Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019

1 Analysis of taxation in the long run

For simplicity of notation, we will assume that $n = g = 0$ and $N = A = 1$. We will assume that in each period the government's budget is balanced, and consider two uses for the tax revenues: lump-sum transfers to households v and government expenditures per person g.

1.1 Households

Utility maximization problem:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \\ \text{subject to} \quad & \dot{a}_t = (1 - \tau_t^a) \, r_t a_t + (1 - \tau_t^w) \, w_t - (1 + \tau_t^c) \, c_t - \tau_t + v_t \end{aligned}$$

where τ^a is a capital gains tax, τ^w is a labor income tax, τ^c is a consumption tax and τ is a lump-sum tax.

Set up the Hamiltonian:

$$\mathcal{H} = e^{-(\rho-n)t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[(1 - \tau_t^a) \, r_t a_t + (1 - \tau_t^w) \, w_t - (1 + \tau_t^c) \, c_t - \tau_t + v_t \right]$$

First order conditions:

$$\begin{aligned} c_t \quad &: \quad e^{-(\rho-n)t} c_t^{-\sigma} - (1 + \tau_t^c) \, \lambda_t = 0 \quad && \rightarrow \quad (1 + \tau_t^c) \, \lambda_t = e^{-\rho t} c_t^{-\sigma} \\ a_t \quad &: \quad \lambda_t \, (1 - \tau_t^a) \, r_t = -\dot{\lambda}_t \quad && \rightarrow \quad -\frac{\dot{\lambda}_t}{\lambda_t} = (1 - \tau_t^a) \, r_t \end{aligned}$$

Transform the FOC for consumption:

$$\begin{aligned} (1 + \tau_t^c) \, \lambda_t &= e^{-\rho t} c_t^{-\sigma} \quad | \quad \ln(\cdot) \\ \ln(1 + \tau_t^c) + \ln \lambda_t &= -\rho t - \sigma \ln c_t \quad | \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ \frac{\dot{(1 + \tau_t^c)}}{(1 + \tau_t^c)} + \frac{\dot{\lambda}_t}{\lambda_t} &= -\rho - \sigma \frac{\dot{c}_t}{c_t} \\ \frac{\dot{c}_t}{c_t} &= -\frac{\dot{(1 + \tau_t^c)}}{(1 + \tau_t^c)} \frac{1}{\sigma} + \frac{-\dot{\lambda}_t/\lambda_t - \rho}{\sigma} \end{aligned}$$

Euler equation:

$$\frac{\dot{c}_t}{c_t} = -\frac{\dot{(1 + \tau_t^c)}}{(1 + \tau_t^c)} \frac{1}{\sigma} + \frac{(1 - \tau_t^a) \, r_t - \rho}{\sigma}$$

If the taxes are constant over time, the Euler equation simplifies to:

$$\frac{\dot{c}_t}{c_t} = \frac{(1 - \tau^a) \, r_t - \rho}{\sigma}$$

1.2 Firms

Profit maximizing problem:

$$\begin{aligned} \max \quad & \Pi_t = (1 - \tau_t^f) \left[F(K_t, L_t) - \delta K_t - w_t L_t \right] - r_t K_t \\ \max \quad & \Pi_t = (1 - \tau_t^f) L_t \left[f(k_t) - \delta k_t - w_t \right] - r_t L_t k_t \end{aligned}$$

where τ^f is a tax on firms' accounting profits, which in principle can differ from the economc profits. Here we are assuming that capital is owned directly by firms, and while the tax code allows for deducting capital depreciation costs, it does not take into account the opportunity cost of holding capital, rK.

First order conditions:

$$\begin{aligned}
k_t &: \quad (1-\tau_t^f)L_t \left[f'(k_t)-\delta\right]-r_tL_t=0 \qquad &\rightarrow \quad & r_t=(1-\tau_t^f)\left[f'(k_t)-\delta\right]\\
L_t &: \quad (1-\tau_t^f)\left[f(k_t)-\delta k_t-w_t\right]-r_tk_t=0 \qquad &\rightarrow \quad & w_t=f(k_t)-\delta k_t-\frac{r_tk_t}{(1-\tau_t^f)}=f(k_t)-f'(k_t)k_t
\end{aligned}$$

The tax on firms' accounting profits lowers the return on capital and incentivises firms to hold less capital. While the tax does not affect wages directly, it has the effect via changes in the capital per worker, as wages depend positively on the level of capital per worker:

$$\frac{\partial w}{\partial k}=f'(k_t)-\left[f''(k_t)k_t+f'(k_t)\right]=-f''(k_t)k_t>0$$

After-tax profits are still zero (because of price taking behavior and constant returns to scale):

$$\begin{aligned}
&(1-\tau_t^f)\left[f(k_t)-\delta k_t-\left[f(k_t)-f'(k_t)k_t\right]\right]-(1-\tau_t^f)\left[f'(k_t)-\delta\right]k_t=\\
&\qquad\qquad=(1-\tau_t^f)\left[-\delta k_t+f'(k_t)k_t\right]-(1-\tau_t^f)\left[f'(k_t)-\delta\right]k_t=0
\end{aligned}$$

Firm accounting profits tax revenue is equal to:

$$\tau_t^f\left[f(k_t)-\delta k_t-\left[f(k_t)-f'(k_t)k_t\right]\right]=\tau_t^f\left[f'(k_t)-\delta\right]k_t$$

1.3 Government sector

The government maintains a balanced budget. In per person terms:

$$\begin{aligned}
g_t+v_t&=\tau_t^f\left[f(k_t)-\delta k_t-w_t\right]+\tau_t^a r_t a_t+\tau_t^w w_t+\tau_t^c c_t+\tau_t\\
g_t+v_t&=\tau_t^f\left[f'(k_t)-\delta\right]k_t+\tau_t^a r_t a_t+\tau_t^w w_t+\tau_t^c c_t+\tau_t\\
v_t&=\tau_t^f\left[f'(k_t)-\delta\right]k_t+\tau_t^a r_t a_t+\tau_t^w w_t+\tau_t^c c_t+\tau_t-g_t
\end{aligned}$$

1.4 General equilibrium

Market clearing for assets market:

$$k_t=a_t$$

Rewrite households' budget constraint to get the resource constraint / capital accumulation equation:

$$\begin{aligned}
\dot{a}_t&=(1-\tau_t^a)r_ta_t+(1-\tau_t^w)w_t-(1+\tau_t^c)c_t-\tau_t+v_t\\
\dot{k}_t&=(1-\tau_t^a)r_tk_t+(1-\tau_t^w)w_t-(1+\tau_t^c)c_t-\tau_t+v_t\\
\dot{k}_t&=r_tk_t+w_t-c_t\qquad -\tau_t^a r_t k_t-\tau_t^w w_t-\tau_t^c c_t-\tau_t\\
&\quad+\tau_t^f\left[f'(k_t)-\delta\right]k_t+\tau_t^a r_t a_t+\tau_t^w w_t+\tau_t^c c_t+\tau_t-g_t\\
\dot{k}_t&=r_tk_t+w_t-c_t+\tau_t^f\left[f'(k_t)-\delta\right]k_t-g_t\\
\dot{k}_t&=(1-\tau_t^f)\left[f'(k_t)-\delta\right]k_t+w_t-c_t+\tau_t^f\left[f'(k_t)-\delta\right]k_t-g_t\\
\dot{k}_t&=\left[f'(k_t)-\delta\right]k_t+w_t-c_t-g_t\\
\dot{k}_t&=\left[f'(k_t)-\delta\right]k_t+f(k_t)-f'(k_t)k_t-c_t-g_t\\
\dot{k}_t&=f(k_t)-\delta k_t-c_t-g_t
\end{aligned}$$

Rewrite the Euler equation (under assumption that taxes stay constant over time):

$$\begin{aligned}
\frac{\dot{c}_t}{c_t}&=\frac{(1-\tau^a)r_t-\rho}{\sigma}\\
\frac{\dot{c}_t}{c_t}&=\frac{(1-\tau^a)(1-\tau^f)(f'(k_t)-\delta)-\rho}{\sigma}
\end{aligned}$$

1.5 Steady state

Under constant tax rates:

$$\dot{k} = 0 \quad \rightarrow \quad 0 = \frac{(1-\tau^a)(1-\tau^f)(f'(k)-\delta)-\rho}{\sigma}$$

$$(1-\tau^a)(1-\tau^f)(f'(k)-\delta)-\rho = 0$$

$$f'(k^*) = \frac{\rho}{(1-\tau^a)(1-\tau^f)} + \delta$$
$$c^* = f(k^*) - \delta k^* - g$$

Government consumption lowers private consumption but does not affect steady state capital per worker.

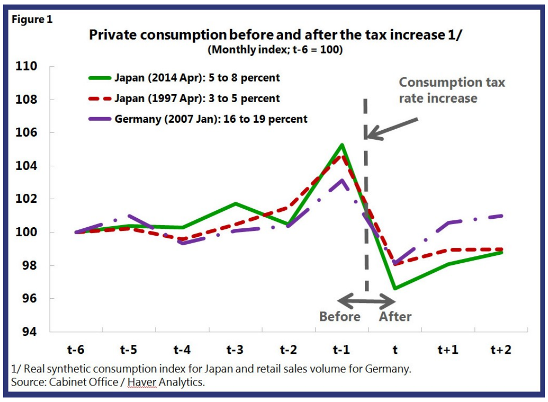
Capital gains and firm earnings taxes lower steady state capital per worker which then translates to a lower steady state private consumption.

1.6 Changes in consumption taxes

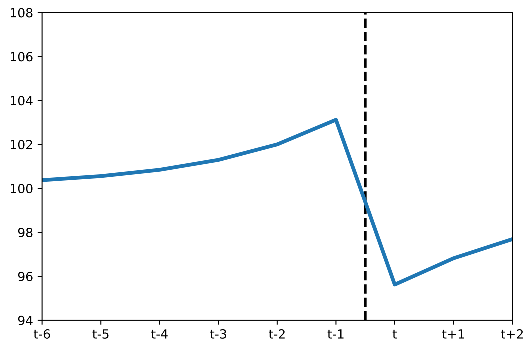
In our analysis above we have for convenience assumed that taxes are constant over time. However, it is interesting to see whether our model can reproduce the responses of households to changes in the consumption tax. Recall the Euler equation:

$$\frac{\dot{c}_t}{c_t} = -\frac{\dot{\overline{(1+\tau_t^c)}}}{(1+\tau_t^c)}\frac{1}{\sigma} + \frac{(1-\tau_t^a)\,r_t-\rho}{\sigma}$$

The Euler equation implies that when consumption taxes are increased, $\dot{\overline{(1+\tau_t^c)}} > 0$, then consumption growth turns negative at the moment of the hike in taxes. Moreover, as the households are forward looking, if they will know of the tax change in advance they will want to enjoy higher consumption when it is cheaper (lower tax). The model can produce the following consumption pattern over time, which is almost identical to the reaction of consumption in Japan and Germany following the VAT hikes:



Source: **Danninger (2014)**



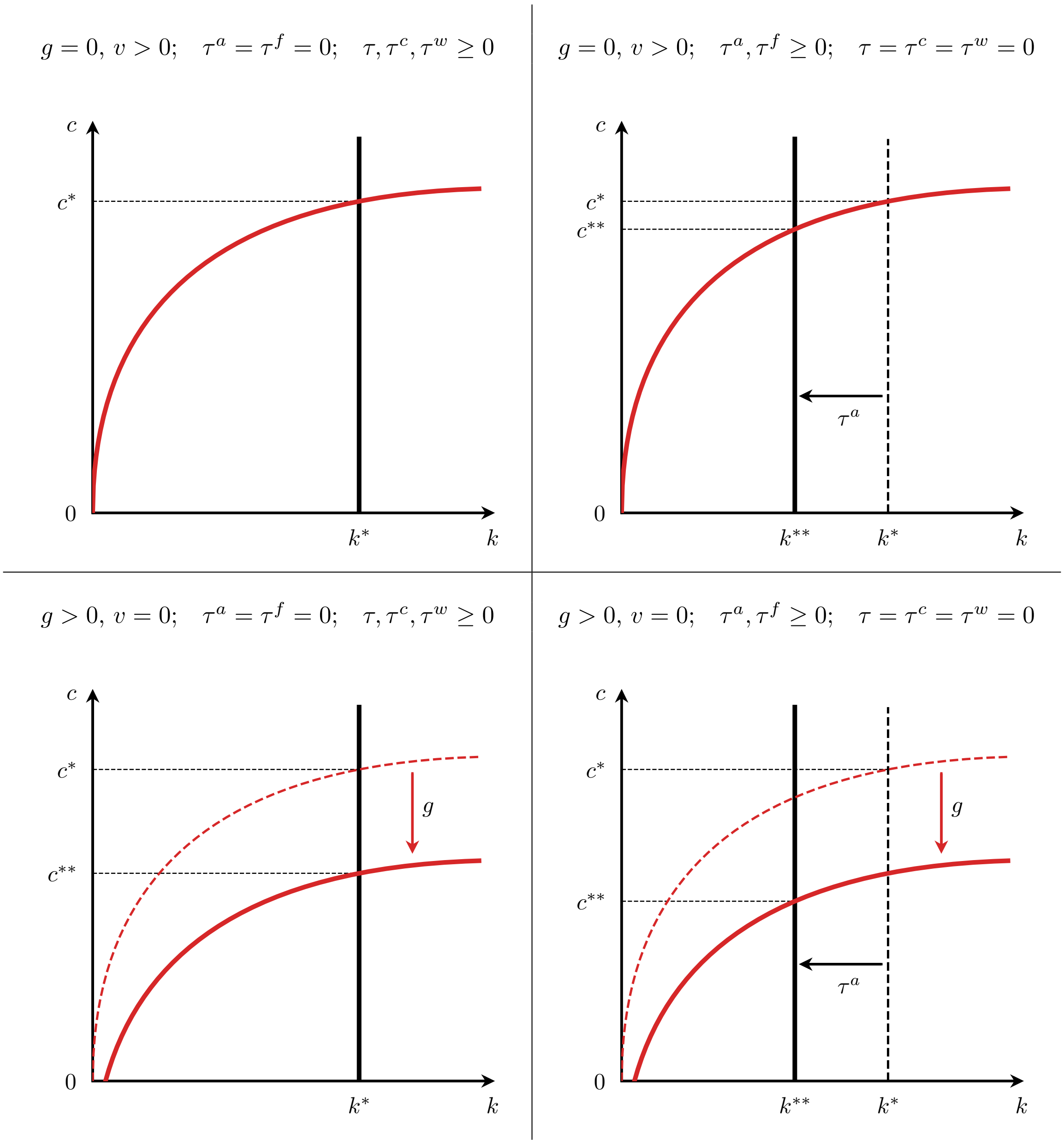
1.7 Effects of taxes on the steady state

From looking at the steady state conditions we notice that not all taxes act the same:

$$f'(k^*) = \frac{\rho}{(1-\tau^a)(1-\tau^f)} + \delta$$
$$c^* = f(k^*) - \delta k^* - g$$

The capital gains tax and the tax on firms' accounting profits affect the incentives to accumulate and hold capital and their presence lowers the level of capital per worker in the steady state. However, the other taxes (lump-sum, consumption and labor income) do not appear in our steady state conditions.

There is also a distinction in whether the government uses the tax revenue to arrange transfers to households v or to finance public g. For a given level of capital per worker public consumption necessarily crowds out private consumption, and can have differing welfare effects depending on households' preferences toward public vs. private consumption.



1.8 Chamley-Judd result – redistribution impossibility theorem

Let us conceptually divide population into two polar groups: workers and capitalists. Workers do not save and consume their wages and any transfers they receive. Capitalists both save and consume. The government wants to redistribute between capitalists and workers. It levies tax on capital gains and distributes the proceeds to workers.

Worker households

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{\left(c_t^w\right)^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \\ \text{subject to} \quad & c_t^w = w_t + v_t \end{aligned}$$

Solution:

$$c_t^w = w_t + v_t$$

Capitalist households

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{\left(c_t^c\right)^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \\ \text{subject to} \quad & \dot{a}_t = \left(1 - \tau_t^a\right) r_t a_t - c_t^c \end{aligned}$$

Solution:

$$\frac{\dot{c}_t^c}{c_t^c} = \frac{\left(1-\tau^a\right) r_t - \rho}{\sigma}$$

Firms

$$\max \quad \Pi_t = L_t \left[f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t\right]$$

Solution:

$$\begin{aligned} r_t &= f'\left(k_t\right) - \delta \\ w_t &= f\left(k_t\right) - f'\left(k_t\right) k_t \end{aligned}$$

Government sector

$$v_t = \frac{N^c}{N^w} \tau^a r_t a_t$$

General equilibrium

Capital market equilibrium:

$$k_t = \frac{N^c}{N^w} a_t \quad \rightarrow \quad v_t = \tau^a r_t k_t$$

Steady state capital per worker:

$$\begin{aligned} 0 &= \left(1-\tau^a\right) r - \rho \\ 0 &= \left(1-\tau^a\right) f'\left(k^*\right) - \delta - \rho \\ f'\left(k^*\right) &= \frac{\rho}{1-\tau^a} + \delta \end{aligned}$$

Steady state workers' consumption:

$$c^{w*} = f\left(k^*\right) - f'\left(k^*\right) \cdot k^* + \tau^a \left[f'\left(k^*\right) - \delta\right] k^*$$

The Chamley-Judd result states that the worker's consumption in the steady state is maximal when $\tau^a = 0$. For an easy version of the proof, let us assume that $f(k) = k^\alpha$ and $\delta = 0$:

$$c^{w*} = (k^*)^\alpha - \alpha (k^*)^\alpha + \tau^a \alpha (k^*)^\alpha = (1 - \alpha + \alpha\tau^a) \left(\frac{\alpha (1 - \tau^a)}{\rho} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\ln c^{w*} = \ln (1 - \alpha + \alpha\tau^a) + \frac{\alpha}{1-\alpha} (\ln \alpha + \ln (1 - \tau^a) - \ln \rho)$$

$$\frac{\partial \ln c^{w*}}{\partial \tau^a} = \frac{\alpha}{1 - \alpha + \alpha\tau^a} + \frac{\alpha}{1-\alpha} \left(-\frac{1}{1-\tau^a} \right) = \frac{\alpha}{1 - \alpha + \alpha\tau^a} - \frac{\alpha}{1 - \alpha + \alpha\tau^a - \tau^a} < 0$$

It turns out that it is impossible to increase steady state consumption of workers by taxing capitalists. Taxing capitalists reduces steady state capital stock and lowers wages. Even if all of the revenue from taxation is given to workers in transfer, the loss in wages is greater than the gain from the transfer.

See e.g. **here** for conditions under which the above result might not hold. For example, **Aiyagari (1995)** shows that with incomplete insurance markets and borrowing constraints, the optimal capital gains tax rate is positive, even in the long run. Also, **Straub and Werning (2014)** show that the Chamley-Judd result depends critically on whether (and how fast) the economy actually converges to the steady state discussed above.

1.9 Taxation in the long run with endogenous labor supply

In the analysis above we have analyzed the impact of taxation in the standard Ramsey model, where households supply one unit of labor inelastically. In such environment, lump-sum, consumption and labor income tax were indistinguishable and had no effects on the steady state of the economy. Here we extend the Ramsey model to incorporate the labor supply choice. Now the effects of the three taxes will not be identical.

Households

For simplicity of analysis we will assume the following "log-log" utility function, which we have encountered in the neoclassical labor markets lecture:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \left[\ln c_t + \phi \ln (1 - h_t) \right] dt \\ \text{subject to} \quad & \dot{a} = (1 - \tau^w) wh + ra - (1 + \tau^c) c - \tau + v \end{aligned}$$

Hamiltonian:

$$\mathcal{H} = e^{-\rho t} \left[\ln c + \phi \ln (1 - h) \right] + \lambda \left[(1 - \tau^w) wh + ra - (1 + \tau^c) c + v \right]$$

First order conditions:

$$\begin{aligned} c &: \quad e^{-\rho t} c^{-1} - (1 + \tau^c) \lambda = 0 \\ h &: \quad e^{-\rho t} \frac{-\phi}{1-h} + \lambda (1 - \tau^w) w = 0 \\ a &: \quad \lambda r = -\dot{\lambda} \end{aligned}$$

FOC for consumption:

$$\begin{aligned} \lambda (1 + \tau^c) = e^{-\rho t} c^{-1} \quad & | \quad \ln \\ \ln \lambda + \ln (1 + \tau^c) = -\rho t - \ln c \quad & | \quad \frac{\mathrm{d}}{\mathrm{d}t} \\ \frac{\dot{\lambda}}{\lambda} = -\rho - \frac{\dot{c}}{c} & \\ \frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda} - \rho & \end{aligned}$$

FOC for assets:

$$-\frac{\dot{\lambda}}{\lambda} = r$$

Join the two for the Euler equation:

$$\frac{\dot{c}}{c} = r - \rho$$

FOC for hours worked:

$$e^{-\rho t}\frac{\phi}{1-h} = \lambda\left(1-\tau^w\right)w$$

Include FOC for consumption:

$$\begin{aligned} e^{-\rho t}\frac{\phi}{1-h} &= e^{-\rho t}c^{-1}\frac{1-\tau^w}{1+\tau^c}w \\ \frac{\phi}{1-h} &= \frac{1-\tau^w}{1+\tau^c}\frac{w}{c} \end{aligned}$$

Firms

Assume the following Cobb-Douglas production function:

$$\begin{aligned} Y &= AK^{\alpha}\left(Lh\right)^{1-\alpha} \\ y &= Ak^{\alpha}h^{1-\alpha} \end{aligned}$$

$$\begin{aligned} w &= \left(1-\alpha\right)Ak^{\alpha}h^{-\alpha} = \left(1-\alpha\right)\frac{y}{h} \\ r &= \alpha Ak^{\alpha-1}h^{1-\alpha} - \delta = \alpha\frac{y}{h} - \delta \end{aligned}$$

Government

Runs balanced budget:

$$v = \tau^w wh + \tau^c c + \tau$$

General equilibrium

$$\begin{aligned} \dot{a} &= \left(1-\tau^w\right)wh + ra - \left(1+\tau^c\right)c - \tau + v \\ \dot{a} &= wh + ra - c \\ \dot{k} &= wh + rk - c \\ \dot{k} &= \left(1-\alpha\right)\frac{y}{h}\cdot h + \left(\alpha\frac{y}{h} - \delta\right)k - c \\ \dot{k} &= y - \delta k - c \end{aligned}$$

System

$$\begin{aligned} \dot{k} &= y - \delta k - c & & & y &= k^{\alpha}h^{1-\alpha} \\ \frac{\dot{c}}{c} &= r - \rho & &\text{where} & r &= \alpha k^{\alpha-1}h^{1-\alpha} - \delta \\ \frac{\phi}{1-h} &= \frac{1-\tau^w}{1+\tau^c}\frac{w}{c} & & & w &= \left(1-\alpha\right)k^{\alpha}h^{-\alpha} \end{aligned}$$

Steady state

Euler equation

$$\dot{c} = 0 \quad \rightarrow \quad r = \rho$$
$$\alpha A k^{\alpha-1} h^{1-\alpha} - \delta = \rho$$
$$\left(\frac{k}{h}\right)^{\alpha-1} = \frac{\delta+\rho}{\alpha A}$$
$$\frac{k}{h} = \left(\frac{\alpha A}{\delta+\rho}\right)^{1/(1-\alpha)}$$

Output per hour

$$\frac{y}{h} = \left(\frac{k}{h}\right)^{\alpha}$$

Wage per hour

$$w = (1-\alpha)\frac{y}{h}$$

Resource constraint

$$\dot{k} = 0 \quad \rightarrow \quad c = y - \delta k$$
$$\frac{c}{h} = \frac{y}{h} - \delta\frac{k}{h}$$

Labor-consumption choice

$$\frac{\phi}{1-h} = \frac{1-\tau^w}{1+\tau^c}\frac{w}{c}$$
$$c = \frac{(1-\tau^w)\,w}{1+\tau^c}\cdot\frac{1-h}{\phi}$$

$$1-h = \phi\frac{(1+\tau^c)\,c}{(1-\tau^w)\,w} \quad | \quad : h$$
$$\frac{1}{h} - 1 = \phi\frac{(1+\tau^c)}{(1-\tau^w)\,w}\frac{c}{h}$$
$$\frac{1}{h} = 1 + \phi\frac{(1+\tau^c)}{(1-\tau^w)\,w}\frac{c}{h}$$

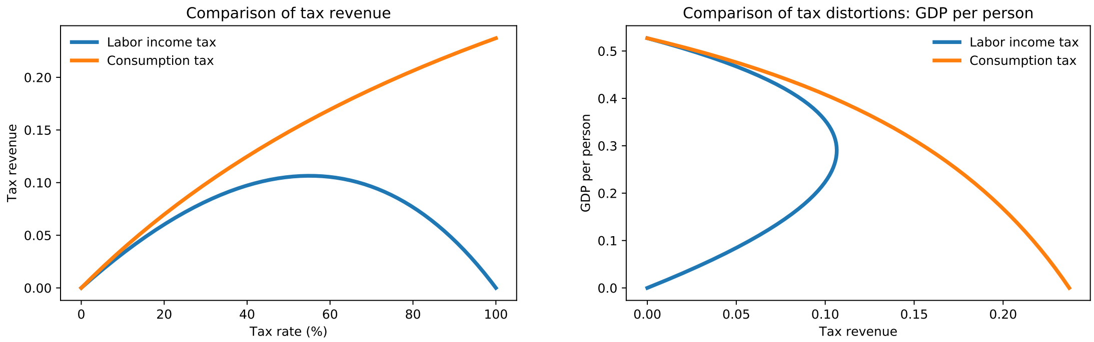
$$h = \left[1 + \phi\frac{(1+\tau^c)}{(1-\tau^w)\,w}\frac{c}{h}\right]^{-1}$$

To note: $\frac{k}{h}$, $\frac{y}{h}$, $\frac{c}{h}$ and w do not depend on either labor income tax or consumption tax. That's why if households do not choose labor input, the effects of these taxes are no different from the effects of a lump-sum tax (i.e. no effect at all). Only if we allow h to be chosen endogenously, then hours worked depend on these taxes.

The labor income tax generates a **Laffer curve**: an "inverted-U" relationship between the tax rate and tax revenue, with a peak somewhere between 0% and 100% rates. **Harald and Uhilg (2009)** find that the current tax rates in the US and Europe are to the left of the peak (which is estimated to be between 60% and 70% rates).

Below I also present the relationship between the tax rates and tax revenue for the consumption tax, which does not exhibit the same behavior as the labor income tax (note also that 100% is not the upper bound for the consumption tax rate). Consumption tax is able to raise more tax revenue and at the same time is less distortionary (right figure) than the labor income tax. This is why many economists advocate for reducing labor income taxes and increasing consumption taxes.

One of the arguments against the consumption tax is that it is hard to make it progressive. See this blog post by **John Cochrane** to read a proposition on how to introduce a progressive consumption tax.



Let us see how our model performs in matching the behavior of real world economies. If you look at the data, the labor productivity (GDP per hor worked) in many Western European countries is virtually indistinguishable from that in the United States. However, United States has higher GDP per person compared to Europe, since Europeans work less than workers in the United States. **Prescott (2004)** suggests that the above pattern is the result of different rates of labor taxation. Below I perform a similar exercise, using the OECD data from 2000 to 2018 (I consider averages over that period).

		Data			Model	
		United States	France	Germany	United States	"Europe"
GDP per hour (PPP $)	y/h	59	56	56	59	56
Average labor tax wedge	τ^w	26%	44%	44%	26%	44%
Average hours worked	h	1790	1530	1400	1790	1430
GDP per worker (PPP $)	y	102 500	86 200	78 200	102 500	79 700