Marcin Bielecki, Advanced Macroeconomics QF, Fall 2018
Homework 1 – deadline: 31st October, 11.30 AM

Problem 1

Consider the following two-period utility maximization problem. This utility function belongs to the CRRA (Constant Relative Risk Aversion) class of functions which can be thought of as generalized logarithmic functions. An agent lives for two periods and in both receives some positive income.

$$\begin{aligned} \max_{c_t, c_{t+1}, a_{t+1}} \quad & U = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & c_t + a_{t+1} = y_t \\ & c_{t+1} = y_{t+1} + (1+r)\, a_{t+1} \end{aligned}$$

where $\sigma \geq 0$,[1] $\beta \in [0, 1]$ and $r \geq -1$.

(a) Rewrite the budget constraints into a single lifetime budget constraint and set up the Lagrangian.

(b) Obtain the first order conditions for c_t and c_{t+1}. Express c_{t+1} as a function of c_t.

(c) Using the lifetime budget constraint obtain the formulas for optimal c_t and c_{t+1}.

(d) Set $\sigma = 1$ and verify that the formulas for optimal c_t and c_{t+1} are identical to the ones we obtained in class for the utility function $U = \ln c_t + \beta \ln c_{t+1}$.

(e) Return to expressions obtained in (c). Assume now that $y_{t+1} = 0$. How does c_t react when interest rate r increases? How does it depend on σ? How does σ impact the relative strength of income and substitution effects?

Problem 2

Consider the following two-period model with production:

$$\begin{aligned} \max_{c_t, c_{t+1}, b_{t+1}, k_{t+1}} \quad & U = \ln c_t + \beta \ln c_{t+1} \\ \text{subject to} \quad & c_t + b_{t+1} + k_{t+1} = y_t \\ & c_{t+1} = 2 \cdot k_{t+1}^{1/2} + (1+r)\, b_{t+1} + (1-\delta)\, k_{t+1} \end{aligned}$$

where b_{t+1} denotes bonds, k_{t+1} denotes physical capital invested in the household's firm and $\delta \in [0, 1]$ stands for capital depreciation rate.

(a) Write down the problem in the form of a Lagrangian.

(b) Find the optimal value of k_{t+1}.

(c) Find the optimal values of c_t, c_{t+1} and b_{t+1}.

(d) Calculate the derivative of optimal k_{t+1} with respect to r. Provide intuition for this result.

(e) In equilibrium the rental rate for capital r^k is equal to the marginal product of capital $\partial y_{t+1} / \partial k_{t+1}$. Show that $r^k - \delta = r$.

[1] For $\sigma = 1$ the CRRA function becomes logarithmic: $U = \ln c_t + \beta \ln c_{t+1}$. This can be easily proven by using the L'Hôpital's rule to compute the following limit: $\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma}$.

Problem 3

In this problem we will show how changes in taxation can change consumption of an agent if she is borrowing-constrained. The agent's income in the first period is 1/3 of her income in the second period. Assume that $\beta(1+r) = 1$ and consider the following utility maximization problem:

$$\begin{aligned} \max_{c_t, c_{t+1}, a_{t+1}} \quad & U = \ln c_t + \beta \ln c_{t+1} \\ \text{subject to} \quad & c_t + a_{t+1} = y/3 \\ & c_{t+1} = y + (1+r)\, a_{t+1} \end{aligned}$$

(a) Using the Lagrangian method find optimal c_t, c_{t+1} and a_{t+1}. Are savings positive or negative?

(b) Assume now that the agent cannot borrow and faces an additional non-borrowing constraint: $a_{t+1} \geq 0$. Using the Lagrangian method find optimal c_t, c_{t+1} and a_{t+1}.

(c) Show graphically in the (c_t, c_{t+1}) space the problem of the agent and especially show that the agent would be on a higher indifference curve were she allowed to borrow.

(d) Suppose that the government arranges a transfer v to this agent by issuing bonds. In the future, the government will tax the agent to be able to buy back the bonds. The new constraints of the agent are:

$$\begin{aligned} & c_t + a_{t+1} = y/3 + v \\ & c_{t+1} = y + (1+r)\, a_{t+1} - (1+r)\, v \\ & a_{t+1} \geq 0 \end{aligned}$$

What is the impact of the government transfer on the agent's first period consumption?

(e) Show graphically in the (c_t, c_{t+1}) space the effect of the transfer scheme from (d).

(f) Suppose that the transfer is large enough so that the agent's savings become positive. What would be the impact of even bigger transfers on first period consumption?