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1 Dynamic consumption choice

1.1 Two-period optimization

An agent solves the following problem:

$$\begin{aligned}
\max_{c_1,c_2,a} \quad & U = \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 + a = y_1 \\
& c_2 = y_2 + (1+r)\, a \\
& \beta \in [0,1]\,, \text{ given} \\
& y_1, y_2 \geq 0, \text{ given} \\
& r \geq -1, \text{ given} \\
& c_1, c_2 > 0
\end{aligned}$$

where c_1 and c_2 denote agent's consumption in the first and second period, y_1 and y_2 denote agent's income in the first and second period, β is the **discount factor**, and a denotes wealth / net assets (possibly negative) held by the agent at the end of period 1 (at the very beginning of period 2), which yield real rate of interest r.

First construct the lifetime budget constraint:

$$a = \frac{c_2 - y_2}{1+r}$$

$$\begin{aligned}
c_1 + \frac{c_2 - y_2}{1+r} &= y_1 \\
c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r}
\end{aligned}$$

The lifetime budget constraint says that the present discounted value (PDV) of income (plus initial wealth which is here assumed to be 0) has to equal the PDV of consumption. Rewrite the budget constraint so that one side of the equation equals 0:

$$y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0$$

Set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

where λ is the Lagrange multiplier with the following economic interpretation: by how much would the agent's utility increase if the budget constraint was marginally relaxed, i.e. the agent had marginally higher PDV of income ($\partial U / \partial y_1$).

The Lagrangian has two choice variables: c_1 and c_2. Therefore, we will need to calculate two first order conditions.

First order conditions (FOCs):

$$\begin{aligned}
c_1 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda\,[-1] = 0 &&\rightarrow \quad \lambda = \frac{1}{c_1} \\
c_2 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda \left[-\frac{1}{1+r} \right] = 0 &&\rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2}
\end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta(1+r)c_1$$

An equation linking in optimum consumption in different time periods is called the intertemporal (from Latin: "between time (periods)") condition or the **Euler equation**.

The Euler equation can be also obtained in the following way. Suppose that the agent has found the optimal solution to the problem. She now considers consuming x units less in the first period so that she can consume $(1+r)x$ units more in the second period. Her utility level is now given by:

$$U = \ln(c_1 - x) + \beta\ln(c_2 + (1+r)x)$$

If the solution was truly optimal, the derivative of the utility function with respect to x, evaluated at $x = 0$, should be equal to 0:

$$\frac{\partial U}{\partial x} = \frac{1}{c_1 - x}\cdot(-1) + \beta\frac{1}{c_2 + (1+r)x}\cdot(1+r)$$
$$\frac{\partial U}{\partial x}|_{x=0} = -\frac{1}{c_1} + \beta(1+r)\frac{1}{c_2} = 0 \quad \rightarrow \quad c_2 = \beta(1+r)c_1$$

Plug the optimality condition into the budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \frac{\beta(1+r)c_1}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \beta c_1 = y_1 + \frac{y_2}{1+r}$$

Finally, we can obtain the optimal levels of consumption and assets at the end of the first period:

$$c_1 = \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right]$$
$$c_2 = \frac{\beta}{1+\beta}\left[(1+r)y_1 + y_2\right]$$
$$a = y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] = \frac{\beta}{1+\beta}y_1 - \frac{1}{1+\beta}\frac{y_2}{1+r}$$

1.1.1 Comparative statics

Let us take a look at how changes in the parameters of our problem affect the agent's choices.

Changes in β
The higher is β, the more patient is our agent. Intuitively, since the agent with higher β cares more about the future, she would save more in the first period (and thus consume less in the first period) in order to consume more in the second period:

$$\frac{\partial c_1}{\partial \beta} = -\frac{1}{(1+\beta)^2}\left[y_1 + \frac{y_2}{1+r}\right] < 0$$
$$\frac{\partial c_2}{\partial \beta} = \frac{1+\beta-\beta}{(1+\beta)^2}\left[(1+r)y_1 + y_2\right] > 0$$
$$\frac{\partial a}{\partial \beta} = \frac{1+\beta-\beta}{(1+\beta)^2}y_1 - \left[-\frac{1}{(1+\beta)^2}\right]\frac{y_2}{1+r} > 0$$

Changes in y_1
An increase in first period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional first period income to second period consumption, savings will also increase:

$$\frac{\partial c_1}{\partial y_1} = \frac{1}{1+\beta} > 0$$
$$\frac{\partial c_2}{\partial y_1} = \frac{\beta}{1+\beta}(1+r) > 0$$
$$\frac{\partial a}{\partial y_1} = \frac{\beta}{1+\beta} > 0$$

Changes in y_2
An increase in second period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional second period income to first period consumption, savings will decrease:

$$\frac{\partial c_1}{\partial y_2} = \frac{1}{1+\beta}\frac{1}{1+r} > 0$$
$$\frac{\partial c_2}{\partial y_2} = \frac{\beta}{1+\beta} > 0$$
$$\frac{\partial a}{\partial y_2} = -\frac{1}{1+\beta}\frac{1}{1+r} < 0$$

Changes in r
An increase in the interest rate generates two effects: substitution effect and income effect. Substitution effect is generated by the change in relative prices of consumption in the first period versus consumption in the second period. It induces the agent to consume more in the second period and consume less in the first period. Income effect is generated by the fact that the PDV of income changes due to the changes in the discounting rate. The sign of the effect depends on whether an agent is a saver or a borrower. An increase in interest rates creates a positive income effect for the saver, increasing her consumption in both periods. Conversely, an increase in interest rates creates a negative income effect for the borrower, decreasing her consumption in both periods.

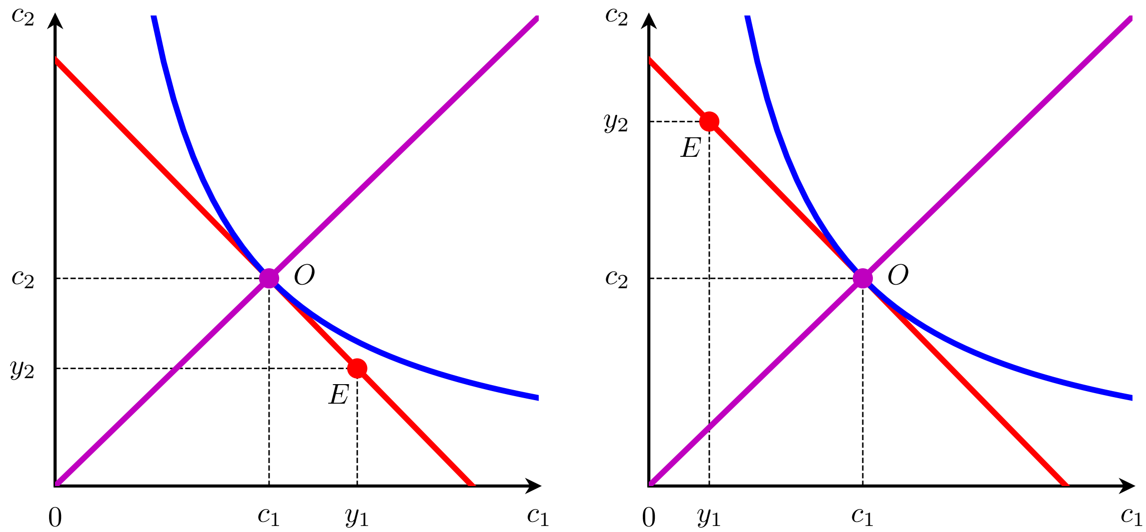
These effects can be summarized by the following table, where pluses and minuses describe the direction of change in a variable for which we can be certain of the direction, regardless of the assumed utility function:

Effects of an increase in r	Saver			Borrower		
	c_1	c_2	a	c_1	c_2	a
Income	+	+	-	-	-	+
Substitution	-	+	+	-	+	+
Net	?	+	?	-	?	+

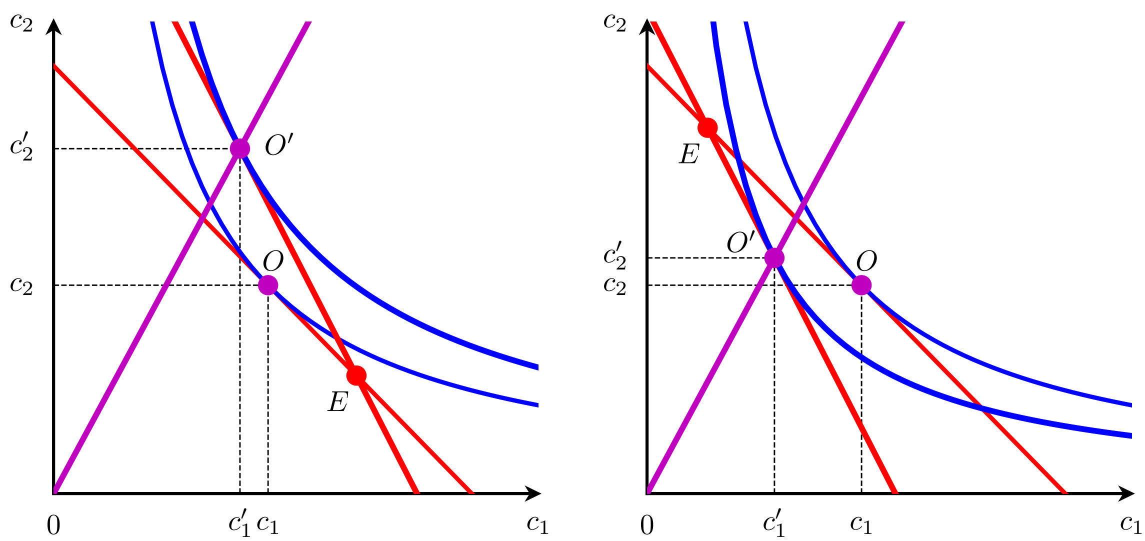
There are however also changes in variables marked with question marks, where the direction of change in a variable depends on the relative strength of substitution and income effects. In the case of a saver, if the income effect is stronger (weaker) than the substitution effect, first period consumption increases (decreases) and savings decrease (increase). In the case of a borrower, if the income effect is stronger (weaker) than the substitution effect, second period consumption decreases (increases).

Saver **Borrower**

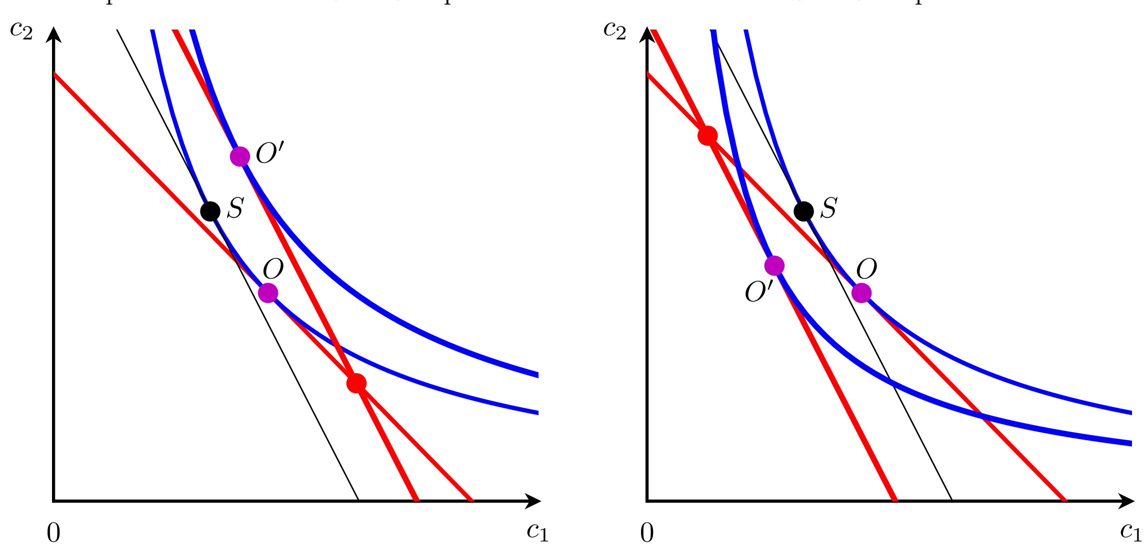
Optimal choice is labeled O. E denotes the initial endowment point.



Reaction to an increase in r. New optimal point is labeled O'.



Decomposition of effects. $O \to S$ is pure substitution effect while $S \to O'$ is pure income effect

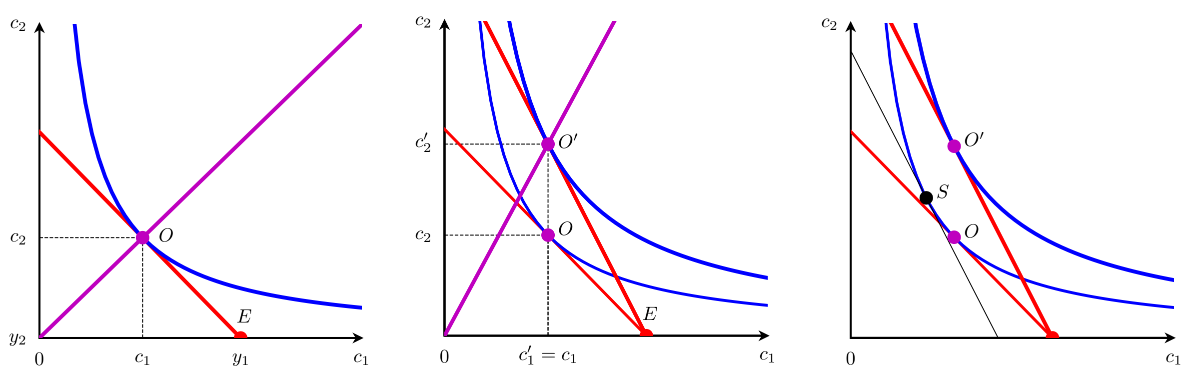


The relative strength of the two effects depends on the shape of the indifference curves. For our agent:

$$\frac{\partial c_1}{\partial r} = \frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] < 0$$
$$\frac{\partial c_2}{\partial r} = \frac{\beta}{1+\beta} y_1 > 0$$
$$\frac{\partial a}{\partial r} = -\frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] > 0$$

In response to the increase in the interest rate, first period consumption decreases (if only $y_2 > 0$), second period consumption increases (if only $y_1 > 0$) and savings increase (if only $y_2 > 0$). The negative relationship between the level of interest rates and consumption is a usual assumption in macroeconomics, and is present in the undergraduate macroeconomic models such as IS-LM and AS-AD.

The case of logarithmic utility and $y_2 = 0$ is a very special case where both first period consumption and savings stay constant while second period consumption increases in response to the increase in the interest rate. The income and substitution effects have exactly equal strength and cancel each other out.



1.2 Inequality constraints: non-borrowing constraint

An agent solves the same problem as before, but now the agent cannot borrow and so $a \geq 0$:

$$\begin{aligned} \max_{c_1, c_2, a} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r)\,a \\ & a \geq 0 \end{aligned}$$

Note: mathematical convention requires that any inequality constraint is written as “bigger than” or “bigger or equal than” so that our multiplier has the correct sign for interpretation purposes.

We will consider now two approaches to solve this problem. In the first approach the level of assets at the end of the first period a will be eliminated and substituted with $y_1 - c_1$, similar to what we have done previously. In the second approach a will remain explicit and will be an additional choice variable over which the agent will optimize.

1.2.1 Approach I: a is eliminated from the constraints

Obtain the lifetime budget constraint:

$$c_1 + a = y_1$$
$$c_2 = y_2 + (1+r)\,a$$
$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

Non-borrowing constraint:

$$a \geq 0 \quad \rightarrow \quad y_1 - c_1 \geq 0$$

Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right] + \mu \left[y_1 - c_1 \right]$$

The Lagrangian has two choice variables: c_1 and c_2, with a given implicitly by $y_1 - c_1$. Therefore, we will need to calculate two first order conditions. Additionally, since one of the constraints is an inequality constraint, we will add a **complementary slackness** constraint that says that either $y_1 - c_1 = 0$ and the constraint is binding ($\mu \geq 0$), or $y_1 \geq c_1$ and the constraint is not binding ($\mu = 0$).

First order conditions (FOCs):

$$c_1 \;:\; \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda\,[-1] + \mu\,[-1] = 0 \quad \rightarrow \quad \lambda = \frac{1}{c_1} - \mu$$
$$c_2 \;:\; \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda \left[-\frac{1}{1+r} \right] = 0 \quad \rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2}$$
$$\text{CS} \;:\; \mu\,[y_1 - c_1] = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad y_1 - c_1 \geq 0$$

Because the non-borrowing constraint might not be binding, we have to consider the following two cases:

Case 1: constraint not binding, $\mu = 0$, $y_1 \geq c_1$:
Resulting optimality condition:

$$\frac{1}{c_1} = \beta\,(1+r)\,\frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta\,(1+r)\,c_1$$

Plug the optimality condition into the budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \frac{\beta\,(1+r)\,c_1}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \beta c_1 = y_1 + \frac{y_2}{1+r}$$

$$c_1 = \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right]$$
$$c_2 = \frac{\beta}{1+\beta} \left[(1+r)\,y_1 + y_2 \right]$$
$$a = y_1 - \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right] = \frac{\beta}{1+\beta} y_1 - \frac{1}{1+\beta} \frac{y_2}{1+r}$$

We should now check whether $a \geq 0$ (or, equivalently, if $c_1 \leq y_1$). If yes, then this is the optimal solution. Otherwise the optimal solution lies in case 2.

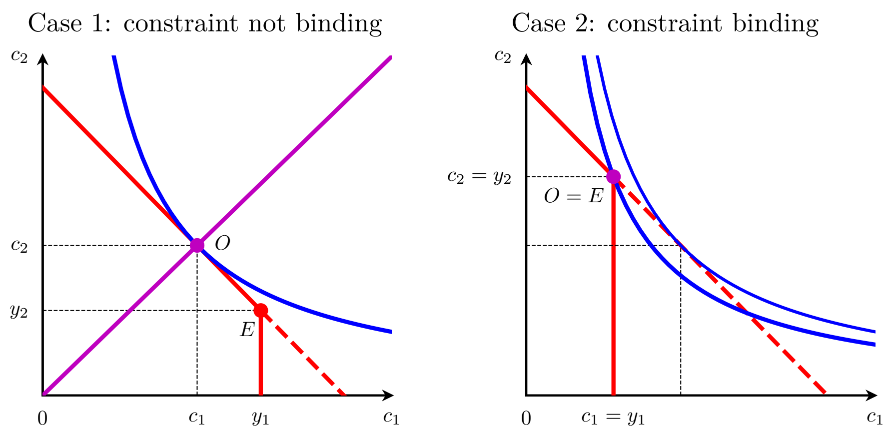
Case 2: constraint binding, $\mu \geq 0$, $y_1 = c_1$:
The solution is now simpler:

$$c_1 = y_1$$
$$a = y_1 - c_1 = 0$$
$$c_2 = y_2 + (1+r)\,a = y_2$$

What remains to find is the value of μ:

$$\mu = \frac{1}{c_1} - \lambda = \frac{1}{c_1} - \beta\,(1+r)\,\frac{1}{c_2} = \frac{1}{y_1} - \beta\,(1+r)\,\frac{1}{y_2}$$

If the agent is indeed constrained, then the multiplier μ should be positive. It has the following interpretation: by how much would the utility of an agent improve if we allowed this agent to borrow a marginal amount. In other words, the higher is μ the more utility is "lost" because of the non-borrowing condition. It is easy to show that μ increases with y_2 and decreases with y_1. Intuitively, the higher is y_2 relative to y_1, the more is the agent induced to decrease savings, which eventually would become negative. At this point, however, the non-borrowing constraint kicks in and the agents "suffers".



1.2.2 Approach II: a remains explicit

Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda_1\,[y_1 - c_1 - a] + \lambda_2\,[y_2 + (1+r)\,a - c_2] + \mu\,[a]$$

The Lagrangian has three choice variables: c_1, c_2 and a. Therefore, we will need to calculate three first order conditions. Additionally, since one of the constraints is an inequality constraint, we will add a complementary slackness (CS) constraint that says that either $a = 0$ and the constraint is binding ($\mu \geq 0$), or $a \geq 0$ and the constraint is not binding ($\mu = 0$).

First order conditions (FOCs):

$$\begin{aligned}
c_1 &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda_1\,[-1] = 0 && \rightarrow \quad \lambda_1 = \frac{1}{c_1} \\
c_2 &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda_2\,[-1] = 0 && \rightarrow \quad \lambda_2 = \frac{\beta}{c_2} \\
a &: \quad \frac{\partial \mathcal{L}}{\partial a} = \lambda_1\,[-1] + \lambda_2\,[1+r] + \mu = 0 && \rightarrow \quad \lambda_1 = \lambda_2\,(1+r) + \mu \\
\text{CS} &: \quad \mu a = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad a \geq 0
\end{aligned}$$

Because of the potentially not binding constraint, we have again the following two cases:

Case 1: constraint nonbinding, $\mu = 0$, $a \geq 0$:

$$\lambda_1 = \frac{1}{c_1}$$
$$\lambda_2 = \frac{\beta}{c_2}$$
$$\lambda_1 = \lambda_2 (1+r)$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta (1+r) \frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta (1+r) c_1$$

Plug the optimality condition into the budget constraint:

$$\begin{aligned}
c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r} \\
c_1 + \frac{\beta (1+r) c_1}{1+r} &= y_1 + \frac{y_2}{1+r} \\
c_1 (1+\beta) &= y_1 + \frac{y_2}{1+r} \\
c_1 &= \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right] \\
c_2 &= \frac{\beta}{1+\beta} \left[(1+r) y_1 + y_2 \right] \\
a &= \frac{\beta}{1+\beta} y_1 - \frac{1}{1+\beta} \frac{y_2}{1+r}
\end{aligned}$$

We should now check whether $a > 0$. If yes, then this is the optimal solution. Otherwise the optimal solution lies in case 2.

Case 2: constraint binding, $\mu \geq 0$, $a = 0$:

The solution is now simpler:

$$\begin{aligned}
a &= 0 \\
c_1 &= y_1 - a = y_1 \\
c_2 &= y_2 + (1+r) a = y_2
\end{aligned}$$

What remains to find is the value of μ:

$$\begin{aligned}
\mu &= \lambda_1 - \lambda_2 (1+r) \\
\mu &= \frac{1}{c_1} - \frac{\beta}{c_2} (1+r) \\
\mu &= \frac{1}{y_1} - \frac{\beta}{y_2} (1+r)
\end{aligned}$$

If the agent is indeed constrained, then the multiplier μ should be positive.

1.3 Choice under uncertainty

Imagine now that second period variables (future income and interest rate) are uncertain and the agent wants to maximize her **expected utility**, given information available to her in the first period:

$$\begin{aligned} \max \quad & U = E_1 \left[u(c_1) + \beta \cdot u(c_2) \right] \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1 + r_2) a \end{aligned}$$

In this example, instead of having a functional form for the utility function, we assume a generic within-period utility function u, satisfying the standard assumptions that $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c \to 0} u'(c) = \infty$.

Lagrangian:

$$\mathcal{L} = E_1 \left[u(c_1) + \beta \cdot u(c_2) + \lambda_1 \left[y_1 - c_1 - a \right] + \lambda_2 \left[y_2 + (1 + r_2) a - c_2 \right] \right]$$

First order conditions:

$$\begin{aligned} c_1 &: \quad E_1 \left[u'(c_1) - \lambda_1 \right] = 0 \\ c_2 &: \quad E_1 \left[\beta \cdot u'(c_2) - \lambda_2 \right] = 0 \\ a &: \quad E_1 \left[-\lambda_1 + \lambda_2 \cdot (1 + r_2) \right] = 0 \end{aligned}$$

Simplify and rewrite (note that first period variables are known with certainty):

$$\begin{aligned} \lambda_1 &= u'(c_1) \\ E_1 \left[\lambda_2 \right] &= E_1 \left[\beta \cdot u'(c_2) \right] \\ \lambda_1 &= E_1 \left[\lambda_2 \cdot (1 + r_2) \right] \end{aligned}$$

Resulting optimality condition:

$$u'(c_1) = E_1 \left[\beta \cdot u'(c_2) \cdot (1 + r_2) \right]$$

Note that, unlike from before, we need to be extra careful not to break any expectation operators. Let us rewrite the Euler equation in the following way:

$$1 = E_1 \left[\beta \frac{u'(c_2)}{u'(c_1)} \cdot (1 + r_2) \right]$$

The above equation can be also interpreted as an asset pricing equation. In this particular example, the price of a unit of savings is one unit of first period consumption. The payoff from holding an asset in the second period will be $(1 + r_2)$ per unit of savings. The term $\beta \cdot u'(c_2) / u'(c_1)$ is called the **stochastic discount factor** and measures the relative marginal utility of consumption across periods.

A general form of consumption-based asset equation can be expressed as $p_t = E_t \left[m_{t+1} \cdot x_{t+1} \right]$, where p_t denotes asset price, m_{t+1} is the stochastic discount factor and x_{t+1} is the future asset payoff. This equation is the basis of John Cochrane's book **Asset Pricing**, which covers a variety of cases:

	Price p_t	Payoff x_{t+1}
Stock	p_t	$p_{t+1} + d_{t+1}$
Return	1	R_{t+1}
Price-dividend ratio	$\frac{p_t}{d_t}$	$\left(\frac{p_{t+1}}{d_{t+1}} + 1 \right) \frac{d_{t+1}}{d_t}$
Excess return	0	$R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$
Managed portfolio	z_t	$z_t R_{t+1}$
Moment condition	$E \left[p_t z_t \right]$	$x_{t+1} z_t$
One-period bond	p_t	1
Risk-free rate	1	R^f
Option	C	$\max \left\{ S_T - K, 0 \right\}$

1.4 Ricardian equivalence

One of many "irrelevance results" in economics, that can be demonstrated using the two-period framework, is the **Ricardian equivalence**. It posits that a decrease in taxation will not result in an increase in private consumption because households know that the resulting debt will have to be repaid by the taxpayers (themselves!) in the next period through higher taxes. Therefore, households instead of consuming more increase their savings to be able to cover expected higher taxes in the future.

Government sector
The government's budget constraints are:

$$
\begin{aligned}
g_1 &= \tau_1 + b \\
g_2 + (1+r)\, b &= \tau_2
\end{aligned}
$$

where g_1 and g_2 denote government spending in the first and second period, b are government bonds issued in the first period that have to be bought back in the second period, and τ_1 and τ_2 denote lump-sum taxes levied on the households in the first and second period.

Here we impose that the government cannot be indebted in the last period, which seems restrictive, but when generalized to the case of longer time horizons it simply posits that the government debt does not explode to infinity and the government does not go bankrupt.

Households
Households solve the following utility maximization problem:

$$
\begin{aligned}
\max_{c_1, c_2, a} \quad & \ln c_1 + \beta \ln c_2 \\
\text{subject to} \quad & c_1 + a = (y_1 - \tau_1) \\
& c_2 = (y_2 - \tau_2) + (1+r)\, a
\end{aligned}
$$

Private savings a are comprised of government bonds b and other assets $\bar{a}$ that pay the same interest rate as bonds. The expression $y_1 - \tau_1$, or income after taxation, is often called the disposable or after-tax income.

Lifetime budget constraint:

$$
c_1 + \frac{c_2}{1+r} = y_1 - \tau_1 + \frac{y_2 - \tau_2}{1+r}
$$

Lagrangian:

$$
\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 - \tau_1 + \frac{y_2 - \tau_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]
$$

FOCs:

$$
\begin{aligned}
c_1 \quad &: \quad \frac{1}{c_1} - \lambda = 0 \qquad \rightarrow \quad \lambda = \frac{1}{c_1} \\
c_2 \quad &: \quad \frac{\beta}{c_2} - \frac{\lambda}{1+r} = 0 \quad \rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2}
\end{aligned}
$$

Euler equation:

$$
c_2 = \beta\,(1+r)\, c_1
$$

In this setup, government bonds and other assets are perfect substitutes, so from the perspective of the households their relative mix is indeterminate. We can pin down the level of savings in other assets by using the households' and government budget constraints in the first time period:

$$\begin{aligned} c_1 + b + \bar{a} &= y_1 - \tau_1 \\ b &= g_1 - \tau_1 \\ c_1 + g_1 - \tau_1 + \bar{a} &= y_1 - \tau_1 \\ \bar{a} &= y_1 - g_1 - c_1 \end{aligned}$$

We can also use the same procedure to rewrite the households' budget constraint in the second time period:

$$\begin{aligned} c_2 &= y_2 - \tau_2 + (1+r)\, b + (1+r)\, \bar{a} \\ b &= \frac{\tau_2 - g_2}{1+r} \\ c_2 &= y_2 - \tau_2 + \tau_2 - g_2 + (1+r)\, \bar{a} \\ c_2 &= y_2 - g_2 + (1+r)\, \bar{a} \end{aligned}$$

Now let us join the budget constraints:

$$\begin{aligned} c_2 &= y_2 - g_2 + (1+r)\,(y_1 - g_1 - c_1) \\ c_1 + \frac{c_2}{1+r} &= y_1 - g_1 + \frac{y_2 - g_2}{1+r} \end{aligned}$$

And plug in the Euler equation:

$$\begin{aligned} c_1 + \frac{\beta\,(1+r)\, c_1}{1+r} &= y_1 - g_1 + \frac{y_2 - g_2}{1+r} \\ (1+\beta)\, c_1 &= y_1 - g_1 + \frac{y_2 - g_2}{1+r} \end{aligned}$$

Finally we can obtain the optimal levels of consumption:

$$\begin{aligned} c_1 &= \frac{1}{1+\beta} \left[y_1 - g_1 + \frac{y_2 - g_2}{1+r} \right] \\ c_2 &= \frac{\beta}{1+\beta} \left[(1+r)\,(y_1 - g_1) + y_2 - g_2 \right] \end{aligned}$$

The levels of asset holdings in the economy are pinned down as follows: government bonds are determined by $b = g_1 - \tau_1$ and other assets by $\bar{a} = y_1 - g_1 - c_1$.

Maybe surprisingly, neither the level of government bonds issuance nor the level of taxation influence the optimal consumption levels and savings in assets other than government bonds. What does influence them, though, is the present discounted value of government expenditures.

1.4.1 Breaking the Ricardian equivalence

Distortionary taxation

All taxes, except lump-sum taxes considered above, create distortions by changing the economic incentives. For example, if income was not exogenous, but determined by the labor supply, and taxes were levied on labor income, then a drop in income taxation in the first time period would result in more labor supplied in the first time period and less labor supplied in the second period, when taxes would be higher. It would affect the PDV of income and also the consumption-labor incentives. Thus, we should then expect changes in consumption levels in response to changes in the bond-taxes mix.

Treating government debt as net wealth

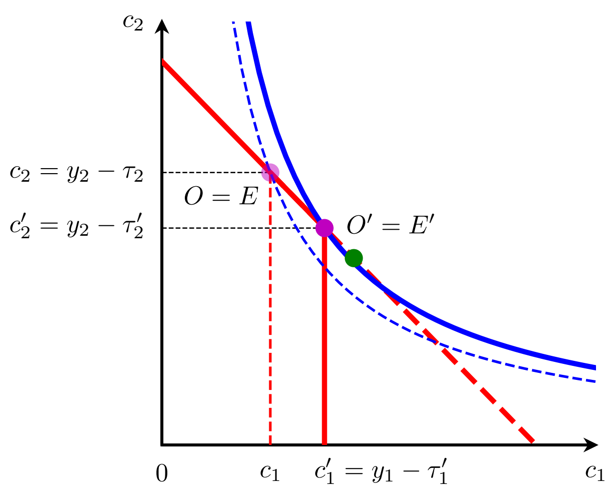
If people treat government bonds as assets, they may think that they are "wealthy" and decide to spend more. However, the government bonds also represent future liabilities, that is, in the future they will have to be repaid with higher taxes. In our above example, the same agent is taxed to repay the government bonds she owns. Thus treating government debt as net wealth can be interpreted as a form of myopic behavior.

Myopy (shortsightedness) or (quasi-)hyperbolic discounting

The above model assumed that people care not only about their current, but also future consumption, and realize the overall structure of the problem they are facing. However, if people do not fully realize how the economy and the intertemporal government budget constraints work (**myopia**), or they discount the near future inconsistently with discounting the far future (**hyperbolic discounting**), then changes in current disposable income will influence consumption.

Incomplete markets (non-borrowing constraints)

If some of the households cannot borrow, they might not be able to reach their optimal consumption point. A drop in taxes in the first period boosts their disposable income and makes the nonborrowing constraint less painful, or even nonbinding. In the example below, a reduction in taxes in the first period, from τ_1 to τ_1', generates an increase in first period consumption, from c_1 to c_1', with $\Delta c_1 = -\Delta\tau_1$. Note however, that if the tax reduction would push the first period disposable income beyond the green point, the household would no longer be borrowing constrained, and further reductions in taxes would have no impact on consumption.



Overlapping generations

In our considerations we have assumed that the households' and government's planning horizons are identical. However, if people have shorter planning horizon than the government, they may expect that the higher taxes will affect the economy after they die and thus a current tax break will boost the PDV of their income.

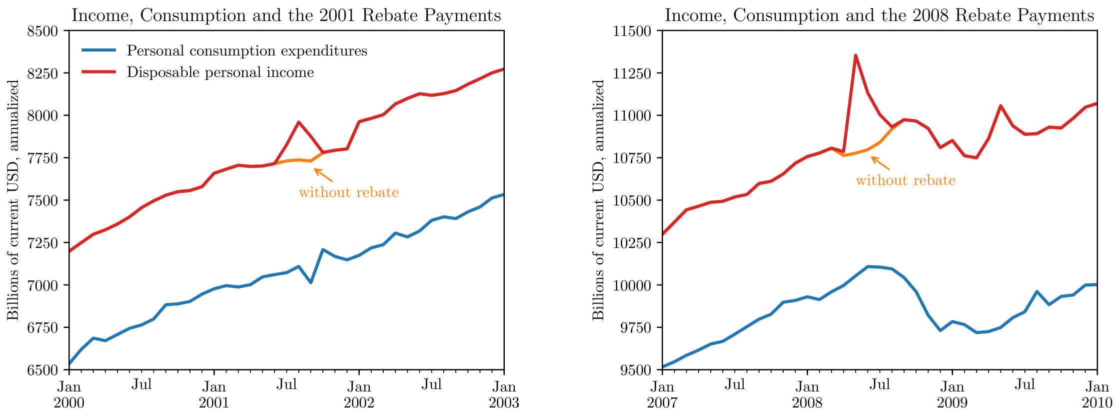
Spending the 2008 Rebate, by Age

Age group	Percent mostly spending
29 or less	11.7
30–39	14.2
40–49	16.9
50–64	19.9
65 or over	28.4

Source: **Shapiro and Slemrod (2009)** Did the 2008 Tax Rebates Stimulate Spending?, *American Economic Review*, Vol. 99(2), pp. 374–379.

1.5 Consumption behavior: empirical studies

1.5.1 Transitory changes in taxes



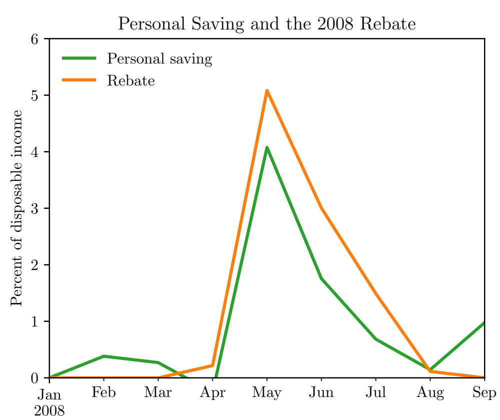
Source: **Taylor (2009)**, US Bureau of Economic Analysis.

PCE Regressions with Rebate Payments

Lagged PCE	0.794	0.832
	(0.057)	(0.056)
Rebate payments	0.048	0.081
	(0.055)	(0.054)
Disposable personal income (w/o rebate)	0.206	0.188
	(0.056)	(0.055)
Oil price ($/bbl lagged 3 months)		-1.007
		(0.325)
R^2	0.999	0.999

Notes: The dependent variable is personal consumption expenditures. Standard errors are reported in parentheses. The oil price is for West Texas Intermediate. The sample period is January 2000 to October 2008.

Source: **Taylor (2009)** The Lack of an Empirical Rationale for a Revival of Discretionary Fiscal Policy, *American Economic Review*, Vol. 99(2), pp. 550–555.



Source: **Shapiro and Slemrod (2009)**, US Bureau of Economic Analysis.

Responses to 2001 and 2008 Rebate Surveys

	2001		2008	
	Number	Percent	Number	Percent
Mostly spend	256	21.8	447	19.9
Mostly save	376	32.0	715	31.8
Mostly pay off debt	544	46.2	1083	48.2
Will not get rebate	223		212	
Don't know / refused	45		61	
Total	1444	100	2518	100

Source: **Shapiro and Slemrod (2009)** and **Shapiro and Slemrod (2003)** Consumer Response to Tax Rebates, *American Economic Review*, Vol. 93(1), pp. 381–396.

1.5.2 Transitory and permanent changes in income

The **life-cycle** / **permanent income** hypothesis of consumer behavior that we have developed makes a number of predictions that we can test in the data. Some of them are:

1. Consumption is forward-looking, depending not just on current income but future income as well.
2. Consumption reacts more to permanent changes in income than transitory changes in income.
3. Current consumption does not react much to predictable, anticipated changes in income.

The results from the literature are mixed, implying that at least a significant fraction of households does not behave as the theory predicts, even when augmented with liquidity constraints.

In one of the first empirical tests of aggregate consumption behavior, **Hall (1978)** postulates that under some additional assumptions the consumption should behave as a random walk process:

$$u'(c_{t-1}) = E_{t-1}\left[\beta u'(c_t)(1+r_t)\right] \quad + \quad \text{additional assumptions} \quad \longrightarrow \quad c_t = c_{t-1} + \varepsilon_t$$

and indeed, this null hypothesis is not rejected. Obviously, what is most interesting is whether some macroeconomic variables have predictive power for the error term ε_t. If the theory is correct, no variable x known in time periods $t-1$ and $t-2$ could be able to predict consumption growth between periods $t-1$ and t, and in the following equation the coefficient α should be indistinguishable from 0:

$$\Delta c_t = \alpha \Delta x_{t-1} + e_t$$

The null hypothesis cannot be rejected for several considered variables, including changes in disposable income. However in some cases, most notably in case of stock prices growth, the null was rejected.

Further research has shown that there are two apparent deviations from our theory: on one hand, consumption exhibits *excess sensitivity* to changes in current income, and on the other, *excess smoothness* to changes in permanent income. This means that consumption reacts stronger to changes in current income than theory predicts, and also weaker to changes in permanent income, which actually is quite volatile. **Flavin (1981)** assumes that changes in income are to some extent autocorrelated, where ε_t is the unpredictable part, and consumption is allowed to react to this innovation:

$$\Delta y_t = \mu + \lambda \Delta y_{t-1} + \varepsilon_t$$
$$\Delta c_t = \beta \Delta y_t + \theta \varepsilon_t + \nu_t$$

If $\beta > 0$, then consumption overreacts to changes in current income (excess sensitivity), and that is what she finds ($\beta \approx 0.36$). On the other hand, **Campbell and Deaton (1989)** find on the basis of a VAR model that consumption exhibits excess smoothness as it depends very strongly on its past values and does not respond much to the estimated permanent income process.

A possible explanation of the above departures from theory is that there are two groups of households: those behaving according to theory, and others being liquidity constrained / mypoic, which could potentially account for both excess sensitivity and smoothness in aggregate data. **Hall and Mishkin (1982)** find that in their sample of 2000 households about 80% behave as if they followed the LC/PIH behavior, and the remaining 20% behave as if they were borrowing constrained or consumed a fixed fraction of their income.

Campbell and Mankiw (1989) test this idea against aggregate data and estimate the following equation:

$$\Delta c_t = \lambda \Delta y_t + (1 - \lambda)\, e_t \equiv \lambda z_t + \nu_t$$

where λ is a fraction of borrowing-constrained households. Since z_t and ν_t are almost surely correlated, estimates of λ are biased upwards. To aleviate this they employ instrumental variables estimation and find that, depending on the specification, $\lambda = 0.42 \pm 0.16$ or $\lambda = 0.52 \pm 0.13$.

Going back to disaggregate data, **Shea (1995)** constructs a sample of of 647 observations, where the union contract provides clear information about the households' future earnings, and regresses consumption growth on this measure of expected wage growth. Our theory predicts that the coefficient should be 0, but the estimated coefficient is in fact 0.89 ± 0.46, a quantitatively large (though only marginally statistically significant) departure from the random-walk prediction. Interestingly, this cannot be explained by borrowing constraints, as the group with positive liquid wealth behaves essentially identically to the low liquid assets group. Moreover, the low-wealth sample is split conditional on the direction of expected wage growth. It turns out that households react asymmetrically in response to expected changes in income, which is inconsistent with "spend-a-fixed-fraction" of income type of myopic behavior, and is also inconsistent with the behavior of borrowing-constrained households, but consistent with behavior characterized by **loss aversion**.

A group of papers also finds departures from our theory in response to predictable government transfers and wage changes. **Parker (1999)** finds that households increase their consumption after they reach the annual cap in social security contributions and their disposable income increases. Similar behavior in response to income tax refunds is found by **Souleles (1999)**. **Shapiro and Slemrod (2003)** point out that apart from transitory tax rebates, the 2001 tax reforms also implied positive changes in permanent income, to which households did not react. On the other hand and using higher quality data, **Johnson et al. (2006)** find that households have spent their 2001 rebate transfers over the course of few months, and they were especially stimulative for low income and low wealth households. More recently, **Evans and Moore (2012)** find that mortality rates decline before the first day of the month and spike after the first, which suggests that the mortality cycle is linked to short-term variation in levels of economic activity as a result of households being liquidity constrained by the end of the month. **Stephens (2003)** makes similar observations regarding the behavior of social security recipients.

In some cases though our theory is corroborated. **Paxson (1993)** shows that Thai households who experience large seasonal variations in their income are able to smooth their consumption over the year. **Browning and Collado (2001)** document the fact that some Spanish workers receive additional bonuses in June and December, but their consumption behavior is indistinguishable from behavior of workers who do not receive bonuses. **Hsieh (2003)** investigates the effects of predictable dividend payments from the **Alaska Permanent Fund** and finds that households behave consistently with PIH, including those households that overreact to income tax refunds. In Hsieh's words, "These two pieces of evidence suggest that bounded rationality, rather than the lack of desire to smooth the marginal utility of consumption, is the source of rejections of the LC/PIH. For households to incorporate anticipated income changes into their chosen consumption paths, these income changes must be large and transparent, and the costs associated with the mental processing of these forecastable income changes must be small relative to the utility gains from consumption smoothing."