Marcin Bielecki, Advanced Macroeconomics QF, Fall 2018

1 Ramsey-Cass-Koopmans (RCK) model

The RCK model is an extension of the Solow-Swan model, pioneered by **Frank Ramsey (1928)** and extended by **David Cass (1965)** and **Tjalling Koopmans (1965)**. Known also as the Neoclassical Growth Model, it is a "workhorse" of modern macroeconomics and forms the basis of the majority of modern works in economic growth and business cycles literature.

In the OLG model there were an infinite number of agents (since in every new period a new generation entered the economy) and they had a finite time horizon. In the Ramsey model the agents solve a problem with an infinite time horizon. This can be justified by the following argument. In the (basic) OLG model parents are "selfish" and leave no assets to their children. However in the real world we see that at least some individuals leave sizeable bequests and support their children financially throughout their lives. This implies that parents care for their children's "utility".

In the RCK model it is assumed that parents care about their children as much as they care about themselves. Moreover, both the labor and capital income are pooled within the household and split equally across its members. Therefore, each household is in fact a "dynasty" of individuals with an infinite planning horizon, and the welfare function of a household is constructed as follows:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where β is a discount factor and $u(\cdot)$ is an instantaneous utility function (felicity function).

Usually this felicity function is represented by a Constant Relative Risk Aversion (CRRA) function:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

where $\sigma > 0$ regulates the degree of risk aversion.

Firms' problem

The problem of the firms will be the same for both simplified and general cases. The firms want to hire optimal quantities of labor and capital to maximize their profits:

$$\max \quad \Pi_t = K_t^\alpha (A_t L_t)^{1-\alpha} - w_t L_t - r_t^k K_t$$

First order conditions (FOCs):

$$\frac{\partial \Pi_t}{\partial L_t} = (1-\alpha) K_t^\alpha A_t^{1-\alpha} L_t^{-\alpha} - w_t = 0$$

$$\frac{\partial \Pi_t}{\partial K_t} = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - r_t^k = 0$$

Simplify the above expressions:

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{A_t L_t}\right)^\alpha$$

$$r_t^k = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1}$$

The above equations determine both the wage w and the capital rental rate r^k as functions of capital (per effective labor). The real interest rate is equal to the capital rental rate, less the capital depreciation rate:

$$r_t = r_t^k - \delta$$

1.1 Simplified model

We will first consider a RCK model with simplifying assumptions that yield an analytical solution, and then consider a general case. For now, we will assume for simplicity that there is no population and technology growth, and without loss of generality we can normalize $A = 1$, so that capital per effective labor and capital per worker are equivalent. We will also assume total depreciation of capital ($\delta = 1$) and that the instantaneous utility function is logarithmic ($\sigma = 1$).

Households' problem

The representative "dynasty" solves the following problem:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{subject to} \quad & c_t + a_{t+1} = w_t + (1 + r_t)\, a_t \end{aligned}$$

Write down the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1} \right]$$

Note that we have an infinity of time periods and Lagrange multipliers. Rewrite the Lagrangian so that it is easier to take derivatives:

$$\begin{aligned} \mathcal{L} = \ldots & + \beta^t \ln c_t + \beta^t \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1} \right] \\ \ldots & + \beta^{t+1} \lambda_{t+1} \left[w_{t+1} + (1 + r_{t+1})\, a_{t+1} - c_{t+1} - a_{t+2} \right] + \ldots \end{aligned}$$

In each period t we can choose period t consumption c_t and end-of-period t assets a_{t+1}.

First order conditions (FOCs):

$$\begin{aligned} c_t \quad : \quad & \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} + \beta^t \lambda_t [-1] = 0 & \rightarrow \quad & \lambda_t = \frac{1}{c_t} \\ a_{t+1} \quad : \quad & \frac{\partial \mathcal{L}}{\partial a_{t+1}} = \beta^t \lambda_t [-1] + \beta^{t+1} \lambda_{t+1} [(1 + r_{t+1})] = 0 & \rightarrow \quad & \lambda_t = \beta (1 + r_{t+1})\, \lambda_{t+1} \end{aligned}$$

Join conditions:

$$\begin{aligned} \frac{1}{c_t} &= \beta (1 + r_{t+1}) \frac{1}{c_{t+1}} \\ c_{t+1} &= \beta (1 + r_{t+1})\, c_t \end{aligned}$$

We have obtained the standard Euler equation.

By using the Euler equation, the budget constraints and after some algebra the optimal consumpion in period t can be expressed as:

$$c_t = \frac{1}{1 - \beta} \left[(1 + r_t)\, a_t + \sum_{i=0}^{\infty} \frac{w_{t+i}}{1 + \bar{r}_{t+i}} \right]$$

where:

$$1 + \bar{r}_{t+i} = \begin{cases} 1 & \text{for } i = 0 \\ (1 + r_{t+1}) \cdot \ldots \cdot (1 + r_{t+i}) & \text{for } i = 1, 2, \ldots \end{cases}$$

As you can see, the optimal behavior of households requires them to formulate accurate forecasts of prices w and r.

General Equilibrium

The above solution of the households' problem is still relatively general, as we haven't made full use of our simplifying assumptions yet. Consider now the general equilibrium, where all markets clear.

Since there is no government and no international trade, domestic capital is the only asset which can be in positive net supply:

$$a_t = k_t \quad \text{for all } t$$

And the budget constraint of the households can be rewritten as:

$$\begin{aligned} c_t + a_{t+1} &= w_t + (1 + r_t)\, a_t \\ c_t + k_{t+1} &= w_t + (1 + r_t)\, k_t \end{aligned}$$

We can also use the expressions for prices (recall that $A = 1$ and $\delta = 1$):

$$\begin{aligned} w_t &= (1-\alpha)\, A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = (1-\alpha)\, k_t^{\alpha} \\ r_t^k &= \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} = \alpha k_t^{\alpha-1} \\ r_t &= r_t^k - \delta = \alpha k_t^{\alpha-1} - 1 \end{aligned}$$

And plug them into the budget constraint:

$$\begin{aligned} c_t + k_{t+1} &= (1-\alpha)\, k_t^{\alpha} + \left(1 + \alpha k_t^{\alpha-1} - 1\right) k_t \\ c_t + k_{t+1} &= (1-\alpha)\, k_t^{\alpha} + \alpha k_t^{\alpha} \\ c_t + k_{t+1} &= k_t^{\alpha} \end{aligned}$$

Closed-form solution

Finally, let us guess-and-verify that this economy behaves as a Solow-Swan economy, and the households save a constant fraction s of their income:

$$\begin{aligned} k_{t+1} &= s y_t = s k_t^{\alpha} \\ c_t &= (1-s)\, y_t = (1-s)\, k_t^{\alpha} \end{aligned}$$

Use the Euler equation and the equation for the interest rate:

$$\begin{aligned} c_{t+1} &= \beta\,(1 + r_{t+1})\, c_t \\ c_{t+1} &= \beta \alpha k_{t+1}^{\alpha-1} c_t \\ (1-s)\, k_{t+1}^{\alpha} &= \beta \alpha k_{t+1}^{\alpha-1} (1-s)\, k_t^{\alpha} \\ k_{t+1} &= \beta \alpha k_t^{\alpha} \end{aligned}$$

Indeed, the saving rate is a constant:

$$s = \alpha\beta$$

Note that $\partial s / \partial \beta > 0$ and that $s \leq \alpha = s_{GR}$, with equality when $\beta = 1$ (households care just as much about the future and the present). This implies that the RCK economy is never dynamically inefficient.

Under our assumptions, we can obtain an analytical, closed form solution of the RCK model:

$$\begin{aligned} c_t &= (1 - \alpha\beta)\, k_t^{\alpha} \\ k_{t+1} &= \alpha\beta k_t^{\alpha} \end{aligned}$$

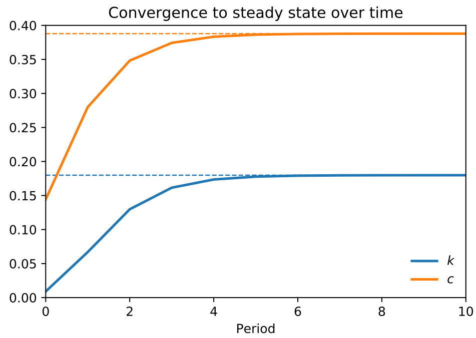
Note that under these assumptions the households do not have to forecast future prices, which simplifies greatly the solution procedure.

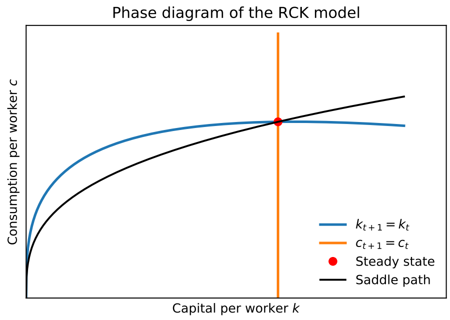
Finally, let us find the steady state of the model:

$$
\begin{aligned}
k^* &= \alpha\beta \left(k^*\right)^{\alpha} \\
\left(k^*\right)^{1-\alpha} &= \alpha\beta \\
k^* &= \left(\alpha\beta\right)^{1/(1-\alpha)} \\
c^* &= \left(1-\alpha\beta\right)\left(\alpha\beta\right)^{\alpha/(1-\alpha)}
\end{aligned}
$$

The first plot below illustrates the process of convergence to the steady state over time. Note that since $\delta = 1$, the convergence is very fast.

The second plot is a new type of plot, called a phase diagram, and will illustrate the convergence to the steady state in the (k, c) space.





The phase diagram shows a number of conditions. The first one, $k_{t+1} = k_t$, describes a set of combinations of k and c for which capital per worker does not change over time. We can find them via the budget/resource constraint:

$$
\begin{aligned}
c_t + k_{t+1} &= k_t^{\alpha} \\
c_t &= k_t^{\alpha} - k_{t+1}
\end{aligned}
$$

If $k_{t+1} = k_t = k$:

$$
c = k^{\alpha} - k
$$

The second condition, $c_{t+1} = c_t$, describes a set of combinations of k and c for which consumption per worker does not change over time. We can find them via the Euler equation:

$$
\begin{aligned}
c_{t+1} &= \beta\left(1 + r_{t+1}\right) c_t \\
1 &= \beta\left(1 + r_{t+1}\right) \\
r_{t+1} &= \frac{1}{\beta} - 1 \\
\alpha k_{t+1}^{\alpha-1} - 1 &= \frac{1}{\beta} - 1 \\
k_{t+1}^{\alpha-1} &= \frac{1}{\alpha\beta} \\
k_{t+1} &= \left(\alpha\beta\right)^{1/(1-\alpha)} = k^*
\end{aligned}
$$

That means that when the capital per worker is at its steady state level, $c_{t+1} = c_t$ for all c.

Obviously, the intersection of $k_{t+1} = k_t$ and $c_{t+1} = c_t$ is by definition the steady state of the model.

Finally, the "saddle path" is our model's solution. Importantly, there is only one path that leads from any possible level of capital per worker k to the steady state. This guarantees that the model has a unique solution.

1.2 Neoclassical Growth Model

We now turn to analyzing the full RCK/NGM model. Population grows at rate n, while technology improves at rate g. The firms' problem is the same as before. What is a bit different is the households' problem.

Households' problem

The representative "dynasty" solves the following problem:

$$\begin{aligned} \max \quad & U=\sum_{t=0}^{\infty}\beta^{t}\cdot\frac{c_{t}^{1-\sigma}-1}{1-\sigma} \\ \text{subject to} \quad & c_{t}+(1+n)\,a_{t+1}=w_{t}+(1+r_{t})\,a_{t} \end{aligned}$$

Lagrangian:

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^{t}\left\{\frac{c_{t}^{1-\sigma}-1}{1-\sigma}+\lambda_{t}\left[w_{t}+(1+r_{t})\,a_{t}-c_{t}-(1+n)\,a_{t+1}\right]\right\}$$

Expand the Lagrangian:

$$\begin{aligned} \mathcal{L}=\ldots+\beta^{t}\frac{c_{t}^{1-\sigma}-1}{1-\sigma}+\beta^{t}\lambda_{t}\left[w_{t}+(1+r_{t})\,a_{t}-c_{t}-(1+n)\,a_{t+1}\right] \\ \ldots+\beta^{t+1}\lambda_{t+1}\left[w_{t+1}+(1+r_{t+1})\,a_{t+1}-c_{t+1}-(1+n)\,a_{t+2}\right]+\ldots \end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned} c_{t} \quad &: \quad \frac{\partial\mathcal{L}}{\partial c_{t}}=\beta^{t}c_{t}^{-\sigma}+\beta^{t}\lambda_{t}\left[-1\right]=0 \\ a_{t+1} \quad &: \quad \frac{\partial\mathcal{L}}{\partial a_{t+1}}=\beta^{t}\lambda_{t}\left[-(1+n)\right]+\beta^{t+1}\lambda_{t+1}\left[(1+r_{t+1})\right]=0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda_{t}&=c_{t}^{-\sigma} \\ (1+n)\,\lambda_{t}&=\beta\,(1+r_{t+1})\,\lambda_{t+1} \end{aligned}$$

Join conditions:

$$\begin{aligned} c_{t}^{-\sigma}&=\frac{\beta\,(1+r_{t+1})}{1+n}c_{t+1}^{-\sigma} \\ \left(\frac{c_{t+1}}{c_{t}}\right)^{\sigma}&=\frac{\beta\,(1+r_{t+1})}{1+n} \\ \frac{c_{t+1}}{c_{t}}&=\left[\frac{\beta\,(1+r_{t+1})}{1+n}\right]^{1/\sigma} \\ c_{t+1}&=\left[\frac{\beta\,(1+r_{t+1})}{1+n}\right]^{1/\sigma}c_{t} \end{aligned}$$

We have obtained the Euler equation. Note that if $n=0$ and $\sigma=1$, we go back to our usual form of Euler equation.

General Equilibrium

Again, since there is no government and no international trade, domestic capital is the only asset which can be in positive net supply:

$$a_t = k_t \quad \text{for all } t$$

And the budget constraint of the households can be rewritten as:

$$\begin{aligned} c_t + (1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t \\ c_t + (1+n)\, k_{t+1} &= w_t + (1+r_t)\, k_t \end{aligned}$$

We can also use the expressions for prices (but this time technology improves over time, so we make use of per effective labor variables):

$$\begin{aligned} w_t &= (1-\alpha)\, A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = (1-\alpha)\, A_t \hat{k}_t^{\alpha} \\ r_t^k &= \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} = \alpha \hat{k}_t^{\alpha-1} \\ r_t &= r_t^k - \delta = \alpha \hat{k}_t^{\alpha-1} - \delta \end{aligned}$$

And plug them into the budget constraint:

$$\begin{aligned} c_t + (1+n)\, k_{t+1} &= (1-\alpha)\, A_t \hat{k}_t^{\alpha} + \left(1 + \alpha \hat{k}_t^{\alpha-1} - \delta\right) k_t \quad | \quad : A_t \\ \hat{c}_t + (1+n)\, \frac{k_{t+1}}{A_t} &= (1-\alpha)\, \hat{k}_t^{\alpha} + \left(1 + \alpha \hat{k}_t^{\alpha-1} - \delta\right) \hat{k}_t \\ \hat{c}_t + (1+n)\, \frac{A_{t+1}}{A_t} \frac{k_{t+1}}{A_{t+1}} &= (1-\alpha)\, \hat{k}_t^{\alpha} + \alpha \hat{k}_t^{\alpha} + (1-\delta)\, \hat{k}_t \\ (1+n)\,(1+g)\, \hat{k}_{t+1} &= \hat{k}_t^{\alpha} - \hat{c}_t + (1-\delta)\, \hat{k}_t \\ \hat{k}_{t+1} &= \frac{\hat{k}_t^{\alpha} - \hat{c}_t + (1-\delta)\, \hat{k}_t}{(1+n)\,(1+g)} \end{aligned}$$

The $\hat{k}_t^{\alpha} - \hat{c}_t$ part represents gross investment, and is the analogue of $s\hat{k}_t^{\alpha}$ in the Solow-Swan and OLG models.

Plug in the interest rate into the Euler equation and rewrite it in per effective labor terms:

$$\begin{aligned} c_{t+1} &= \left(\beta \frac{1+r_{t+1}}{1+n}\right)^{1/\sigma} c_t \quad | \quad : A_t \\ \frac{c_{t+1}}{A_t} &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n}\right)^{1/\sigma} \frac{c_t}{A_t} \\ \frac{A_{t+1}}{A_t} \frac{c_{t+1}}{A_{t+1}} &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n}\right)^{1/\sigma} \hat{c}_t \\ (1+g)\, \hat{c}_{t+1} &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n}\right)^{1/\sigma} \hat{c}_t \\ \hat{c}_{t+1} &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n}\right)^{1/\sigma} \frac{\hat{c}_t}{(1+g)} \end{aligned}$$

Steady state

If $\hat{c}_{t+1} = \hat{c}_t$, then:

$$
\begin{aligned}
(1+g) &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n} \right)^{1/\sigma} \\
(1+g)^{\sigma} &= \beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n} \\
\frac{(1+g)^{\sigma}(1+n)}{\beta} &= 1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta \\
\alpha \hat{k}_{t+1}^{\alpha-1} &= \frac{(1+g)^{\sigma}(1+n)}{\beta} - (1-\delta) \\
\hat{k}_{t+1}^{\alpha-1} &= \frac{(1+g)^{\sigma}(1+n) - \beta(1-\delta)}{\alpha\beta} \\
\hat{k}^{*} &= \left[\frac{\alpha\beta}{(1+g)^{\sigma}(1+n) - \beta(1-\delta)} \right]^{1/(1-\alpha)}
\end{aligned}
$$

Note that if $\beta = 1$ and $\sigma = 1$, then we can express $\hat{k}^*$ as:

$$
\hat{k}^{*} = \left[\frac{\alpha}{(1+g)(1+n) - (1-\delta)} \right]^{1/(1-\alpha)} = \left[\frac{\alpha}{\delta + n + g + ng} \right]^{1/(1-\alpha)}
$$

which is equivalent to the Solow-Swan model solution, given that the saving rate is at its golden rule level.

If $\hat{k}_{t+1} = \hat{k}_t = \hat{k}$, then:

$$
\begin{aligned}
(1+n)(1+g)\hat{k}_{t+1} &= \hat{k}_t^{\alpha} - \hat{c}_t + (1-\delta)\hat{k}_t \\
(1+n)(1+g)\hat{k} &= \hat{k}^{\alpha} - \hat{c} + (1-\delta)\hat{k}
\end{aligned}
$$

$$
\begin{aligned}
\hat{c} &= \hat{k}^{\alpha} + (1-\delta)\hat{k} - (1+n)(1+g)\hat{k} \\
\hat{c} &= \hat{k}^{\alpha} - (\delta + n + g + ng)\hat{k}
\end{aligned}
$$

And the steady state level of consumption per effective labor is given by:

$$
\hat{c}^{*} = (\hat{k}^{*})^{\alpha} - (\delta + n + g + ng)\hat{k}^{*}
$$

Transition dynamics (saddle path)

Along the transition to the steady state, the model variables have to obey the following conditions:

$$
\begin{aligned}
\hat{k}_{t+1} &= \frac{\hat{k}_t^{\alpha} - \hat{c}_t + (1-\delta)\hat{k}_t}{(1+n)(1+g)} \\
\hat{c}_{t+1} &= \left(\beta \frac{1 + \alpha \hat{k}_{t+1}^{\alpha-1} - \delta}{1+n} \right)^{1/\sigma} \frac{\hat{c}_t}{(1+g)}
\end{aligned}
$$

Solving the RCK model is equivalent to finding, for some initial level of capital per effective labor $\hat{k}_0$, such $\hat{c}_0$ that the above forward equations bring $\hat{k}$ and $\hat{c}$ to the steady state.

Since the general formulation of the model yields no closed-form solution, the saddle path can be found by using one of many numerical solution procedures. A very simple numerical procedure is the shooting algorithm.

2 Optimal taxation in the long run

For simplicity, we will assume $n = g = 0$ and $N = A = 1$. We will consider two uses for the taxes: lump-sum transfers to households v and useless from the point of view of households government consumption g (note the notation change!). We assume that in each period the government's budget is balanced.

Households

Utility maximization problem:

$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & a_{t+1} = (1 + (1 - \tau_t^a) r_t) a_t + (1 - \tau_t^w) w_t - (1 + \tau_t^c) c_t - \tau_t + v_t \end{aligned}$$

where τ^a is capital gains tax, τ^w is labor income tax, τ^c is consumption tax and τ is lump-sum tax.

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[(1 + (1 - \tau_t^a) r_t) a_t + (1 - \tau_t^w) w_t - (1 + \tau_t^c) c_t - \tau_t + v_t - a_{t+1} \right] \right\}$$

FOCs:

$$\begin{aligned} c_t \quad &: \quad \beta^t \left\{ c_t^{-\sigma} - \lambda_t (1 + \tau_t^c) \right\} = 0 \qquad \rightarrow \qquad \lambda_t = \frac{c_t^{-\sigma}}{1 + \tau_t^c} \\ a_{t+1} \quad &: \quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left(1 + \left(1 - \tau_{t+1}^a \right) r_{t+1} \right) = 0 \end{aligned}$$

Euler equation:

$$\begin{aligned} \frac{c_t^{-\sigma}}{1 + \tau_t^c} &= \beta \frac{c_{t+1}^{-\sigma}}{1 + \tau_{t+1}^c} \left(1 + \left(1 - \tau_{t+1}^a \right) r_{t+1} \right) \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} \beta \left(1 + \left(1 - \tau_{t+1}^a \right) r_{t+1} \right) \end{aligned}$$

Firms

Profit maximizing problem:

$$\max \quad \Pi_t = (1 - \tau_t^f) \left[K_t^{\alpha} L_t^{1-\alpha} - \delta K_t - w_t L_t \right] - r_t K_t$$

where τ^f is firm earnings tax.

FOCs:

$$\begin{aligned} K_t \quad &: \quad (1 - \tau_t^f) \left[\alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta \right] - r_t = 0 \qquad \rightarrow \qquad r_t = (1 - \tau_t^f) \left(\alpha k_t^{\alpha-1} - \delta \right) \\ L_t \quad &: \quad (1 - \tau_t^f) \left[(1 - \alpha) K_t^{\alpha} L_t^{-\alpha} - w_t \right] = 0 \qquad \rightarrow \qquad w_t = (1 - \alpha) k_t^{\alpha} \end{aligned}$$

The tax on firm's earnings lowers the return on capital and incentivises firms to hold less capital.

Firm earnings tax revenue is equal to:

$$\tau_t^f \left[K_t^{\alpha} L_t^{1-\alpha} - \delta K_t - w_t L_t \right] = L_t \cdot \tau_t^f \left[k_t^{\alpha} - \delta k_t - (1 - \alpha) k_t^{\alpha} \right] = L_t \cdot \tau_t^f \left(\alpha k_t^{\alpha} - \delta k_t \right)$$

Government sector

The government maintains balanced budget. In per capita terms:

$$\begin{aligned} g_t + v_t &= \tau_t^f \left(\alpha k_t^{\alpha} - \delta k_t \right) + \tau_t^a r_t a_t + \tau_t^c c_t + \tau_t^w w_t + \tau_t \\ v_t &= \tau_t^f \left(\alpha k_t^{\alpha} - \delta k_t \right) + \tau_t^a r_t a_t + \tau_t^c c_t + \tau_t^w w_t + \tau_t - g_t \end{aligned}$$

General equilibrium

Market clearing for capital:

$$k_t = a_t$$

Rewrite households' budget constraint to get capital accumulation equation:

$$\begin{aligned}
k_{t+1} &= \left(1 + \left(1 - \tau_t^a\right) r_t\right) k_t + \left(1 - \tau_t^w\right) w_t - \left(1 + \tau_t^c\right) c_t - \tau_t + v_t \\
k_{t+1} &= \left(1 + r_t\right) k_t + w_t - c_t - \left(\tau_t^a r_t k_t + \tau_t^w w_t + \tau_t^c c_t + \tau_t - v_t\right) \\
k_{t+1} &= \left[1 + \left(1 - \tau_t^f\right)\left(\alpha k_t^{\alpha-1} - \delta\right)\right] k_t + \left(1 - \alpha\right) k_t^\alpha - c_t - \left[g_t - \tau_t^f \left(\alpha k_t^\alpha - \delta k_t\right)\right] \\
k_{t+1} &= \left[1 + \alpha k_t^{\alpha-1} - \delta\right] k_t + \left(1 - \alpha\right) k_t^\alpha - c_t - g_t \\
k_{t+1} &= k_t^\alpha + \left(1 - \delta\right) k_t - c_t - g_t
\end{aligned}$$

Rewrite the Euler equation:

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} \beta \left[1 + \left(1 - \tau_{t+1}^a\right)\left(1 - \tau_{t+1}^f\right)\left(\alpha k_t^{\alpha-1} - \delta\right)\right]$$

Steady state

Assume constant tax rates:

$$\begin{aligned}
1 &= \beta \left[1 + \left(1 - \tau^a\right)\left(1 - \tau^f\right)\left(\alpha k^{\alpha-1} - \delta\right)\right] \\
\frac{1}{\beta} - 1 &= \left(1 - \tau^a\right)\left(1 - \tau^f\right)\left(\alpha k^{\alpha-1} - \delta\right) \\
\frac{\rho}{\left(1 - \tau^a\right)\left(1 - \tau^f\right)} &= \left(\alpha k^{\alpha-1} - \delta\right) \\
\alpha k^{\alpha-1} &= \frac{\rho}{\left(1 - \tau^a\right)\left(1 - \tau_t^f\right)} + \delta \\
k^* &= \left(\frac{\alpha}{\rho / \left[\left(1 - \tau^a\right)\left(1 - \tau_t^f\right)\right] + \delta}\right)^{1/(1-\alpha)}
\end{aligned}$$

$$k_{t+1} = k_t = k^* \quad \rightarrow \quad c^* = \left(k^*\right)^\alpha - \delta k^* - g$$

Government consumption lowers private consumption but does not affect steady state capital per worker.

Capital gains and firm earnings taxes lower steady state capital per worker which then translates to lower steady state private consumption.

2.1 Chamley (1986)-Judd (1985): redistribution impossibility theorem

Again assume $n = g = 0$ and $N = A = 1$ for simplicity. Divide population into two groups: workers and capitalists. Workers do not save and consume their wages and any transfers they receive. Capitalists both save and consume. The government wants to redistribute between capitalists and workers. It levies tax on capital gains and distributes the proceeds to workers.

Worker households

$$\begin{aligned}
\max \quad & U_0^w = \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^w\right)^{1-\sigma} - 1}{1 - \sigma} \\
\text{subject to} \quad & c_t^w = w_t + v_t
\end{aligned}$$

Solution:

$$c_t^w = w_t + v_t$$

Capitalist households

$$\begin{aligned} \max \quad & U_0^c = \sum_{t=0}^{\infty} \beta^t \frac{(c_t^c)^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & a_{t+1} = (1 + (1-\tau^a) r_t) a_t - c_t^c \end{aligned}$$

Solution:

$$\left(\frac{c_{t+1}^c}{c_t^c}\right)^{\sigma} = \beta (1 + (1-\tau^a) r_{t+1})$$

Firms

$$\max \quad \Pi_t = K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - r_t^k K_t$$

Solution:

$$\begin{aligned} r_t^k &= \alpha K_t^{\alpha-1} L_t^{1-\alpha} && \rightarrow \quad r_t = \alpha k_t^{\alpha-1} - \delta \\ w_t &= (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) k_t^{\alpha} \end{aligned}$$

Government sector

$$N^w v_t = N^c \tau^a r_{t+1} a_{t+1}$$

General equilibrium

Capital market equilibrium:

$$N^w k_t = N^c a_t \quad \rightarrow \quad v_t = \tau^a r_{t+1} k_{t+1}$$

Steady state capital per worker:

$$\begin{aligned} 1 &= \beta \left(1 + (1-\tau^a)\left(\alpha k^{\alpha-1} - \delta\right)\right) \\ \alpha k^{\alpha-1} &= \frac{\rho}{1-\tau^a} + \delta \\ k &= \left(\frac{\alpha}{\rho/(1-\tau^a) + \delta}\right)^{1/(1-\alpha)} \end{aligned}$$

Steady state workers' consumption:

$$c^w = (1-\alpha) k^{\alpha} + \tau^a \left(\alpha k^{\alpha-1} - \delta\right) k$$

For an easy proof, let us assume that $\delta = 0$:

$$\begin{aligned} c^w &= (1-\alpha) k^{\alpha} + \tau^a \alpha k^{\alpha} = (1-\alpha+\alpha\tau^a)\left(\frac{\alpha(1-\tau^a)}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \\ \ln c^w &= \ln(1-\alpha+\alpha\tau^a) + \frac{\alpha}{1-\alpha}\left(\ln\alpha + \ln(1-\tau^a) - \ln\rho\right) \\ \frac{\partial \ln c^w}{\partial \tau^a} &= \frac{\alpha}{1-\alpha+\alpha\tau^a} + \frac{\alpha}{1-\alpha}\left(-\frac{1}{1-\tau^a}\right) = \frac{\alpha}{1-\alpha+\alpha\tau^a} - \frac{\alpha}{1-\alpha+\alpha\tau^a-\tau^a} < 0 \end{aligned}$$

It turns out that it is impossible to increase steady state consumption of workers by taxing capitalists. Taxing capitalists reduces steady state capital stock and lowers wages. Even if all of the revenue from taxation is given to workers in transfer, the loss in wages is greater than the gain from the transfer.

See **here** for conditions under which the above result might not hold. For example, **Aiyagari (1995)** and **Chamley (2001)** show that under incomplete markets and borrowing constraints, the optimal long-run capital gains tax rate is positive. Also, **Straub and Werning (2014)** show that the Chamley-Judd result depends critically on whether the economy actually converges to the steady state discussed above.