Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019

1 Ramsey-Cass-Koopmans model, continuous time version

Assumptions similar to the Solow-Swan model:

- Closed economy
- No government (for now)
- Single, homogenous final good with its price normalized to 1 in each period (all variables are expressed in real terms)
- All households supply a unit of labor, number of people equal to number of workers
- Output is produced according to a neoclassical production function
- Two types of representative, optimizing agents:
  - Firms
  - Households
- Households are solving a utility maximizing problem – we have a well defined welfare measure

1.1 Firms

Representative firms produce according to the neoclassical production function:

$$Y_t = F\left(K_t, L_t, A_t\right)$$

where Y is real GDP, K is capital stock, L is employment and A denotes the technology/productivity level. Firms want to maximize their profits:

$$\max \quad \Pi_t = F\left(K_t, L_t, A_t\right) - w_t L_t - \left(r_t + \delta\right) K_t$$

Since the neoclassical production function F has the property of constant returns to scale in capital and labor, we can rewrite the production function in intensive form f (per worker):

$$y_t \equiv \frac{Y_t}{L_t} = \frac{1}{L_t} F\left(K_t, L_t, A_t\right) = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}, A_t\right) = F\left(k_t, 1, A_t\right) \equiv f\left(k_t\right)$$

where $y \equiv Y/L$ denotes GDP per worker and $k \equiv K/L$ denotes capital stock per worker.

The profit maximization problem using per worker variables is:

$$\max \quad \Pi_t = L_t\left[f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t\right]$$

Firms will choose employment level L and capital per worker k given wage w and real interest rate r.

First order conditions:

$$\begin{aligned}
k_t &: \quad L_t\left[f'\left(k_t\right) - \left(r_t + \delta\right)\right] = 0 \quad &\rightarrow \quad & r_t = f'\left(k_t\right) - \delta \\
L_t &: \quad f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t = 0 \quad &\rightarrow \quad & w_t = f\left(k_t\right) - \left(r_t + \delta\right) k_t = f\left(k_t\right) - f'\left(k_t\right) \cdot k_t
\end{aligned}$$

We can verify that due to perfect competition and constant returns to scale economic profits are zero:

$$\Pi = L_t\left[f\left(k_t\right) - w_t - \left(r_t + \delta\right) k_t\right] = L_t\left[f\left(k_t\right) - \left[f\left(k_t\right) - f'\left(k_t\right) \cdot k_t\right] - f'\left(k_t\right) \cdot k_t\right] = 0$$

1.2 Households

Representative, infinitely lived households (dynasties) solve the following utility maximization problem:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} N_t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \,\mathrm{d}t \\ \text{subject to} \quad & \dot{Assets}_t = w_t L_t + (1 + r_t)\, Assets_t - C_t \end{aligned}$$

where ρ is the discount rate, N denotes population, c stands for consumption per person, $\sigma > 0$ is a parameter of the CRRA utility function, *Assets* denote total assets of the household sector, r is the real interest rate, w denotes the wage earned by workers and $C = c \cdot N$ is the total consumption of households. To simplify notation we will also assume that all agents are working, so that $L_t = N_t$.

We will assume that the population grows at a constant rate n, and without loss of generality we will normalize the initial population size to 1 ($N_0 = 1$):

$$\frac{\dot{N}_t}{N_t} = n \quad \to \quad N_t = e^{nt} N_0 \quad \to \quad N_t = e^{nt}$$

The utility function can then be rewritten as:

$$U = \int_0^\infty e^{-\rho t} e^{nt} \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} \,\mathrm{d}t = \int_0^\infty e^{-(\rho - n)t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} \,\mathrm{d}t$$

For the utility function to be bounded, we need to assume that $\rho > n$.

The budget constraint is also easy to reformulate:

$$\begin{aligned} \dot{Assets}_t &= w_t L_t + r_t Assets_t - C_t \quad | \quad : N_t \\ \frac{\dot{Assets}_t}{N_t} &= \frac{w_t L_t}{N_t} + \frac{r_t Assets_t}{N_t} - \frac{C_t}{N_t} \\ \frac{\dot{Assets}_t}{N_t} &= w_t + (1 + r_t)\, a_t - c_t \end{aligned}$$

where a denotes assets per person. Now we need to derive the expression for $\dot{Assets}_t / N_t$. Let us do this by looking at a related expression:

$$\dot{a}_t \equiv \frac{\mathrm{d}\,(Assets_t / N_t)}{\mathrm{d}t} = \frac{\dot{Assets}_t \cdot N_t - Assets_t \cdot \dot{N}_t}{N_t^2} = \frac{\dot{Assets}_t}{N_t} - \frac{Assets_t}{N_t} \cdot \frac{\dot{N}_t}{N_t} = \frac{\dot{Assets}_t}{N_t} - a_t n$$

Notice how $\dot{Assets}_t / N_t \neq \dot{a}_t$, as we have to take into account the effects population growth:

$$\frac{\dot{Assets}_t}{N_t} = \dot{a}_t + a_t n$$

We can now go back to our budget constraint:

$$\begin{aligned} \frac{\dot{Assets}_t}{N_t} &= w_t + r_t a_t - c_t \\ \dot{a}_t + a_t n &= w_t + r_t a_t - c_t \\ \dot{a}_t &= w_t + (r_t - n)\, a_t - c_t \end{aligned}$$

We are now ready to solve the problem using the Hamiltonian:

$$\mathcal{H} = e^{-(\rho - n)t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[w_t + (r_t - n)\, a_t - c_t \right]$$

where λ_t has the interpretation of a co-state variable, associated with the state variable a.

First order conditions (note well that $\partial\mathcal{H}/\partial a = -\dot{\lambda}$):

$$\begin{aligned} \frac{\partial\mathcal{H}}{\partial c_t} = 0 \quad &: \quad e^{-(\rho-n)t}c_t^{-\sigma} - \lambda_t = 0 \quad &\rightarrow \quad &\lambda_t = e^{-(\rho-n)t}c_t^{-\sigma} \\ \frac{\partial\mathcal{H}}{\partial a_t} = -\dot{\lambda}_t \quad &: \quad \lambda_t\left(r_t - n\right) = -\dot{\lambda}_t \quad &\rightarrow \quad &-\frac{\dot{\lambda}_t}{\lambda_t} = r_t - n \end{aligned}$$

Now transform the first order condition for optimal consumption:[1]

$$\begin{aligned} \lambda_t &= e^{-(\rho-n)t}c_t^{-\sigma} \quad | \quad \ln(\cdot) \\ \ln\lambda_t &= -\left(\rho - n\right)t - \sigma\ln c_t \quad | \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ \frac{\dot{\lambda}_t}{\lambda_t} &= -\left(\rho - n\right) - \sigma\frac{\dot{c}_t}{c_t} \\ \frac{\dot{c}_t}{c_t} &= \frac{-\dot{\lambda}_t/\lambda_t - \rho + n}{\sigma} \end{aligned}$$

By plugging in the transformed FOC w.r.t. assets we arrive at the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - n - \rho + n}{\sigma} = \frac{r_t - \rho}{\sigma}$$

The discount rate can be interpreted as a "psychological" interest rate of an agent and it implies that consumption increases over time if the real interest rate exceeds the "psychological" interest rate. In the situation where $r_t > \rho$, an agent is willing to save and it means that she consumes less in the present to consume more in the future. The parameter σ will affect how strongly the difference between real and "psychological" interest rates influences changes in consumption. The higher the σ is, the smaller are the changes of consumption in response to the difference between the interest rates.

1.3 General Equilibrium

In a closed economy, the only asset in positive net supply is capital, because all the borrowing and lending must cancel out within the economy. Hence, equilibrium in the asset market requires $a = k$. We will modify the households' budget constraint accordingly and in the next step plug in the expressions for prices to get the resource constraint:

$$\begin{aligned} \dot{a}_t &= w_t + \left(r_t - n\right)a_t - c_t \\ \dot{k}_t &= w_t + \left(r_t - n\right)k_t - c_t \\ \dot{k}_t &= f\left(k_t\right) - f'\left(k_t\right)\cdot k_t + \left(f'\left(k_t\right) - \delta - n\right)k_t - c_t \\ \dot{k}_t &= f\left(k_t\right) - \left(\delta + n\right)k_t - c_t \end{aligned}$$

If we go back to the Euler equation, we can replace the real interest rate with the marginal product of capital net of depreciation:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{r_t - \rho}{\sigma} \\ \frac{\dot{c}_t}{c_t} &= \frac{f'\left(k_t\right) - \delta - \rho}{\sigma} \end{aligned}$$

The dynamics of the entire economy are summed up by the following two equations:

$$\begin{aligned} \text{Euler equation} \quad &: \quad \frac{\dot{c}_t}{c_t} = \frac{f'\left(k_t\right) - \delta - \rho}{\sigma} \\ \text{Resource constraint} \quad &: \quad \dot{k}_t = f\left(k_t\right) - \left(\delta + n\right)k_t - c_t \end{aligned}$$

[1] Here we are using the following property:

$$\frac{\mathrm{d}\left(\ln x\left(t\right)\right)}{\mathrm{d}t} = \frac{\mathrm{d}\ln x}{\mathrm{d}x}\cdot\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} = \frac{1}{x}\cdot\dot{x} = \frac{\dot{x}}{x}$$

1.4 Steady state and transition dynamics

The steady state of an economic system is a situation where all variables grow at constant rates. Often economists distinguish between the steady state proper and balanced growth path situations, where the former exhibits zero growth in income per person, while the latter exhibits constant positive growth.

Without technology improvements, variables per person stabilize over time. We can easily find values of c and k that put the system at rest. Start with the Euler equation:

$$0 = \frac{f'(k) - \delta - \rho}{\sigma}$$
$$f'(k^*) = \delta + \rho$$

The steady state level of capital per worker is found by equating the marginal product of capital with the sum of depreciation rate δ and households' discount rate ρ.

If we assume a specific production function, we can obtain an explicit formula for the steady state level of capital per worker. For the following Cobb-Douglas production function:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$
$$f(k) = Ak^{\alpha}$$
$$f'(k) = \alpha Ak^{\alpha-1}$$

$$\alpha Ak^{\alpha-1} = \delta + \rho$$
$$k^* = \left(\frac{\delta+\rho}{\alpha A}\right)^{1/(\alpha-1)} = \left(\frac{\alpha A}{\delta+\rho}\right)^{1/(1-\alpha)}$$

If we go back to the Euler equation in the form relating real interest rate and discount rate of households:

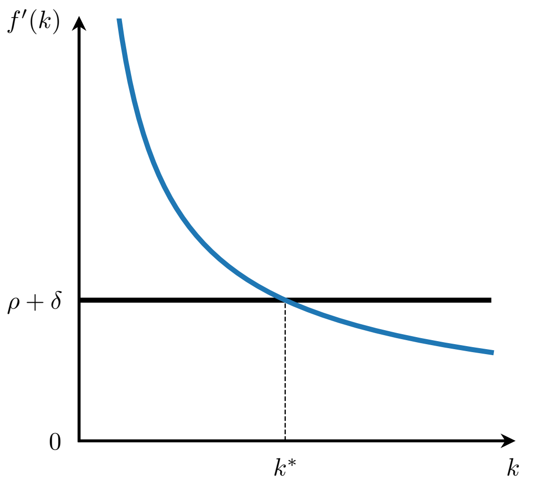
$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\sigma}$$

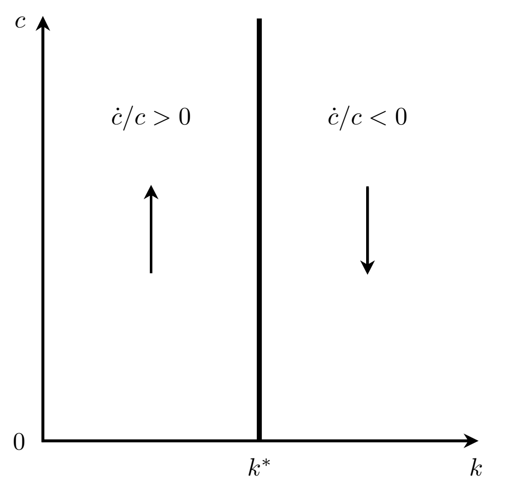
we can see that consumption will increase over time whenever $r > \rho$:

$$r = f'(k) - \delta$$

$$r > \rho \quad \rightarrow \quad f'(k) - \delta > \rho \quad \rightarrow \quad f'(k) > \delta + \rho$$

When the marginal product of capital exceeds its steady state value $(\delta + \rho)$, consumption increases over time. A glance at the figures below reveals that consumption increases over time for $k < k^*$ and by analogy decreases over time for $k > k^*$:





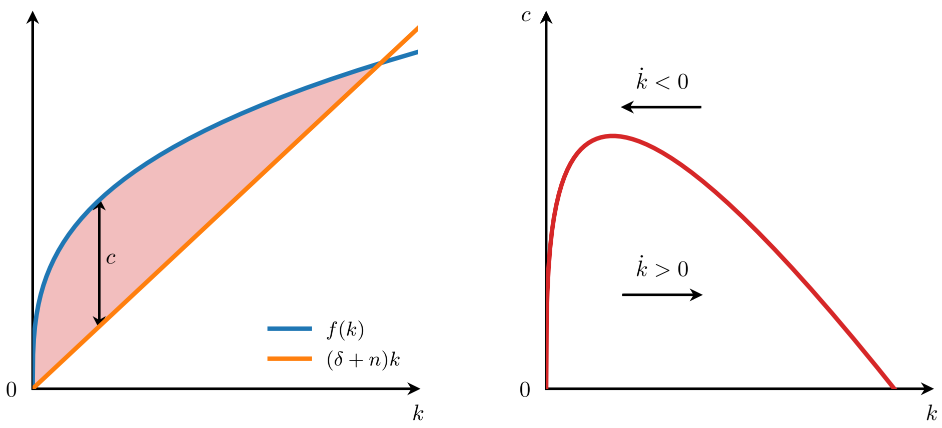
If we rearrange the resource constraint, we will find the expression for c given that the capital stock does not change over time:

$$\begin{aligned} \dot{k}_t &= f\left(k_t\right) - \left(\delta + n\right) k_t - c_t \\ 0 &= f\left(k\right) - \left(\delta + n\right) k - c \\ c &= f\left(k\right) - \left(\delta + n\right) k \end{aligned}$$

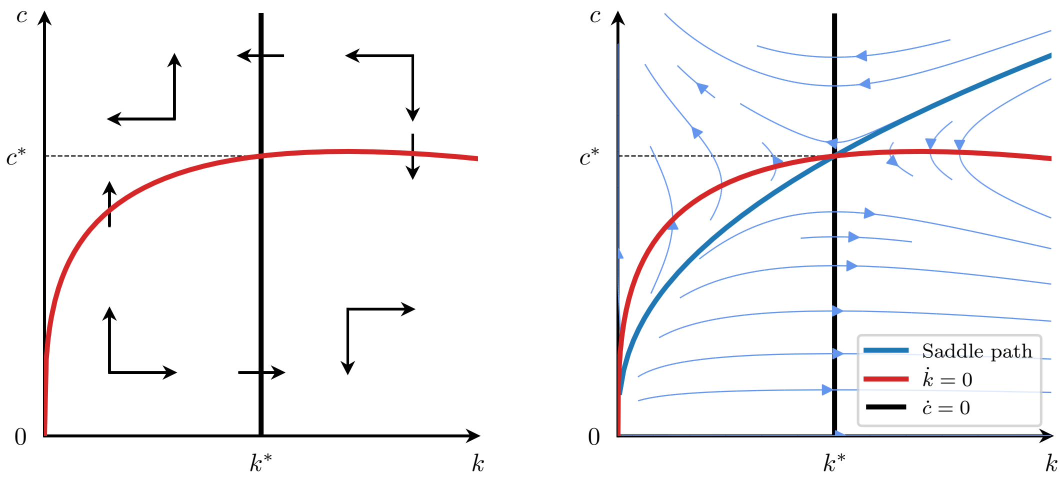
Let us see what would happen if the consumption was chosen above the level guaranteeing stable capital over time:

$$\begin{aligned} c_t &> f\left(k_t\right) - \left(\delta + n\right) k_t \\ \dot{k}_t &= f\left(k_t\right) - \left(\delta + n\right) k_t - c_t \\ \dot{k}_t &< f\left(k_t\right) + \left(\delta + n\right) k_t - \left[f\left(k_t\right) - \left(\delta + n\right) k_t\right] \\ \dot{k}_t &< 0 \end{aligned}$$

If consumption is chosen above the quantity that implies constant capital, capital decreases over time as investment is lower than effective depreciation. If consumption is chosen below the constant capital schedule, capital increases over time. The relationship is displayed in the graphs below:



We can now join the two schedules and produce the full phase diagram:



1.5 Social planner's solution

The social planner is a hypothetical dictator who aims at maximizing the households' utility subject to the technological constraints of the economy. The social planner's solution is equivalent to the first best allocation that the economy can achieve from the point of view of households.

We may want to compute this solution for two reasons. First, in some model economies the decentralized (market / private) solution is not equivalent to the first best allocation. Then we might consider how the government could influence the choices of private agents to bring their decisions closer to the first best allocation. Second, in model economies where social planner's and decentralized solutions coincide, often the social planner's problem is easier or faster to solve for and can be simply used as a model solution technique.

As we will shortly see for ourselves, in the Ramsey model the social planner's solution is identical to the decentralized solution.

The social planner aims here to maximize households' utility subject to the resource constraint:

$$\begin{aligned} \max \quad U &= \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma}-1}{1-\sigma}\, \mathrm{d}t \\ \text{subject to} \quad \dot{k}_t &= f(k_t) - (\delta+n)\, k_t - c_t \end{aligned}$$

Hamiltonian:

$$\mathcal{H} = e^{-(\rho-n)t} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \lambda_t \left[f(k_t) - (\delta+n)\, k_t - c_t \right]$$

First order conditions (note well that $\partial\mathcal{H}/\partial k = -\dot{\lambda}$):

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial c_t} = 0 \quad &: \quad e^{-(\rho-n)t} c_t^{-\sigma} - \lambda_t = 0 \qquad \rightarrow \quad \lambda_t = e^{-(\rho-n)t} c_t^{-\sigma} \\ \frac{\partial \mathcal{H}}{\partial k_t} = -\dot{\lambda}_t \quad &: \quad \lambda_t \left[f'(k_t) - (\delta+n) \right] = -\dot{\lambda}_t \quad \rightarrow \quad -\frac{\dot{\lambda}_t}{\lambda_t} = f'(k_t) - (\delta+n) \end{aligned}$$

By log-differentiating the FOC w.r.t. consumption we arrive at the standard relationship:

$$\begin{aligned} \lambda_t &= e^{-(\rho-n)t} c_t^{-\sigma} \quad | \quad \ln(\cdot) \\ \ln \lambda_t &= -(\rho-n)\, t - \sigma \ln c_t \quad | \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ \frac{\dot{\lambda}_t}{\lambda_t} &= -(\rho-n) - \sigma \frac{\dot{c}_t}{c_t} \\ \frac{\dot{c}_t}{c_t} &= \frac{-\dot{\lambda}_t/\lambda_t - \rho + n}{\sigma} \end{aligned}$$

Now we can plug in the FOC w.r.t. capital:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{f'(k_t) - (\delta+n) - \rho + n}{\sigma} \\ \frac{\dot{c}_t}{c_t} &= \frac{f'(k_t) - \delta - \rho}{\sigma} \end{aligned}$$

The social planner's Euler equation is identical to one obtained for the general equilibrium of the private, decentralized economy.

The above result also implies that in this model economy the government can do nothing to improve the welfare of the agents. Keep this result in mind when we will analyze the effects of taxation.

1.6 Shape of the saddle path

In general, the solution to the Ramsey model cannot be found analytically, but can be obtained using a wide variety of numerical methods. There is however one case for which an analytical solution exists.

We need to assume that the production function is of the Cobb-Douglas type:

$$f(k) = Ak^{\alpha} \quad \rightarrow \quad f'(k) = \alpha A k^{\alpha-1}$$

and that the CRRA parameter σ is equal to the elasticity of output with respect to capital α:

$$\sigma = \alpha$$

Then we can "guess-and-verify" that the saddle path is a straight line, that is that the consumption to capital ratio is constant, which in turn imples that the growth rates of consumption and capital are identical:

$$\frac{c}{k} = const. \quad \rightarrow \quad \frac{\dot{c}}{c} = \frac{\dot{k}}{k}$$

The growth rate of consumption is given by the Euler equation:

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\sigma} = \frac{\alpha A k^{\alpha-1} - \delta - \rho}{\sigma}$$

To obtain the growth rate of capital we need to divide the resource constraint by k:

$$\frac{\dot{k}}{k} = \frac{f(k) - (\delta + n)k - c}{k} = Ak^{\alpha-1} - (\delta + n) - \frac{c}{k}$$

Finally, we can use the assumption that $\sigma = \alpha$ and compare the two equations for growth rates:

$$\begin{aligned} \frac{\alpha A k^{\alpha-1} - \delta - \rho}{\alpha} &= Ak^{\alpha-1} - (\delta + n) - \frac{c}{k} \\ -\frac{\delta + \rho}{\alpha} &= -(\delta + n) - \frac{c}{k} \\ \frac{c}{k} &= \frac{\delta + \rho}{\alpha} - \delta - n \end{aligned}$$

We can indeed verify that the ratio of consumption to capital is given by constant model parameters. The analytical solution of the model is then given by:

$$\begin{aligned} c_t &= \left(\frac{\delta + \rho}{\alpha} - \delta - n\right) k_t \\ \dot{k}_t &= Ak_t^{\alpha} - (\delta + n) k_t - c_t \end{aligned}$$

Since empirical studies imply that $\sigma \approx 2$ and $\alpha \approx 1/3$, the above example cannot be considered realistic. However, it helps us derive the following results (see **Barro and Sala-i-Martin (2004)** for proofs):

- If $\sigma < \alpha$ (another unrealistic case) then the saddle path is convex and runs below the straight line connecting $(0, 0)$ with (k^*, c^*).
- If $\sigma > \alpha$ then the saddle path is concave and runs above the straight line connecting $(0, 0)$ with (k^*, c^*) and below the $\dot{k} = 0$ curve. Additionally, if the economy experiences a one-off increase in technology level A then consumption increases on impact (this will be important to recall when studying the Real Business Cycles model).

In the economy where there is no sustained technological progress, parameter σ only matters for the shape of the saddle path, but does not influence the steady state level of the economy. When we allow for technological progress, σ will also impact the steady state.

1.7 Behavior of the saving rate

One of the key advantages of the Ramsey model over the Solow model is that here the saving rate arises from the optimal behavior of the households, and can vary over time. There is however another case for which an analytical solution exists, and it involves a constant saving rate. After we demonstrate this result we will discuss under which conditions saving rate increases or decreases as the economy converges towards its steady state.

Again, we need to assume that the production function is of the Cobb-Douglas type. Additionally, suppose that the following relationship holds:

$$\sigma = 1/s^*$$

where s^* denotes the saving rate in the steady state. Let us first derive its value:

$$s \equiv \frac{y-c}{y} \quad \rightarrow \quad s^* = \frac{y^* - (y^* - \delta k^*)}{y^*} = \frac{\delta k^*}{y^*}$$

For the Cobb-Douglas production function we have that (for simplicity $A = 1$):

$$k^* = \left(\frac{\alpha}{\delta + \rho}\right)^{1/(1-\alpha)} \quad \text{and} \quad y = k^\alpha \quad \rightarrow \quad s^* = \delta \left(k^*\right)^{1-\alpha} = \frac{\delta\alpha}{\delta + \rho}$$

Now we can "guess-and-verify" that under our assumptions the saving rate is indeed constant. First let us rewrite our two key equations, taking into account our assumptions:

$$\dot{k}_t = (y_t - c_t) - \delta k_t = s^* \cdot f\left(k_t\right) - \delta k_t = s^* \cdot k_t^\alpha - \delta k_t \quad \rightarrow \quad \frac{\dot{k}_t}{k_t} = s^* \cdot k_t^{\alpha-1} - \delta$$

$$\frac{\dot{c}_t}{c_t} = \frac{f'\left(k_t\right) - \delta - \rho}{\sigma} = \frac{\alpha k_t^{\alpha-1} - \delta - \rho}{\sigma} = s^*\left(\alpha k_t^{\alpha-1} - \delta - \rho\right)$$

And now let us take the time derivative of the consumption to output ratio. If the saving rate is indeed constant, then this time derivative should be equal to 0:

$$\frac{\mathrm{d}\left(c_t/y_t\right)}{\mathrm{d}t} = \frac{\dot{c}_t}{c_t} - \frac{\dot{y}_t}{y_t} = \frac{\dot{c}_t}{c_t} - \alpha\frac{\dot{k}_t}{k_t} = s^*\left(\alpha k_t^{\alpha-1} - \delta - \rho\right) - \alpha\left[s^* \cdot k_t^{\alpha-1} - \delta\right] = \alpha\delta - s^*\left(\delta + \rho\right) = 0$$

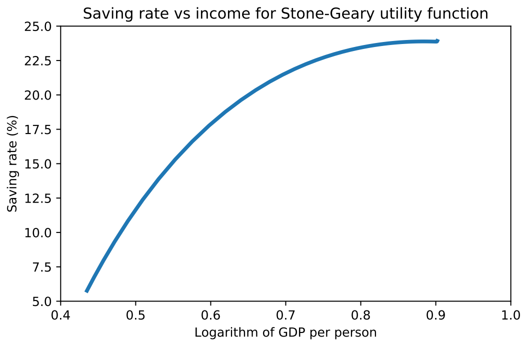
which turns out to be the case. Again, since empirical studies imply that $\sigma \approx 2$ but observed saving (investment) rates are usually smaller than 30%, the above example cannot be considered realistic. However, it helps us derive the following results (see **Barro and Sala-i-Martin (2004)** for proofs):

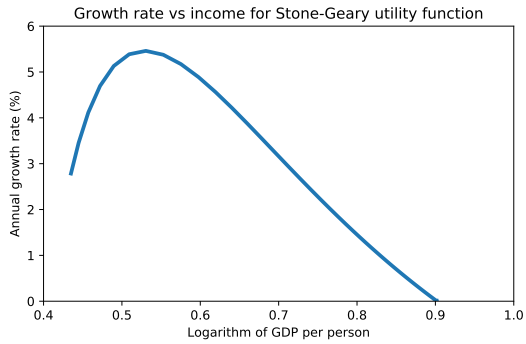
- If $\sigma > 1/s^*$ (another unrealistic case) then the saving rate is initially lower but increases as the economy converges to the steady state.
- If $\sigma < 1/s^*$ then the saving rate is initially higher but decreases as the economy converges to the steady state.

The above result however implies that the speed of convergence is even higher than in the Solow model, which was already too high compared to what we observe in the data. One can however introduce a small modification to the utility function. Suppose that there is some level of consumption $\bar{c}$ which is necessary for a human to survive. Then the utility function assumes the following **Stone-Geary** form:

$$u\left(c\right) = \frac{\left(c - \bar{c}\right)^{1-\sigma} - 1}{1 - \sigma}$$

In economies where $y \approx \bar{c}$ (output per person is close to the survival requirement) households will save very little. Only when output per person becomes sufficiently high, those households will start saving more (even if $\sigma < 1/s^*$). This generates an "inverse-U" convergence pattern: the least developed countries exhibit relatively low growth rates, while the middle income countries grow the fastest.





1.8 Allowing for technological progress

Assume that technology A grows at a constant rate g:

$$\frac{\dot{A}_t}{A_t} = g \quad \rightarrow \quad A_t = e^{gt} A_0$$

Since technology grows over time, both consumption and capital per worker will grow over time. To find the balanced growth path of the system, we need to utilize variables per effective units of labor:

$$\begin{aligned}
\text{Capital per effective labor} \quad &: \quad \hat{k}_t \equiv \frac{K_t}{A_t L_t} = \frac{k_t}{A_t} \\
\text{Consumption per effective labor} \quad &: \quad \hat{c}_t \equiv \frac{C_t}{A_t L_t} = \frac{c_t}{A_t} \\
\text{Output per effective labor} \quad &: \quad \hat{y}_t \equiv \frac{Y_t}{A_t L_t} = \frac{y_t}{A_t} \\
\text{Wage per effective labor} \quad &: \quad \hat{w}_t \equiv \frac{w_t}{A_t}
\end{aligned}$$

The problem of the households is unchanged, but now we want to rewrite the Euler equation in terms of per effective labor variables. Consider the definitional relationship between consumption per person c and per effective labor $\hat{c}$:

$$\begin{aligned}
c_t &= \hat{c}_t \cdot A_t \quad | \quad \ln \\
\ln c_t &= \ln \hat{c}_t + \ln A_t \quad | \quad \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\
\frac{\dot{c}_t}{c_t} &= \frac{\dot{\hat{c}}_t}{\hat{c}_t} + \frac{\dot{A}_t}{A_t} = \frac{\dot{\hat{c}}_t}{\hat{c}_t} + g
\end{aligned}$$

We can now use the Euler equation:

$$\begin{aligned}
\frac{\dot{\hat{c}}_t}{\hat{c}_t} &= \frac{\dot{c}_t}{c_t} - g = \frac{r_t - \rho}{\sigma} - g \\
\frac{\dot{\hat{c}}_t}{\hat{c}_t} &= \frac{r_t - \rho - \sigma g}{\sigma}
\end{aligned}$$

Now we have to restate the problem of the firm to obtain prices in terms of per effective labor variables:

$$\begin{aligned}
\max \quad \Pi_t &= F\left(K_t, A_t L_t\right) - w_t L_t - \left(r_t + \delta\right) K_t \\
\max \quad \Pi_t &= A_t L_t \left[F\left(\frac{K_t}{A_t L_t}, 1\right) - \frac{w_t}{A_t} - \left(r_t + \delta\right) \frac{K_t}{A_t L_t} \right] \\
\max \quad \Pi_t &= A_t L_t \left[f(\hat{k}_t) - \hat{w}_t - \left(r_t + \delta\right) \hat{k}_t \right]
\end{aligned}$$

where $\hat{y}_t = f(\hat{k}_t) \equiv F\left(\frac{K_t}{A_t L_t}, 1\right)$.

First order conditions:

$$\begin{aligned}
\hat{k}_t \quad &: \quad A_t L_t \left[f'(\hat{k}_t) - (r_t + \delta) \right] = 0 \qquad \rightarrow \quad r_t = f'(\hat{k}_t) - \delta \\
L_t \quad &: \quad A_t \left[f(\hat{k}_t) - \hat{w}_t - (r_t + \delta)\, \hat{k}_t \right] = 0 \quad \rightarrow \quad \hat{w}_t = f(\hat{k}_t) - (r_t + \delta)\, \hat{k}_t = f(\hat{k}_t) - f'(\hat{k}_t) \cdot \hat{k}_t
\end{aligned}$$

To find the general equilibrium of the economy, start with the households' budget constraint in per capita terms and rewrite in per effective labor terms:

$$\begin{aligned}
\dot{a}_t &= w_t + (r_t - n)\, a_t - c_t \\
\dot{k}_t &= w_t + (r_t - n)\, k_t - c_t \quad | \quad : A_t \\
\frac{\dot{k}_t}{A_t} &= \frac{w_t}{A_t} + (r_t - n)\, \frac{k_t}{A_t} - \frac{c_t}{A_t} \\
\frac{\dot{k}_t}{A_t} &= \hat{w}_t + (r_t - n)\, \hat{k}_t - \hat{c}_t
\end{aligned}$$

Let us now find the expression for $\mathrm{d}\hat{k}_t / \mathrm{d}t$:

$$\dot{\hat{k}}_t = \frac{\mathrm{d}\,(k_t / A_t)}{\mathrm{d}t} = \frac{\dot{k}_t \cdot A_t - k_t \cdot \dot{A}_t}{A_t^2} = \frac{\dot{k}_t}{A_t} - \hat{k}_t g \quad \rightarrow \quad \frac{\dot{k}_t}{A_t} = \dot{\hat{k}}_t + g\hat{k}_t$$

Now let us use the above result in the resource constraint:

$$\begin{aligned}
\dot{\hat{k}}_t + g\hat{k}_t &= \hat{w}_t + (r_t - n)\, \hat{k}_t - \hat{c}_t \\
\dot{\hat{k}}_t &= \hat{w}_t + (r_t - n - g)\, \hat{k}_t - \hat{c}_t
\end{aligned}$$

Finally, plug in prices:

$$\begin{aligned}
\dot{\hat{k}}_t &= f(\hat{k}_t) - f'(\hat{k}_t) \cdot \hat{k}_t + (f'(\hat{k}_t) - \delta - n - g)\hat{k}_t - \hat{c}_t \\
\dot{\hat{k}}_t &= f(\hat{k}_t) - (\delta + n + g)\, \hat{k}_t - \hat{c}_t
\end{aligned}$$

And do the same in the Euler equation:

$$\begin{aligned}
\frac{\dot{\hat{c}}_t}{\hat{c}_t} &= \frac{r_t - \rho - \sigma g}{\sigma} \\
\frac{\dot{\hat{c}}_t}{\hat{c}_t} &= \frac{f'(\hat{k}_t) - \delta - \rho - \sigma g}{\sigma}
\end{aligned}$$

The procedure for finding the balanced growth path is very similar to the one before. From the Euler equation we can obtain the balanced growth path level of capital per effective labor:

$$\begin{aligned}
0 &= \frac{f'(\hat{k}_t) - \delta - \rho - \sigma g}{\sigma} \\
f'(\hat{k}^*) &= \delta + \rho + \sigma g
\end{aligned}$$

From the resource constraint we can obtain the balanced growth path level of consumption per effective labor:

$$\begin{aligned}
0 &= f(\hat{k}) - (\delta + n + g)\, \hat{k} - \hat{c} \\
\hat{c}^* &= f(\hat{k}^*) - (\delta + n + g)\, \hat{k}^*
\end{aligned}$$

To relate it to our previous results, imagine that $g = 0$ and technology is normalized to 1 ($A = 1$). Then $\hat{k} = k$ and the new expressions reduce to the expressions for the zero technology growth case.

Note how higher rate of technology growth lowers both $\hat{k}^*$ and $\hat{c}^*$. However, you need to remember that those variables do not impact the welfare of agents directly, as households care for their consumption per capita c, which increases faster with higher g. Therefore, higher g increases the welfare of agents.