t+1}}{1+r} \end{aligned}$$

$$\begin{aligned} c_t &= \frac{1}{1 + \beta^{\frac{1}{\sigma}}\, (1+r)^{\frac{1}{\sigma}-1}} \left[y_t + \frac{y_{t+1}}{1+r} \right] \\ c_{t+1} &= \frac{\beta^{\frac{1}{\sigma}}\, (1+r)^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}}\, (1+r)^{\frac{1}{\sigma}-1}} \left[y_t + \frac{y_{t+1}}{1+r} \right] \end{aligned}$$

(d) Set $\sigma = 1$ and verify that the formulas for optimal c_t and c_{t+1} are identical to the ones we obtained in class for the utility function $U = \log c_t + \beta \log c_{t+1}$.

$$c_t = \frac{1}{1+\beta}\left[y_t + \frac{y_{t+1}}{1+r}\right]$$
$$c_{t+1} = \frac{\beta(1+r)}{1+\beta}\left[y_t + \frac{y_{t+1}}{1+r}\right]$$

(e) Assume now that $y_{t+1} = 0$. How does c_t react when interest rate r increases? How does it depend on σ? How σ impacts the relative strength of income and substitution effects?

$$c_t = \frac{1}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}} y_t$$

For $\sigma > 1$, the expression depends positively on r, meaning that the income effect is stronger than the substitution effect. For $\sigma < 1$, the expression depends negatively on r, meaning that the income effect is weaker than the substitution effect. For $\sigma = 1$, the expression does not depend on r, meaning that the income effect and the substitution effect have equal strength.

Problem 3

Suppose you have a two-period OLG model. L_t agents are born in time t, where $L_t = (1+n)^t$, with $n > 0$. Preferences of a young agent born in time t are given by:

$$U_t = \log c_t^y + \beta \log c_{t+1}^o$$

The old agents want to consume as much as possible. Each young agent is endowed with y units of the consumption good. The old have no endowment whatsoever. There is a storage technology that allows to convert one unit of period t goods into $1+r$ units of period $t+1$ goods. There is a social security system that is "pay-as-you-go". In each period t the government taxes the young and uses the receipts to make transfers to the old, denoted with p_t. We consider a lump-sum (per capita) tax on the young that is constant over time, i.e. $\tau_t = \tau$ for all $t = 0, 1, 2, \ldots$

(a) Write down the government balanced budget constraint in period t. Find the level of pension benefits p_t.

$$p_t L_{t-1} = \tau L_t$$
$$p_t = \tau \frac{L_t}{L_{t-1}}$$
$$p_t = \tau(1+n)$$

(b) Solve for the optimal consumption levels of the agent born in time t. Find the level of savings of the young agent.
Budget constraints:

$$c_t^y + a_{t+1} = y - \tau$$
$$c_{t+1}^o = (1+r)a_{t+1} + p_{t+1}$$

Lifetime budget constraint:

$$a_{t+1} = \frac{c_{t+1}^o - p_{t+1}}{1+r}$$
$$c_t^y + \frac{c_{t+1}^o}{1+r} = y - \tau + \frac{p_{t+1}}{1+r}$$

Plugging in pension benefits:

$$c_t^y + \frac{c_{t+1}^o}{1+r} = y - \tau + \tau\frac{1+n}{1+r}$$
$$c_t^y + \frac{c_{t+1}^o}{1+r} = y + \tau\left(\frac{1+n}{1+r} - 1\right)$$

Lagrangian:

$$\mathcal{L} = \ln c_t^y + \beta \ln c_{t+1}^o + \lambda\left[y + \tau\left(\frac{1+n}{1+r} - 1\right) - c_t^y - \frac{c_{t+1}^o}{1+r}\right]$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t^y} = \frac{1}{c_t^y} + \lambda\left[-1\right] = 0$$
$$\frac{\partial \mathcal{L}}{\partial c_t^y} = \beta\frac{1}{c_{t+1}^o} + \lambda\left[-\frac{1}{1+r}\right] = 0$$

Simplify and rewrite:

$$\lambda = \frac{1}{c_t^y}$$
$$\lambda = \beta\left(1+r\right)\frac{1}{c_{t+1}^o}$$

Join conditions to get the Euler equation:

$$\frac{1}{c_t^y} = \beta\left(1+r\right)\frac{1}{c_{t+1}^o}$$
$$c_{t+1}^o = \beta\left(1+r\right)c_t^y$$

Use the lifetime budget constraint:

$$c_t^y + \frac{c_{t+1}^o}{1+r} = y + \tau\left(\frac{1+n}{1+r} - 1\right)$$
$$c_t^y + \frac{\beta\left(1+r\right)c_t^y}{1+r} = y + \tau\left(\frac{1+n}{1+r} - 1\right)$$
$$c_t^y\left(1+\beta\right) = y + \tau\left(\frac{1+n}{1+r} - 1\right)$$

$$c_t^y = \frac{1}{1+\beta}\left[y + \tau\left(\frac{1+n}{1+r} - 1\right)\right]$$
$$c_{t+1}^o = \frac{\beta\left(1+r\right)}{1+\beta}\left[y + \tau\left(\frac{1+n}{1+r} - 1\right)\right]$$

Level of savings:

$$a_{t+1} = y - c_t^y = y - \frac{1}{1+\beta}\left[y + \tau\left(\frac{1+n}{1+r} - 1\right)\right] = \frac{\beta}{1+\beta}y - \frac{1}{1+\beta}\tau\left(\frac{1+n}{1+r} - 1\right)$$

(c) What is the effect of an increase in τ on savings of the young agent?

$$\frac{\partial a_{t+1}}{\partial \tau} = -\frac{1}{1+\beta}\left(\frac{1+n}{1+r} - 1\right)$$

When $n > r$, this effect is negative and when $n < r$, this effect is positive.

(d) What is the optimal tax rate τ if $n < r$? Explain why.
If $n < r$, the pension system lowers the lifetime income of the agents. In this case, the optimal tax rate is 0.

Problem 4

Consider a Ramsey-Cass-Koopmans economy where for simplicity we assume $g = 0$ and $A = 1$. The representative households solve the following utility maximization problem:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \log c_t \\ \text{subject to} \quad & (1+n)\, a_{t+1} = (1+r_t)\, a_t + w_t - c_t + v_t \end{aligned}$$

where v is the lump-sum transfer from the government to households.

The representative firm solves the following profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_t = (1-\tau_y)\, Y_t - (r_t+\delta)\, K_t - w_t L_t \\ \text{subject to} \quad & Y_t = K_t^{\alpha} L_t^{1-\alpha} \end{aligned}$$

where τ_y is the firm revenue tax.

(a) Derive the Euler equation of the households.
Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \lambda_t \left[(1+r_t)\, a_t + w_t - c_t + v_t - (1+n)\, a_{t+1}\right]$$

Expand the Lagrangian to derive the first order conditions easier:

$$\begin{aligned} \mathcal{L} = \ldots + \beta^t \ln c_t + \ldots + \lambda_t \left[(1+r_t)\, a_t + w_t - c_t + v_t - (1+n)\, a_{t+1}\right] \\ + \lambda_{t+1}\left[(1+r_{t+1})\, a_{t+1} + w_{t+1} - c_{t+1} + v_{t+1} - (1+n)\, a_{t+2}\right] + \ldots \end{aligned}$$

Derive the first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \frac{1}{c_t} + \lambda_t\,[-1] = 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+1}} &= \lambda_t\,[-(1+n)] + \lambda_{t+1}\,[(1+r_{t+1})] = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda_t &= \beta^t \frac{1}{c_t} \\ (1+n)\, \lambda_t &= \lambda_{t+1}\,(1+r_{t+1}) \end{aligned}$$

Join conditions to get the Euler equation:

$$\begin{aligned} (1+n)\, \beta^t \frac{1}{c_t} &= \beta^{t+1} \frac{1}{c_{t+1}}\,(1+r_{t+1}) \\ c_{t+1} &= \frac{\beta\,(1+r_{t+1})}{1+n} c_t \end{aligned}$$

(b) Derive the first order conditions of the firm. Express resulting factor prices in terms of variables per worker.
Restate the problem of the firm:

$$\max \quad \Pi_t = (1-\tau_y)\, K_t^{\alpha} L_t^{1-\alpha} - (r_t+\delta)\, K_t - w_t L_t$$

Derive the first order conditions:

$$\frac{\partial \Pi_t}{\partial K_t} = (1-\tau_y)\,\alpha K_t^{\alpha-1} L_t^{1-\alpha} - (r_t+\delta) = 0$$
$$\frac{\partial \Pi_t}{\partial L_t} = (1-\tau_y)(1-\alpha) K_t^{\alpha} L_t^{-\alpha} - w_t = 0$$

Simplify and rewrite:

$$r_t = (1-\tau_y)\,\alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta = (1-\tau_y)\,\alpha k_t^{\alpha-1} - \delta$$
$$w_t = (1-\tau_y)(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\tau_y)(1-\alpha)\,k_t^{\alpha}$$

(c) Write down the government budget constraint. Using the assumptions of closed economy and balanced government budget, find the conditions for general equilibrium in this economy.
Government balanced budget constraint:

$$v_t L_t = \tau_y Y_t \quad \longrightarrow \quad v_t = \tau_y \frac{Y_t}{L_t} = \tau_y K_t^{\alpha} L_t^{-\alpha} = \tau_y k_t^{\alpha}$$

General equilibrium
Start with the households' budget constraint. Replace a with k and plug in prices and taxes/transfers:

$$(1+n)\,a_{t+1} = (1+r_t)\,a_t + w_t - c_t + v_t$$
$$(1+n)\,k_{t+1} = (1+r_t)\,k_t + w_t - c_t + v_t$$
$$(1+n)\,k_{t+1} = \left(1+(1-\tau_y)\,\alpha k_t^{\alpha-1} - \delta\right) k_t + (1-\tau_y)(1-\alpha)\,k_t^{\alpha} - c_t + \tau_y k_t^{\alpha}$$
$$(1+n)\,k_{t+1} = (1-\delta)\,k_t + (1-\tau_y)\,\alpha k_t^{\alpha} + (1-\tau_y)(1-\alpha)\,k_t^{\alpha} - c_t + \tau_y k_t^{\alpha}$$
$$(1+n)\,k_{t+1} = (1-\delta)\,k_t + k_t^{\alpha} - \tau_y k_t^{\alpha} - c_t + \tau_y k_t^{\alpha}$$
$$(1+n)\,k_{t+1} = (1-\delta)\,k_t + k_t^{\alpha} - c_t$$

Now plug in prices into the Euler equation:

$$c_{t+1} = \frac{\beta\,(1+r_{t+1})}{1+n} c_t$$
$$c_{t+1} = \frac{\beta\left(1+(1-\tau_y)\,\alpha k_{t+1}^{\alpha-1} - \delta\right)}{1+n} c_t$$

(d) Find the steady state level of capital per worker k^* and consumption per worker c^* in this economy. Discuss how they depend on the tax rate τ_y.
Start with the Euler equation. For $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$ we have:

$$c^* = \frac{\beta\left(1+(1-\tau_y)\,\alpha\,(k^*)^{\alpha-1} - \delta\right)}{1+n} c^*$$
$$\frac{1+n}{\beta} = 1 + (1-\tau_y)\,\alpha\,(k^*)^{\alpha-1} - \delta$$
$$\frac{1+n}{\beta} - (1-\delta) = (1-\tau_y)\,\alpha\,(k^*)^{\alpha-1}$$
$$(k^*)^{1-\alpha} = \frac{(1-\tau_y)\,\alpha}{\frac{1+n}{\beta} - (1-\delta)}$$
$$k^* = \left[\frac{(1-\tau_y)\,\alpha}{\frac{1+n}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\alpha}}$$

Steady state level of capital per worker k^* depends negatively on τ_y
Now examine the resource constraint:

$$(1+n)\,k^* = (1-\delta)\,k^* + (k^*)^\alpha - c^*$$
$$c^* = (k^*)^\alpha - (\delta + n)\,k^*$$

Steady state level of consumption per worker c^* depends negatively on τ_y, through the negative effect of the tax on the steady state level of capital per worker k^*.

Problem 5

Consider the following variant of the RBC model, where households always supply one unit of labor. Households maximize expected utility subject to the standard budget constraint:

$$\max \quad U_t = E_t\left[\sum_{i=0}^{\infty} \beta^i \log c_{t+i}\right]$$
$$\text{subject to} \quad c_t + k_{t+1} = w_t + (1+r_t)\,k_t + d_t$$

Firms maximize profits subject to the Cobb-Douglas production function (and they treat h as a choice variable, although $h = 1$ in equilibrium):

$$\max \quad d_t = y_t - w_t h_t - (r_t + \delta)\,k_t$$
$$\text{subject to} \quad y_t = z_t k_t^\alpha h_t^{1-\alpha}$$

where $\delta = 1$ (capital deprecates fully). The technology constant z evolves according to the process:

$$z_t = (1-\rho_z) + \rho_z z_{t-1} + \varepsilon_{z,t}$$

(a) Derive the first order conditions of the households.

Lagrangian:

$$\begin{aligned}\mathcal{L} &= \sum_{i=0}^{\infty} \beta^i E_t\left[\log c_{t+i} + \lambda_{t+i}\left(w_{t+i} + (1+r_{t+i})\,k_{t+i} + d_{t+i} - c_{t+i} - k_{t+1+i}\right)\right] \\ &= \log c_t + \lambda_t\left(w_t + (1+r_t)\,k_t + d_t - c_t - k_{t+1}\right) \\ &\quad + \beta E_t\left[\log c_{t+1} + \lambda_{t+1}\left(w_{t+1} + (1+r_{t+1})\,k_{t+1} + d_{t+1} - c_{t+1} - k_{t+2}\right)\right] + \dots\end{aligned}$$

First order conditions:

$$\begin{aligned} c_t &: \quad \frac{1}{c_t} + \lambda_t(-1) = 0 \quad \rightarrow \quad \lambda_t = \frac{1}{c_t} \\ k_{t+1} &: \quad \lambda_t(-1) + \beta E_t\left[\lambda_{t+1}(1+r_{t+1})\right] = 0 \\ &\quad \rightarrow \quad \lambda_t = \beta E_t\left[\lambda_{t+1}(1+r_{t+1})\right]\end{aligned}$$

Euler equation:

$$\lambda_t = \beta E_t\left[\lambda_{t+1}(1+r_{t+1})\right]$$
$$\frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right]$$

(b) Derive the first order conditions of the firm.

Plug in the production function into the profit equation:

$$d_t = z_t k_t^\alpha h_t^{1-\alpha} - w_t h_t - (r_t + \delta)\,k_t$$

First order conditions:

$$h_t \quad : \quad (1-\alpha) z_t k_t^\alpha h_t^{-\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha) z_t k_t^\alpha h_t^{-\alpha} = (1-\alpha) \frac{y_t}{h_t}$$
$$k_t \quad : \quad \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - (r_t + \delta) = 0 \quad \rightarrow \quad r_t = \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - \delta = \alpha \frac{y_t}{k_t} - \delta$$

Economic profits are equal to zero:

$$d_t = y_t - w_t h_t - (r_t + \delta) k_t = y_t - (1-\alpha) \frac{y_t}{h_t} \cdot h_t - \alpha \frac{y_t}{k_t} \cdot k_t = 0$$

Knowing that $h_t = 1$ and $\delta = 1$:

$$w_t = (1-\alpha) z_t k_t^\alpha$$
$$r_t = \alpha z_t k_t^{\alpha-1} - 1$$

(c) Find the steady state of the system.

List all the equations (knowing that $h_t = 1$ and $\delta = 1$):

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1}) \right]$$
$$c_t + k_{t+1} = w_t + (1 + r_t) k_t$$
$$y_t = z_t k_t^\alpha$$
$$w_t = (1-\alpha) z_t k_t^\alpha$$
$$r_t = \alpha z_t k_t^{\alpha-1} - 1$$
$$z_t = (1-\rho_z) + \rho_z z_{t-1} + \varepsilon_{z,t}$$

Steady state value of z:

$$z = (1-\rho_z) + \rho_z z \quad \rightarrow \quad z = 1$$

Euler equation:

$$\frac{1}{c} = \beta \frac{1}{c} (1 + r) \quad \rightarrow \quad r = \frac{1}{\beta} - 1$$

Interest rate equation:

$$r = \alpha k^{\alpha-1} - 1 \quad \rightarrow \quad k = \left(\frac{\alpha}{1+r} \right)^{1/(1-\alpha)}$$

Output:

$$y = k^\alpha$$

Consumption:

$$c + k = w + (1+r) k$$
$$c + k = (1-\alpha) k^\alpha + \alpha k^{\alpha-1} \cdot k = k^\alpha$$
$$c = k^\alpha - k$$

(d) Assuming that household behavior can be expressed as $c_t = (1-s) y_t$ where s is a constant, find the value of s as a function of model parameters. Hint: use the Euler equation.

$$c_t + k_{t+1} = y_t$$

$$c_t = (1-s) y_t \quad \rightarrow \quad k_{t+1} = s y_t$$

$$
\begin{aligned}
\frac{1}{c_t} &= \beta E_t\left[\frac{1}{c_{t+1}}\left(1+r_{t+1}\right)\right] \\
\frac{1}{(1-s)\,y_t} &= \beta E_t\left[\frac{1}{(1-s)\,y_{t+1}}\alpha z_{t+1}k_{t+1}^{\alpha-1}\right] \quad | \quad \cdot(1-s) \\
\frac{1}{z_t k_t^{\alpha}} &= \beta E_t\left[\frac{1}{z_{t+1}k_{t+1}^{\alpha}}\alpha z_{t+1}k_{t+1}^{\alpha-1}\right] \\
\frac{1}{z_t k_t^{\alpha}} &= \beta E_t\left[\frac{\alpha}{k_{t+1}}\right] \\
\frac{1}{z_t k_t^{\alpha}} &= \beta E_t\left[\frac{\alpha}{s y_t}\right] \\
\frac{1}{z_t k_t^{\alpha}} &= \beta\frac{\alpha}{s z_t k_t^{\alpha}} \quad | \quad \cdot s z_t k_t^{\alpha} \\
s &= \alpha\beta
\end{aligned}
$$

(e) Find the expression for k_{t+1} as a function of variables at time t.

$$k_{t+1} = s y_t = \alpha\beta\cdot z_t k_t^{\alpha}$$

Problem 6

In class we considered a model where permanent changes to marginal productivity of labor reduced the unemployment rate. This would imply that with trend productivity growth unemployment would disappear over time. Let's modify our model so that it is consistent with stationary unemployment in face of trend productivity growth. Suppose that the flow cost of a vacancy κ and the imputed value of free time b are functions of the wage rate w (instead being exogenous). In particular, assume that $\kappa_t = \kappa_0 w_t$ and $b_t = b_0 w_t$.

(a) Determine the formula for job creation and wage setting along the balanced growth path (steady state).

Standard formulas:

$$
\begin{aligned}
JC &\quad : \quad w = mpn - (r+s)\frac{\kappa}{q(\theta)} \\
W &\quad : \quad w = \gamma b + (1-\gamma)(mpn + \kappa\theta)
\end{aligned}
$$

Assuming that $\kappa_t = \kappa_0 w_t$ and $b_t = b_0 w_t$:

$$
\begin{aligned}
JC &\quad : \quad w = mpn - (r+s)\frac{\kappa_0 w}{q(\theta)} \quad \rightarrow \quad w = \frac{mpn}{1+(r+s)\kappa_0/q(\theta)} \\
W &\quad : \quad w = \gamma b_0 w + (1-\gamma)(mpn + \kappa_0 w\cdot\theta) \quad \rightarrow \quad w = \frac{(1-\gamma)\,mpn}{1-\gamma b_0 - (1-\gamma)\kappa_0\theta}
\end{aligned}
$$

(b) How do θ and wages along the balanced growth path react to productivity changes?

Consider the right and sides of job creation and wage setting equations:

$$
\begin{aligned}
\frac{mpn}{1+(r+s)\kappa_0/q(\theta)} &= \frac{(1-\gamma)\,mpn}{1-\gamma b_0-(1-\gamma)\kappa_0\theta} \quad | \quad : mpn \\
\frac{1}{1+(r+s)\kappa_0/q(\theta)} &= \frac{(1-\gamma)}{1-\gamma b_0-(1-\gamma)\kappa_0\theta}
\end{aligned}
$$

The result is an equation implicit in θ, but importantly it does not depend on the value of mpn. Therefore, labor market tightness θ does not react at all to changes in productivity, while wages move in the direction of productivity changes.

(c) Does a continuous growth of productivity lead to a decrease in the long run unemployment rate?

Since both the labor market tightness θ and the Beveridge curve do not depend on the level of productivity, a continuous growth of productivity does not lead to a decrease in the long run unemployment rate.

Problem 7

Suppose that you have the following simplified log-linearized (around 0 inflation steady state) New Keynesian model. The two main non-policy equations of the model can be written:

$$x_t = E_t x_{t+1} - (i_t - E_t \pi_{t+1})$$
$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}$$

where x is output gap, i is the nominal interest rate, π is inflation rate, β is the households' discount rate and $\kappa > 0$ is a constant that depends on model parameters. The central bank obeys a strict inflation targeting rule. In particular, let π_t^* be an exogenous inflation target. The central bank will adjust i_t so that $\pi_t = \pi_t^*$ is consistent with these equations holding. Assume that the inflation target follows an exogenous AR(1) process:

$$\pi_t^* = \rho_\pi \pi_{t-1}^* + \varepsilon_{\pi,t}, \quad \rho_\pi \in (0,1)$$

(a) Derive an analytic expression for i_t as a function of π_t^*.

Start with expectations:

$$E_t \pi_{t+1}^* = E_t \left[\rho_\pi \pi_t^* + \varepsilon_{\pi,t+1}\right] = \rho_\pi \pi_t^*$$

We also know that $\pi_t = \pi_t^*$ at all times, hence:

$$E_t \pi_{t+1} = \rho_\pi \pi_t^*$$

Start with the NKPC:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}$$
$$\pi_t^* = \kappa x_t + \beta \rho_\pi \pi_t^*$$

$$x_t = \frac{1 - \beta\rho_\pi}{\kappa} \pi_t^*$$

Consider now NKIS:

$$x_t = E_t x_{t+1} - (i_t - E_t \pi_{t+1})$$
$$\frac{1 - \beta\rho_\pi}{\kappa} \pi_t^* = \frac{1 - \beta\rho_\pi}{\kappa} E_t \pi_{t+1}^* - (i_t - \rho_\pi \pi_t^*)$$

$$i_t = \frac{\rho_\pi - \beta\rho_\pi^2}{\kappa} \pi_t^* - \frac{1 - \beta\rho_\pi}{\kappa} \pi_t^* + \rho_\pi \pi_t^*$$

(b) Suppose that ρ_π is 0. In which direction must the central bank adjust i_t in order to achieve an increase in π_t^*?

For $\rho_\pi = 0$ we get:

$$i_t = -\frac{1}{\kappa} \pi_t^*$$

which means that the central bank needs to decrease i_t in order to achieve an increase in π_t^*.

(c) If ρ_π is sufficiently close to 1, is it possible that i_t must increase in order to implement an increase in π_t^*? How does this answer depend on the value of κ?

For $\rho_\pi = 1$ we get:

$$i_t = \frac{1-\beta}{\kappa}\pi_t^* - \frac{1-\beta}{\kappa}\pi_t^* + \pi_t^* = \pi_t^*$$

which means that now the central bank needs to increase i_t in order to achieve an increase in π_t^* and this answer does not depend on the value of κ.

(d) Provide intuition behind the difference in results in (b) and (c).

The problem in (b) represents the case of a central bank wanting to engineer a one-period increase in inflation (for whatever reason) while the inflation target is not changed in the long run. This case represents "regular" monetary policymaking case where a decrease in the nominal interest rate generates a decrease in the inflation rate by moving the output gap in the negative direction.

The problem in (c) represents the case of permanently moving the inflation target in the long run and it implies that an economy with higher inflation target will also have higher nominal interest rates. This is the so-called Fisher effect.