Marcin Bielecki, Advanced Macroeconomics QF, Fall 2018

1 Investment

1.1 Neoclassical investment theory

Households

The economy is populated by N identical, infinitely lived households that solve the following problem:

$$\max_{c_t, b_{t+1}, s_{t+1}} \quad U = \sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$
$$\text{subject to} \quad c_t + b_{t+1} + p_t s_{t+1} = w_t + (1+r) b_t + (d_t + p_t) s_t$$

where b denotes holding of corporate bonds that pay real interest rate r, w denotes the wage that the household receives for supplying labor, p is the price of a single share of a representative firm, s is the number of shares owned by the household and d is the dividend per share from the firm. We assume that the dividends are paid at the end of a time period and only after that the trade in firm shares takes place.

Lagrangian:

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta^i \left\{ u(c_{t+i}) + \lambda_{t+i} \left[w_{t+i} + (1+r) b_{t+i} + (d_{t+i} + p_{t+i}) s_{t+i} - c_{t+i} - b_{t+i+1} - p_{t+i} s_{t+i+1} \right] \right\}$$

Let us expand the Lagrangian so that taking FOCs will be easier:

$$\begin{aligned}
\mathcal{L} &= u(c_t) + \lambda_t \left[w_t + (1+r) b_t + (d_t + p_t) s_t - c_t - b_{t+1} - p_t s_{t+1} \right] \\
&+ \beta \left\{ u(c_{t+1}) + \lambda_{t+1} \left[w_{t+1} + (1+r) b_{t+1} + (d_{t+1} + p_{t+1}) s_{t+1} - c_{t+1} - b_{t+2} - p_{t+1} s_{t+2} \right] \right\} \\
&+ \sum_{i=2}^{\infty} \beta^i \left\{ u(c_{t+i}) + \lambda_{t+i} \left[w_{t+i} + (1+r) b_{t+i} + (d_{t+i} + p_{t+i}) s_{t+i} - c_{t+i} - b_{t+i+1} - p_{t+i} s_{t+i+1} \right] \right\}
\end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned}
c_t &: \quad u'(c_t) - \lambda_t = 0 && \rightarrow \quad \lambda_t = u'(c_t) \\
b_{t+1} &: \quad \underbrace{-\lambda_t}_{i=0} + \underbrace{\beta \lambda_{t+1} (1+r)}_{i=1} = 0 && \rightarrow \quad \lambda_t = \beta \lambda_{t+1} (1+r) \\
s_{t+1} &: \quad \underbrace{-\lambda_t p_t}_{i=0} + \underbrace{\beta \lambda_{t+1} (d_{t+1} + p_{t+1})}_{i=1} = 0 && \rightarrow \quad \lambda_t p_t = \beta \lambda_{t+1} (d_{t+1} + p_{t+1})
\end{aligned}$$

Combining the FOCs for consumption and bonds, we get the usual Euler equation:

$$u'(c_t) = \beta (1+r) u'(c_{t+1})$$

Combining the FOCs for bonds and shares, we get the fundamental pricing equation:

$$p_t = \frac{d_{t+1} + p_{t+1}}{1+r}$$

Denote with S the entire stock of firm shares. Then, total dividend $D = d \cdot S$ and total market value of the firm $V = p \cdot S$. The fundamental pricing equation can be rewritten as:

$$V_t = p_t S = \frac{d_{t+1} S + p_{t+1} S}{1+r} = \frac{D_{t+1} + V_{t+1}}{1+r} = \frac{D_{t+1} + \frac{D_{t+2} + V_{t+2}}{1+r}}{1+r} = \frac{D_{t+1}}{1+r} + \frac{D_{t+2} + V_{t+2}}{(1+r)^2} = \ldots$$

By iterating the above formula ad infinitum one concludes that the fundamental value of the firm can be expressed as the PDV sum of the future dividend flows:[1]

$$V_t = \sum_{i=1}^{\infty} \frac{D_{t+i}}{(1+r)^i}$$

[1] Provided that the value of the firm does not increase faster than the discount factor: $\lim_{i \to \infty} V_{t+i} / (1+r)^i = 0$.

Firms

There is a single representative firm that converts inputs (capital K and labor L) into output Y according to the neoclassical production function:

$$Y_t = F\left(K_t, L_t\right)$$

The firm buys investment goods to increase its capital stock. Its dividend flow can be expressed as:

$$D_t = F\left(K_t, L_t\right) - w_t L_t - \delta K_t - I_t^n + B_{t+1} - (1+r)\, B_t$$

where δ denotes the rate of depreciation of physical capital during the production process, I^n is net investment, and B is the stock of corporate bonds issued by the firm.

Assume that firm managers want to maximize the current dividend flow and value of the firm, which is consistent with shareholders' preferences. Since the current value of the firm is the PDV sum of future dividend flows, the objective function is then the PDV sum of current and future profit flows:

$$\max \quad (D_t + V_t) = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} \left[F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - \delta K_{t+i} - I_{t+i}^n + B_{t+i+1} - (1+r)\, B_{t+i}\right]$$

subject to $\quad K_{t+1} = K_t + I_t^n$

Lagrangian:

$$\mathcal{L} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} \left[\begin{array}{c} F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - \delta K_{t+i} - I_{t+i}^n + B_{t+i+1} - (1+r)\, B_{t+i} \\ +q_{t+i}\left[K_{t+i} + I_{t+i}^n - K_{t+i+1}\right] \end{array} \right]$$

First order conditions (FOCs):

$$\begin{array}{rcll}
L_t & : & F_L\left(K_t, L_t\right) - w_t = 0 & \rightarrow \quad F_L\left(K_t, L_t\right) = w_t \\
I_t^n & : & -1 + q_t = 0 & \rightarrow \quad q_t = 1 \\
K_{t+1} & : & -q_t + \dfrac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) - \delta + q_{t+1}\right] = 0 & \rightarrow \quad q_t = \dfrac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) - \delta + q_{t+1}\right] \\
B_{t+1} & : & 1 + \dfrac{1}{1+r}\left[-(1+r)\right] = 0 & \rightarrow \quad 1 = \dfrac{1+r}{1+r}
\end{array}$$

where F_L and F_K denote the derivatives of the production function w.r.t. employment and capital.

The FOC for employment says that in optimum the firm should equate the marginal product of labor with the market wage. The FOC for net investment says that the Lagrange multiplier q is always equal to 1, which can be interpreted that the cost of a marginal unit of investment is exactly equal to the PDV sum of additional future profits generated by this extra investment. Since $q_t = q_{t+1} = 1$, we can rewrite the FOC for capital as:

$$F_K\left(K_{t+1}, L_{t+1}\right) = r + \delta$$

which says that the future marginal product of capital has to equal its acquisition cost (interest rate) plus the depreciation rate.

The FOC for bonds is satisfied always, independently from the level of B. That means that any amount of firm debt is consistent with profit-maximizing behavior (i.e. leverage does not matter) and the firm can finance investment equally well either through debt or through retained earnings, because the internal and external cost of capital are equal. This is a version of the **Modigliani-Miller (1958) theorem**.

If we assume for simplicity full employment and constant population ($L_t = L_{t+1} = N$), the desired capital stock can be expressed as a function of the interest rate, $K_{t+1}^* = K(r)$ and net investment equals $I_t^n = K_{t+1}^*(r) - K_t$. Thus, if the interest rate increases (decreases), investment decreases (increases).

The simple neoclassical setup has a problem, in that it predicts very large (infinite in the continuous time version of the model) fluctuations of investment in response to a change in interest rates. To amend this, we turn to the q theory of investment.

1.2 Capital adjustment costs: q theory of investment

Previously we have assumed that the only cost related with new capital is its acquisition cost. Here we will assume that there is an adjustment cost, which will be proportional to the net investment/capital stock ratio:

$$D_t = F\left(K_t, L_t\right) - w_t L_t - \delta K_t - I_t^n \left(1 + \frac{\chi}{2}\frac{I_t^n}{K_t}\right) + B_{t+1} - (1+r) B_t$$

Lagrangian:

$$\mathcal{L} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} \left[\begin{array}{c} F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - \delta K_{t+i} - \left(I_{t+i}^n + \frac{\chi}{2}\frac{\left(I_{t+i}^n\right)^2}{K_{t+i}}\right) + B_{t+i+1} - (1+r) B_{t+i} \\ + q_{t+i}\left[K_{t+i} + I_{t+i}^n - K_{t+i+1}\right] \end{array} \right]$$

First order conditions:

$$\begin{aligned}
L_t &: \quad F_L\left(K_t, L_t\right) - w_t = 0 \\
I_t^n &: \quad -\left(1 + \chi \frac{I_t^n}{K_t}\right) + q_t = 0 \\
K_{t+1} &: \quad -q_t + \frac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) - \delta + \frac{\chi}{2}\left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2 + q_{t+1}\right] = 0 \\
B_{t+1} &: \quad 1 + \frac{1}{1+r}\left[-(1+r)\right] = 0
\end{aligned}$$

Since the FOCs for employment and bonds are the same as previously, we can focus solely on the changes in FOCs for investment and capital stock. We can see now that q is not always equal to 1 but is related to net investment/capital ratio. We can express q as the PDV sum of future marginal products of capital net of depreciation and gains from increased capital stock which makes future investment less costly:

$$q_t = \sum_{i=1}^{\infty} \frac{1}{(1+r)^i}\left[F_K\left(K_{t+i}, L_{t+i}\right) - \delta + \frac{\chi}{2}\left(\frac{I_{t+i}^n}{K_{t+i}}\right)^2\right]$$

The system is now dynamic in the sense that decisions in the current period will leave lasting effects on the future (unlike in the neoclassical example). We want to find the steady state of the system and describe its dynamics. To do that, we shall find the expressions for $\Delta K_{t+1} = K_{t+1} - K_t$ and $\Delta q_{t+1} = q_{t+1} - q_t$. For simplicity, assume $L_t = L_{t+1} = N$. Summarizing our knowledge thus far, we have:

$$\begin{aligned}
q_t &= 1 + \chi \frac{I_t^n}{K_t} \\
K_{t+1} &= K_t + I_t^n \\
q_t &= \frac{1}{1+r}\left[F_K\left(K_{t+1}, N\right) - \delta + \frac{\chi}{2}\left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2 + q_{t+1}\right]
\end{aligned}$$

We can rewrite the above conditions in the difference equations form:

$$\begin{aligned}
q_t &= 1 + \chi \frac{I_t^n}{K_t} \\
\Delta K_{t+1} &= I_t^n \\
\Delta q_{t+1} &= r q_t - \left[F_K\left(K_{t+1}, N\right) - \delta + \frac{\chi}{2}\left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2\right]
\end{aligned}$$

And get rid of I_t^n and K_{t+1} by substitutions:

$$\begin{aligned}
I_t^n &= \frac{q_t - 1}{\chi} K_t \\
K_{t+1} &= K_t\left(1 + \frac{q_t - 1}{\chi}\right)
\end{aligned}$$

Then:

$$\Delta K_{t+1} = \frac{q_t - 1}{\chi} K_t$$

$$\Delta q_{t+1} = r q_t - \left[F_K \left(K_t \left(1 + \frac{q_t - 1}{\chi} \right), N \right) - \delta + \frac{\chi}{2} \left(\frac{q_{t+1} - 1}{\chi} \right)^2 \right]$$

The steady state of the system is found by dropping the time subscripts and setting ΔK and Δq to 0:

$$\Delta K = 0 \quad \rightarrow \quad \frac{q-1}{\chi} K = 0 \quad \rightarrow \quad q^* = 1$$

$$\Delta q = 0 \quad \rightarrow \quad rq = F_K \left(K \left(1 + \frac{q-1}{\chi} \right), N \right) - \delta + \frac{\chi}{2} \left(\frac{q-1}{\chi} \right)^2 \quad \underbrace{\rightarrow}_{q = q^* = 1} \quad F_K \left(K^*, N \right) = r + \delta$$

The steady state replicates the neoclassical case.

Now let us analyze the dynamics of the system. Since the equation for Δq_{t+1} is non-linear, it will be much easier to consider its linear approximation around the steady state:

$$\begin{aligned}
\Delta q_{t+1} &\approx \left[\frac{\partial \left(\Delta q_{t+1} \right)}{\partial q_t} \right]^* (q_t - q^*) + \left[\frac{\partial \left(\Delta q_{t+1} \right)}{\partial K_t} \right]^* (K_t - K^*) \\
&\approx \left[r - F_{KK} \left(K^* \left(1 + \frac{q^* - 1}{\chi} \right), N \right) \cdot \frac{K^*}{\chi} + \delta - \frac{\chi}{2} \left(\frac{q^* - 1}{\chi} \right)^2 \right] (q_t - q^*) \\
&\quad + \left[-F_{KK} \left(K^* \left(1 + \frac{q^* - 1}{\chi} \right), N \right) \cdot \left(1 + \frac{q^* - 1}{\chi} \right) \right] (K_t - K^*) \\
&\approx \left[r + \delta - F_{KK} \left(K^*, N \right) \cdot \frac{K^*}{\chi} \right] (q_t - 1) + \left[-F_{KK} \left(K^*, N \right) \right] (K_t - K^*) \\
&\approx A_q \left(q_t - 1 \right) + A_K \left(K_t - K^* \right)
\end{aligned}$$

Both "steady-state constants" A_q and A_K are positive, because $F_{KK} < 0$.

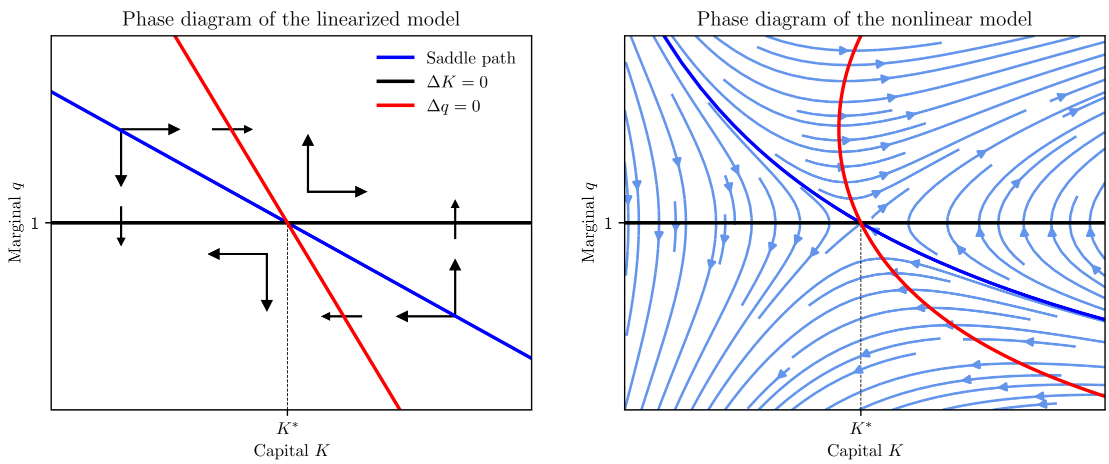
Now we will find functions in the (K, q) space corresponding to $\Delta K_{t+1} = 0$ and $\Delta q_{t+1} = 0$:

$$\Delta K_{t+1} = 0 \quad \rightarrow \quad q = 1$$

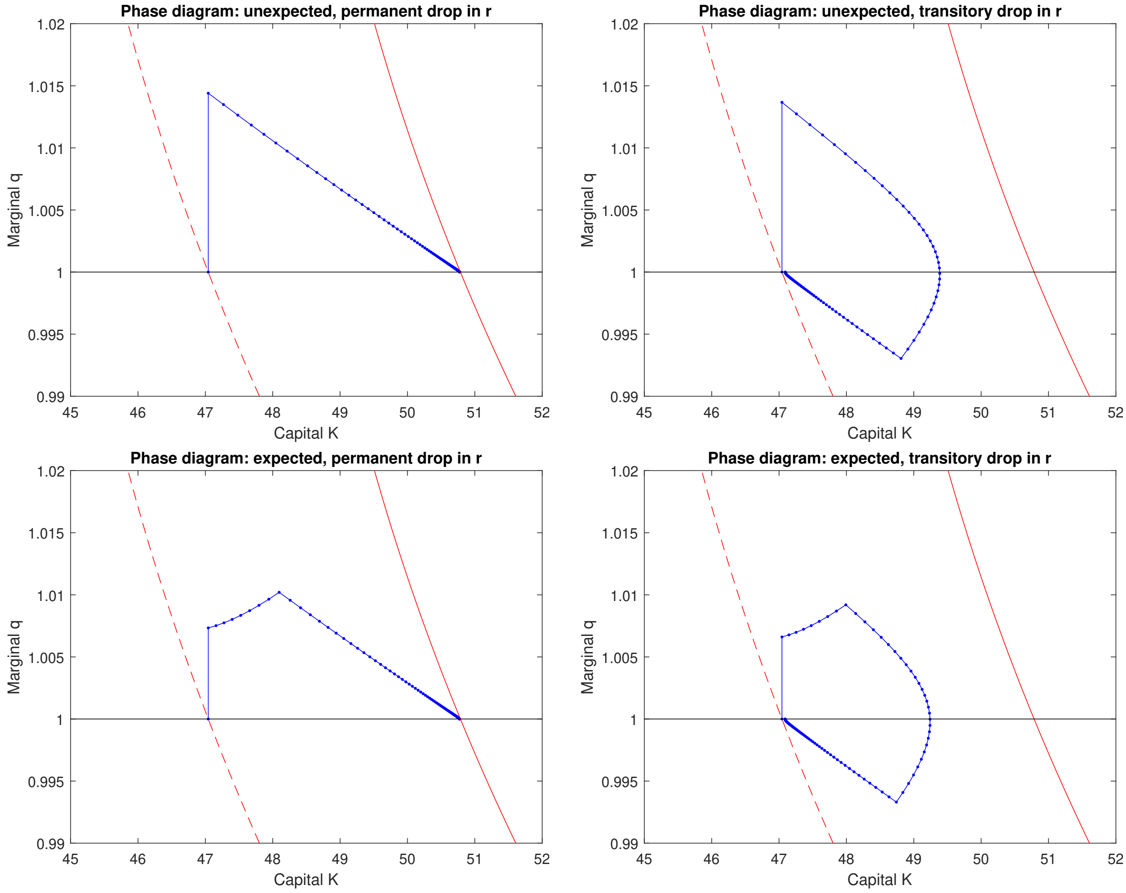
$$\Delta q_{t+1} = 0 \quad \rightarrow \quad A_q \left(q_t - 1 \right) + A_K \left(K_t - K^* \right) = 0 \quad \rightarrow \quad q = 1 - \frac{A_q}{A_K} \left(K_t - K^* \right)$$

The $\Delta q_{t+1} = 0$ function is downward sloping in the (K, q) space (at least close to the steady state).

We can summarize our considerations on a phase diagram:



We can also analyze the dynamic effects of changes in the interest rate r:



1.3 Tobin's (1969) q and the stock market value

From the point of view of the firm manager, since q_t is the shadow price of capital installed at the end of period t in the firm, a natural approach to value the firm is $V_t = q_t K_{t+1}$. From the outsiders' standpoint, q is unobservable. But we have some expectations about future dividend flows, reflected in the stock-market valuation of the firm. It turns out that we can get useful information from the stock market value. Let us start with the expression for q_t:

$$q_t = \frac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) - \delta + \frac{\chi}{2}\left(\frac{I_{t+1}^n}{K_{t+1}}\right)^2 + q_{t+1}\right] \quad \mid \quad \cdot K_{t+1} = K_{t+2} - I_{t+1}^n$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) K_{t+1} - \delta K_{t+1} + \frac{\chi}{2}\frac{\left(I_{t+1}^n\right)^2}{K_{t+1}} + q_{t+1}\left(K_{t+2} - I_{t+1}^n\right)\right]$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) K_{t+1} - \delta K_{t+1} + \frac{\chi}{2}\frac{\left(I_{t+1}^n\right)^2}{K_{t+1}} - I_{t+1}^n\left(1 + \chi\frac{I_{t+1}^n}{K_{t+1}}\right) + q_{t+1}K_{t+2}\right]$$

$$q_t K_{t+1} = \frac{1}{1+r}\left[F_K\left(K_{t+1}, L_{t+1}\right) K_{t+1} - \delta K_{t+1} - I_{t+1}^n\left(1 + \frac{\chi}{2}\frac{I_{t+1}^n}{K_{t+1}}\right) + q_{t+1}K_{t+2}\right]$$

$$q_t K_{t+1} = \sum_{i=1}^{\infty}\frac{1}{(1+r)^i}\left[F_K\left(K_{t+i}, L_{t+i}\right) K_{t+i} - \delta K_{t+i} - I_{t+i}^n\left(1 + \frac{\chi}{2}\frac{I_{t+i}^n}{K_{t+i}}\right)\right]$$

Recall that:

$$V_t = \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} \left[F(K_{t+i}, L_{t+i}) - w_{t+i}L_{t+i} - \delta K_{t+i} - I_{t+i}^n \left(1 + \frac{\chi}{2} \frac{I_{t+i}^n}{K_{t+i}}\right) + B_{t+i+1} - (1+r) B_{t+i} \right]$$

We can without loss of generality ignore B and focus on the case where the firm never issues bonds (recall the Modigliani-Miller theorem). We will also make use of the properties of neoclassical production functions and of the FOC for employment:

$$F(K,L) = F_K(K,L) \cdot K + F_L(K,L) \cdot L \quad \rightarrow \quad F_K(K,L) \cdot K = F(K,L) - wL$$

Then we can write:

$$q_t K_{t+1} = \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} \left[F_K(K_{t+i}, L_{t+i}) K_{t+i} - \delta K_{t+i} - I_{t+i}^n \left(1 + \frac{\chi}{2} \frac{I_{t+i}^n}{K_{t+i}}\right) \right]$$

So it would appear that we could extract q_t using the formula: $q_t = V_t / K_{t+1}$. Is that true?

1.4 Hayashi's (1982) theorem

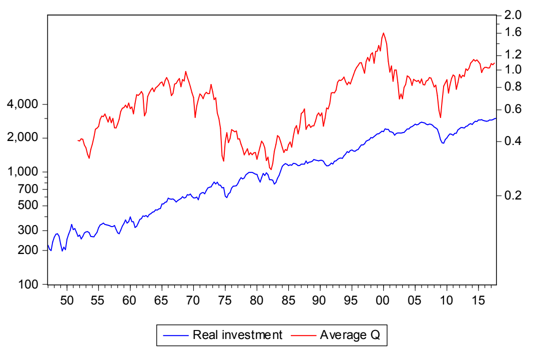
In general, V_t/K_{t+1} is the average Q, not marginal q. However, under certain (very restrictive) assumptions those two concepts coincide:

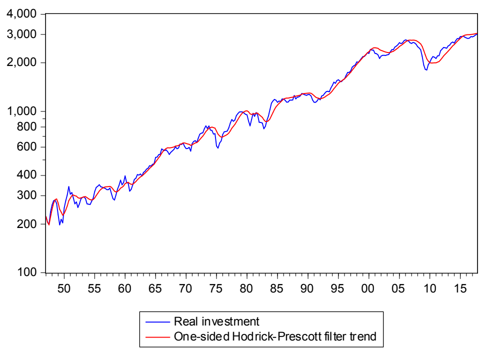
1. Production function and total adjustment cost function exhibit constant returns to scale.
2. Capital goods are homogeneous.
3. Stock market is efficient (uses fundamental pricing).

If these assumptions are satisfied, the firm should invest whenever $V/K > 1$ and disinvest when $V/K < 1$.

1.5 Using average Q to forecast real aggregate investment

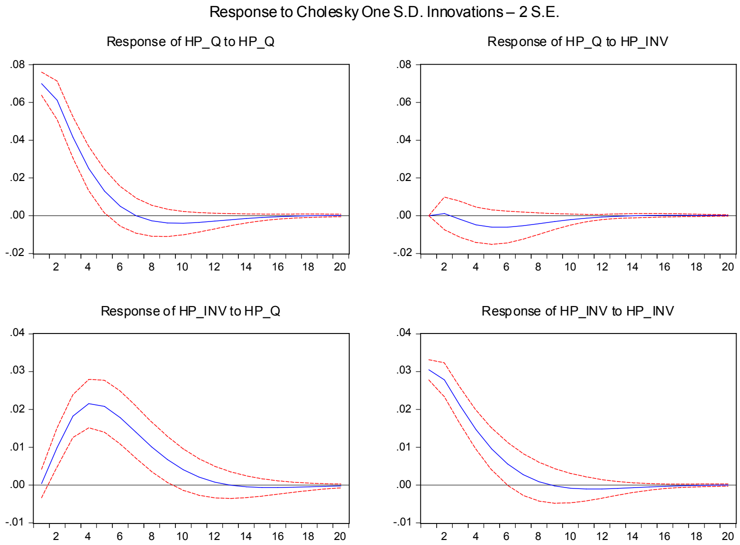
Even if the assumptions of the Hayashi's theorem are not satisfied, average Q does provide information that can be used in forecasting investment. The following graph shows the relevant time series for the US at quarterly frequency.[2] As can be easily seen, real investment is increasing over time and average Q is usually far from 1. However, one can notice that significant drops in Q usually translate to drops in investment a few quarters later. To perform formal analysis, I first detrend (the logarithms of) both time series using the one-sided **Hodrick-Prescott (1981) filter**. This filter can be thought of as a special moving average filter that isolates deviations from trend at business cycle frequency:





[2] **US real investment data**. Average Q is approximated by the ratio between **nonfinancial corporate business equities** and **net worth**.

Next I set up a simple VAR model on HP-deviations from trend of investment and Q. Using a variety of lag selection criteria I decide to include two lags. Then I estimate the VAR model and produce the Impulse Response Functions plot, seen on the right. The HP-deviations of investment and average Q exhibit significant autocorrelation. While the shocks to investment do not translate to the changes in Q, shocks to Q indeed translate to changes in investment, influencing it over the horizon of up to 3 years, and having the strongest impact for 4-6 quarters after the initial shock to Q.



Finally, I use my VAR model to produce one quarter ahead forecasts for HP-deviations of investment, which I combine with the previously isolated HP trend. Basing on the data up to 2017Q3, I forecasted that US investment in 2017Q4 would be equal to 3009 billions of 2009 dollars, with the standard deviation of forecast error of 3%. The actual 2017Q4 real investment was equal to 3002 billions of 2009 dollars and the actual forecast error was smaller than 0.3%.