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In the exogenous growth models, ongoing increases in technology are necessary to sustain long-run growth. However, in these models technological progress is assumed, and not explained within the model (that's why they are named exogenous growth models). In this lecture we will encounter models where long-run growth arises endogenously within the model, and thus are potentially more attractive to economists, as they allow to ask questions on which factors can potentially affect the rate of growth of economy in the long run.

1 AK model

One way to generate the possibility of sustained long-run growth is to eliminate the diminishing returns to capital stock. One way to do it is to assume that the production function is linear in capital, hence the name AK. In this model the concept of capital is broad, as it encompasses human capital, knowledge, public infrastructure, and so on. In the next section we will encounter a model with human capital that makes this interpretation explicit. For now, let us work with the simplified setup.

Households

As usual, households want to maximize their utility, subject to the budget constraint:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma}-1}{1-\sigma} \,\mathrm{d}t \\ \text{subject to} \quad & \dot{a}_t = w_t + (r_t - n)\, a_t - c_t \end{aligned}$$

Hamiltonian:

$$\mathcal{H} = e^{-(\rho-n)t} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \lambda_t \left[w_t + (r_t - n)\, a_t - c_t \right]$$

where λ_t is the co-state variable associated with the state variable a.

First order conditions:

$$\begin{aligned} c_t \quad &: \quad e^{-(\rho-n)t} c_t^{-\sigma} - \lambda_t = 0 \qquad \rightarrow \qquad \lambda_t = e^{-(\rho-n)t} c_t^{-\sigma} \\ a_t \quad &: \quad \lambda_t (r_t - n) = -\dot{\lambda}_t \qquad\qquad \rightarrow \qquad -\frac{\dot{\lambda}_t}{\lambda_t} = r_t - n \end{aligned}$$

Now transform the first order condition for optimal consumption:

$$\begin{aligned} \lambda_t &= e^{-(\rho-n)t} c_t^{-\sigma} \quad \Big| \quad \ln(\cdot) \\ \ln \lambda_t &= -(\rho - n)\, t - \sigma \ln c_t \quad \Big| \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ \frac{\dot{\lambda}_t}{\lambda_t} &= -(\rho-n) - \sigma \frac{\dot{c}_t}{c_t} \\ \frac{\dot{c}_t}{c_t} &= \frac{-\dot{\lambda}_t/\lambda_t - \rho + n}{\sigma} \end{aligned}$$

Resulting Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - n - \rho + n}{\sigma} = \frac{r_t - \rho}{\sigma}$$

Firms

The production function is now assumed to be of AK form, and the profit maximization problem of the firms is stated as:

$$\begin{aligned} \max \quad & Y_t - (r_t + \delta) K_t \\ \text{subject to} \quad & Y_t = AK_t \end{aligned}$$

Rewrite:

$$\max \quad AK_t - (r_t + \delta) K_t$$

First order condition:

$$K_t \quad : \quad A - (r_t + \delta) = 0 \quad \rightarrow \quad r_t = A - \delta$$

Note that since the production function is linear in capital, the real interest rate is a constant, independent on the level of capital stock. Also, since raw labor is useless in production process, wages are equal to 0.

General Equilibrium

Again, since the economy is closed and there is no government, $a = k$. Prices are given by $r = A - \delta$ and $w = 0$.

We can transform the budget constraint into the resource constraint / capital accumulation equation:

$$\begin{aligned} \dot{a} &= w + (r - n) a - c \\ \dot{k} &= w + (r - n) k - c \\ \dot{k} &= (A - \delta - n) k - c \end{aligned}$$

And we can plug in the interest rate into the Euler equation:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{r - \rho}{\sigma} \\ g_c = \frac{\dot{c}}{c} &= \frac{A - \delta - \rho}{\sigma} \end{aligned}$$

As long as $A > \delta + \rho$, the growth rate of consumption per capita is positive and constant. The capital can continue accumulating forever, without diminishing returns:

$$g_k = \frac{\dot{k}}{k} = (A - \delta - n) - \frac{c}{k}$$

Along the Balanced Growth Path (BGP) the growth rate of capital per capita is assumed to be constant. This requires the c/k ratio to be constant as well and thus $g_c = g_k = g$. The model has a closed-form solution, and there is no transitional dynamics, as the economy is always at the BGP:

$$g = \frac{A - \delta - \rho}{\sigma}$$

$$g = (A - \delta - n) - \frac{c}{k} \quad \rightarrow \quad c = (A - \delta - n - g) k$$

Note that the model generates strong predictions about the determinants of the growth rate. For example, a decrease in households' impatience ρ or their risk aversion σ permanently raises the economy's growth rate.

1.1 Human capital in a one sector economy

Assume now that the production requires the use of physical and human capital:

$$Y = AK^{\alpha}H^{1-\alpha}$$

Both the physical and human capital accumulate through investment:

$$\begin{aligned}\dot{K} &= I_K - \delta K \\ \dot{H} &= I_H - \delta H\end{aligned}$$

For simplicity, from now on we will assume that population is constant, so maximizing aggregate consumption is equivalent to maximizing consumption per capita:

$$\begin{aligned}\max \quad & U = \int_0^{\infty} e^{-\rho t}\,\frac{C^{1-\sigma}-1}{1-\sigma}\,\mathrm{d}t \\ \text{subject to} \quad & AK^{\alpha}H^{1-\alpha} = C + I_K + I_H \\ & \dot{K} = I_K - \delta K \\ & \dot{H} = I_H - \delta H\end{aligned}$$

Hamiltonian:

$$\begin{aligned}\mathcal{H} = e^{-\rho t}\,&\frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\left[AK^{\alpha}H^{1-\alpha} - C - I_K - I_H\right] \\ &+ \mu_K\left[I_K - \delta K\right] + \mu_H\left[I_H - \delta H\right]\end{aligned}$$

First order conditions:

$$\begin{aligned}C \quad &: \quad e^{-\rho t}C^{-\sigma} - \lambda = 0 \quad \longrightarrow \quad \frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma} \\ I_K \quad &: \quad -\lambda + \mu_K = 0 \\ I_H \quad &: \quad -\lambda + \mu_H = 0 \\ K \quad &: \quad \lambda\alpha AK^{\alpha-1}H^{1-\alpha} - \mu_K\delta = -\dot{\mu}_K \\ H \quad &: \quad \lambda\left(1-\alpha\right)AK^{\alpha}H^{-\alpha} - \mu_H\delta = -\dot{\mu}_H\end{aligned}$$

Since $\lambda = \mu_K = \mu_H$, we have:

$$\begin{aligned}\alpha AK^{\alpha-1}H^{1-\alpha} - \delta &= \left(1-\alpha\right)AK^{\alpha}H^{-\alpha} - \delta \\ \frac{K}{H} &= \frac{\alpha}{1-\alpha}\end{aligned}$$

If we assume (for now) that investment in both types of capital can be negative, the ratio of physical to human capital is constant at all times and has the value derived above. We can then rewrite the production function in the AK form:

$$Y = AK^{\alpha}H^{1-\alpha} = A\left(\frac{K}{H}\right)^{\alpha-1}K = A\left(\frac{\alpha}{1-\alpha}\right)^{\alpha-1}K$$

The rate of growth of the economy can be determined by obtaining the rate of growth of λ from e.g. the FOC for physical capital:

$$\begin{aligned}-\frac{\dot{\lambda}}{\lambda} &= \alpha AK^{\alpha-1}H^{1-\alpha} - \delta = \alpha A\left(K/H\right)^{\alpha-1} - \delta \\ g = g_C = \frac{\dot{C}}{C} &= \frac{\alpha A\left(K/H\right)^{\alpha-1} - \delta - \rho}{\sigma} = \frac{A\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha} - \delta - \rho}{\sigma}\end{aligned}$$

1.1.1 Non-negativity constraints on investment

Suppose now that we add a condition that gross investment cannot be negative, so you cannot transform one type of capital to the other after it has been already built: $I_K \geq 0$, $I_H \geq 0$. If an economy has imbalanced quantities of physical or human capital, it has to accumulate the less abundant one, and let the other depreciate over time. Consider now an economy with overabundance of human capital, but low levels of physical capital:

$$\frac{K}{H} < \left(\frac{K}{H}\right)^* = \frac{\alpha}{1-\alpha}$$

Although it seems that a problem with nonnegativity constraints would be mathematically more difficult to tackle, in this case it is actually simpler, as it is optimal to set $I_H = 0$ and let it depreciate until the BGP ratio of K/H is reached:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}-1}{1-\sigma} \, \mathrm{d}t \\ \text{subject to} \quad & AK^\alpha H^{1-\alpha} = C + I_K + 0 \\ & \dot{K} = I_K - \delta K \\ & \dot{H} = 0 - \delta H \end{aligned}$$

The capital accumulation equation can now be rewritten as:

$$\dot{K} = AK^\alpha H^{1-\alpha} - C - \delta K$$

Hamiltonian:

$$\mathcal{H} = e^{-\rho t} \frac{C^{1-\sigma}-1}{1-\sigma} + \lambda \left[AK^\alpha H^{1-\alpha} - C - \delta K\right]$$

First order conditions:

$$\begin{aligned} C \quad &: \quad e^{-\rho t} C^{-\sigma} - \lambda = 0 \quad \rightarrow \quad \frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma} \\ K \quad &: \quad \lambda\left[\alpha A K^{\alpha-1} H^{1-\alpha} - \delta\right] = -\dot{\lambda} \end{aligned}$$

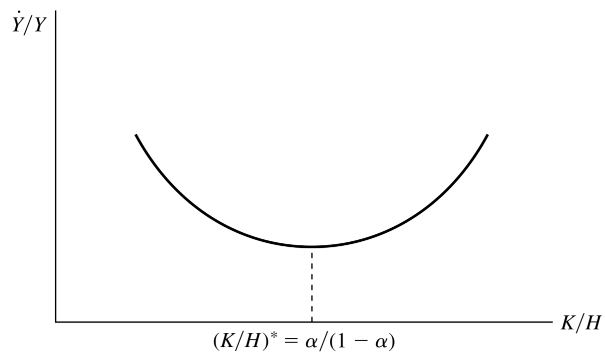
The economy now behaves as a Ramsey-Cass-Koopmans economy, and its growth rate depends on the relative abundance of different capital types:

$$g = g_C = \frac{\dot{C}}{C} = \frac{\alpha A \left(K/H\right)^{\alpha-1} - \delta - \rho}{\sigma} = \frac{\alpha A \left(H/K\right)^{1-\alpha} - \delta - \rho}{\sigma}$$

We can also demonstrate that such economy grows faster than along its BGP:

$$\frac{K}{H} < \left(\frac{K}{H}\right)^* \quad \rightarrow \quad \frac{H}{K} > \left(\frac{H}{K}\right)^* \quad \rightarrow \quad g > g^*$$

This faster-than-BGP growth is possible due to the imbalance effect: while one factor of production slowly depreciates, the other is accumulated faster than along the BGP and both output and consumption increase at a faster rate:



1.2 Externalities in AK models

1.2.1 Learning-by-doing and knowledge spillovers

Consider now an economy with many firms. Each of them hires capital and labor to produce final output. What is now different is that technology level A increases when any firm invests, although individual firms treat A as a number that they cannot influence. To be more specific, the production function of an i-th firm is given by:

$$Y_i = K_i^\alpha \left(AL_i\right)^{1-\alpha}$$

The aggregate capital stock is equal to the sum of capital across firms:

$$K = \sum_i K_i$$

and the level of technology is equal to the aggregate capital stock:

$$A = K$$

Such an economy experiences a positive externality from capital accumulation, but a decentralized, private economy will underinvest and the growth rate will be lower than is socially optimal.

Each firm solves the following profit maximization problem:

$$\max \quad K_i^\alpha \left(AL_i\right)^{1-\alpha} - wL_i - (r+\delta) K_i$$

First order conditions:

$$\begin{aligned} L_i \quad &: \quad (1-\alpha) K_i^\alpha A^{1-\alpha} L_i^{-\alpha} - w = 0 \qquad \rightarrow \qquad w = (1-\alpha) A^{1-\alpha} k_i^\alpha \\ K_i \quad &: \quad \alpha K_i^{\alpha-1} A^{1-\alpha} L_i^{1-\alpha} - (r+\delta) = 0 \quad \rightarrow \quad r = \alpha A^{1-\alpha} k_i^{\alpha-1} - \delta \end{aligned}$$

Since individual firms treat factor prices as given, they all choose the same capital to labor ratio, k. Therefore the aggregate production function can be written as:

$$Y = \sum_i Y_i = \sum_i K_i^\alpha \left(AL_i\right)^{1-\alpha} = A^{1-\alpha} \sum_i \left(\frac{K_i}{L_i}\right)^\alpha L_i = A^{1-\alpha} k^\alpha \sum_i L_i = A^{1-\alpha} k^\alpha L$$

Now we use the assumption that $A = K$ to rewrite the production function in the AK form:

$$Y = K^{1-\alpha} \left(\frac{K}{L}\right)^\alpha L = KL^{1-\alpha}$$

Rewrite the interest rate:

$$r = \alpha K^{1-\alpha} \left(\frac{K}{L}\right)^{\alpha-1} - \delta = \alpha L^{1-\alpha} - \delta$$

Plug in the interest rate into the Euler equation:

$$g = \frac{\dot{C}}{C} = \frac{r-\rho}{\sigma} = \frac{\alpha L^{1-\alpha} - \delta - \rho}{\sigma}$$

Social planner's solution

The social planner maximizes households' welfare given the resource constraints and is aware of the externality which the private sector ignores:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{C^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \\ \text{subject to} \quad & \dot{K} = KL^{1-\alpha} - \delta K - C \end{aligned}$$

Hamiltonian:

$$\mathcal{H} = e^{-\rho t}\,\frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\left[KL^{1-\alpha} - \delta K - C\right]$$

First order conditions:

$$\begin{aligned}
C &: \quad e^{-\rho t}C^{-\sigma} - \lambda = 0 \qquad \rightarrow \quad \frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma} \\
K &: \quad \lambda\left[L^{1-\alpha} - \delta\right] = -\dot{\lambda} \quad \rightarrow \quad -\frac{\dot{\lambda}}{\lambda} = L^{1-\alpha} - \delta
\end{aligned}$$

The rate of growth of the social planner's economy exceeds the rate of growth of the decentralized one:

$$g^{sp} = \left(\frac{\dot{C}}{C}\right)^{sp} = \frac{L^{1-\alpha} - \delta - \rho}{\sigma} > \frac{\alpha L^{1-\alpha} - \delta - \rho}{\sigma} = g^{dec}$$

1.2.2 A Congestion Model

Now we will analyze a case of a negative externality. We will assume that the production function depends positively on government expenditures to GDP ratio:

$$Y = AK \cdot p\left(\frac{G}{Y}\right)$$

To simplify notation, let this G/Y ratio be denoted as ω. The negative externality arises since output of any individual firm lowers this ratio and reduces the economy's efficiency. You may find it convenient to think of this model as a "traffic jam model": the more cars are driving along the same road, the lower can be the average speed due to congestion, hence the name. We will first solve the problem of the social planner, and then demonstrate that the only way the private economy is induced to produce the socially optimal outcome is through income taxation.

Social planner

$$\begin{aligned}
\max \quad & U = \int_0^\infty e^{-\rho t}\,\frac{C^{1-\sigma}-1}{1-\sigma}\,\mathrm{d}t \\
\text{subject to} \quad & \dot{K} = AK \cdot p\,(G/Y) - \delta K - C - G
\end{aligned}$$

The constraint is easier to tackle when the notation change to ω is used:

$$\dot{K} = AK \cdot p\,(\omega) - \delta K - C - \omega Y = (1-\omega)\,AK \cdot p\,(\tau) - \delta K - C$$

Hamiltonian:

$$\mathcal{H} = e^{-\rho t}\,\frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\left[(1-\omega)\,AK \cdot p\,(\omega) - \delta K - C\right]$$

First order conditions:

$$\begin{aligned}
C &: \quad e^{-\rho t}C^{-\sigma} - \lambda = 0 \quad \rightarrow \quad \frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma} \\
\omega &: \quad \lambda\left[-AK \cdot p\,(\omega) + (1-\omega)\,AK \cdot p'\,(\omega)\right] = 0 \\
K &: \quad \lambda\left[(1-\omega)\,A \cdot p\,(\omega) - \delta\right] = -\dot{\lambda}
\end{aligned}$$

The FOC for ω results in the (implicit) condition for optimal share of government expenditures in GDP:

$$p\,(\omega) = (1-\omega) \cdot p'\,(\omega) \quad \rightarrow \quad \omega^* = 1 - \frac{p\,(\omega^*)}{p'\,(\omega^*)}$$

The rate of growth of the economy under social planner's decisions:

$$g^{sp} = \left(\dot{C}/C\right)^{sp} = \frac{(1-\omega^*)\,A \cdot p\,(\omega^*) - \delta - \rho}{\sigma}$$

Decentralized solution

Let's assume that the government has two tools available to finance the public expenditures: lump-sum taxes τ and income taxes τ_y. It will be more convenient to model income taxes as taxes on firm production. Recall that in the RCK model taxes on income decreased capital accumulation and thus generated undesirable side effects. However, in this case, where production generates negative externalities, the presence of income taxes will allow the economy to achieve the socially optimal allocation.

The problem of the households is standard, and results in the familiar Euler equation:

$$\frac{\dot{C}}{C} = \frac{r - \rho}{\sigma}$$

When solving the problem of the firm, we make use of the assumption that firm production is taxed:

$$\max \quad (1 - \tau_y) AK \cdot p(\omega) - (r + \delta) K$$

First order conditions:

$$(1 - \tau_y) A \cdot p(\omega) - (r + \delta) = 0 \quad \rightarrow \quad r = (1 - \tau_y) A \cdot p(\omega) - \delta$$

As expected, the tax on income lowers the equilibrium real interest rate. The decentralized economy grows at a rate:

$$g^{dec} = \left(\dot{C}/C\right)^{dec} = \frac{(1 - \tau_y) A \cdot p(\omega) - \delta - \rho}{\sigma}$$

Consider now the following two cases. If the government expenditures are fully financed by lump-sum taxes, then $\tau_y = 0$ and:

$$g^{dec} = \frac{A \cdot p(\omega) - \delta - \rho}{\sigma} > \frac{(1 - \omega^*) A \cdot p(\omega^*) - \delta - \rho}{\sigma} = g^{sp}$$

In this case the decentralized economy grows faster than is socially optimal! You might think that such a result is weird, as intuitively the faster the economy grows, the better. However here the faster growing economy experiences a negative level effect, and the actual level of output is always lower than is socially optimal.

The other case is when the government expenditures are fully financed by income taxes. Then $\tau_y = \omega$ and the economy grows at a socially optimal growth rate:

$$g^{dec} = \frac{(1 - \omega) A \cdot p(\omega) - \delta - \rho}{\sigma} = \frac{(1 - \omega^*) A \cdot p(\omega^*) - \delta - \rho}{\sigma} = g^{sp}$$

While this setup here might seem very unrealistic to you, it conveys an important message: when certain activities generate negative externalities, it may be socially beneficial to tax them, even by using taxes that in the RCK model were very distortionary.

2 Human capital in a two sector economy (Lucas-Uzawa model)

In this model the economy consists of two sector. The first one produces physical goods. The second is responsible for human capital accumulation. Let u denote the share of human capital employed in the physical goods producing setor, and $(1-u)$ denote the share of human capital employed in the human capital producing setor. The problem can be stated as:

$$\begin{aligned}
\max \quad & \int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}-1}{1-\sigma}\mathrm{d}t \\
\text{subject to} \quad & \dot{K} = AK^\alpha (uH)^{1-\alpha} - \delta K - C \\
& \dot{H} = B(1-u)H - \delta H
\end{aligned}$$

Hamiltonian:

$$\mathcal{H} = e^{-\rho t}\frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\left[AK^\alpha (uH)^{1-\alpha} - \delta K - C\right] + \mu\left[B(1-u)H - \delta H\right]$$

First order conditions:

$$C \;:\; e^{-\rho t}C^{-\sigma} - \lambda = 0 \quad \rightarrow \quad \frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma} \tag{1}$$

$$u \;:\; \lambda\left[(1-\alpha)AK^\alpha u^{-\alpha}H^{1-\alpha}\right] + \mu\left[-BH\right] = 0 \tag{2}$$

$$K \;:\; \lambda\left[\alpha AK^{\alpha-1}(uH)^{1-\alpha} - \delta\right] = -\dot{\lambda} \tag{3}$$

$$H \;:\; \lambda\left[(1-\alpha)AK^\alpha u^{1-\alpha}H^{-\alpha}\right] + \mu\left[B(1-u) - \delta\right] = -\dot{\mu} \tag{4}$$

The idea behind the solution procedure is as follows. To obtain the rate of growth of the economy, we need to know the rate of growth of λ. But it will be much easier to obtain the rate of growth of μ. If we will be able to show that λ and μ grow at the same rates, we will be done.

Consider first (2):

$$\begin{aligned}
\lambda\left[(1-\alpha)AK^\alpha u^{-\alpha}H^{1-\alpha}\right] &= \mu\left[BH\right] \quad \Big| \quad \cdot\frac{u}{H} \\
\lambda\left[(1-\alpha)AK^\alpha u^{1-\alpha}H^{-\alpha}\right] &= \mu\left[Bu\right]
\end{aligned}$$

Then compare with (4):

$$\begin{aligned}
\lambda\left[(1-\alpha)AK^\alpha u^{1-\alpha}H^{-\alpha}\right] + \mu\left[B(1-u) - \delta\right] &= -\dot{\mu} \\
\mu\left[Bu\right] + \mu\left[B(1-u) - \delta\right] &= -\dot{\mu}
\end{aligned}$$

$$-\frac{\dot{\mu}}{\mu} = B - \delta$$

Use (1):

$$\frac{\dot{C}}{C} = \frac{-\dot{\lambda}/\lambda - \rho}{\sigma}$$

If $\dot{\lambda}/\lambda = \dot{\mu}/\mu$ then:

$$g_C = \frac{\dot{C}}{C} = \frac{B - \delta - \rho}{\sigma}$$

To demonstrate that, consider (3):

$$-\frac{\dot{\lambda}}{\lambda} = \alpha A\left(\frac{K}{uH}\right)^{\alpha-1} - \delta$$

$$g_C = \frac{\dot{C}}{C} = \frac{\alpha A\left(\frac{K}{uH}\right)^{\alpha-1} - \delta - \rho}{\sigma}$$

Along the BGP the share of human capital employed in the physical goods producing setor has to be constant:

$$g_u^* = 0$$

As a consequence:

$$\left(\frac{K}{uH}\right)^* = const \quad \rightarrow \quad g_K^* = g_H^*$$

Use now the capital accumulation equation to show that along the BGP all main macroeconomics variables grow at the same rate:

$$\dot{K} = AK^{\alpha}\left(uH\right)^{1-\alpha} - \delta K - C$$
$$g_K = \frac{\dot{K}}{K} = A\left(\frac{K}{uH}\right)^{\alpha-1} - \delta - \frac{C}{K}$$

$$g_K^* = g_C^* = g_H^* = g_Y^*$$

Consider again (2) and prove that along the BGP λ and μ grow at the same rate:

$$\lambda\left[\left(1-\alpha\right)AK^{\alpha}u^{-\alpha}H^{1-\alpha}\right] = \mu\left[BH\right] \quad | \quad \ln\left(\cdot\right) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}$$
$$g_{\lambda} + \alpha g_K + \left(1-\alpha\right)g_H = g_{\mu} + g_H$$
$$g_{\lambda}^* = g_{\mu}^*$$

As a final step, find u^*:

$$\dot{H} = B\left(1-u\right)H - \delta H$$
$$g_H = \frac{\dot{H}}{H} = B\left(1-u\right) - \delta$$

$$1 - u^* = \frac{g^* + \delta}{B}$$
$$u^* = 1 - \frac{g^* + \delta}{B}$$

So at the end we have that the growth rate of the economy is in the long run determined by the efficiency of the human capital accumulation sector:

$$g^* = \frac{B - \delta - \rho}{\sigma}$$