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1 Ramsey-Cass-Koopmans model

The Ramsey model is an extension of the Solow-Swan model, pioneered by **Frank Ramsey (1928)** and extended by **David Cass (1965)** and **Tjalling Koopmans (1965)**. Known also as the Neoclassical Growth Model, it is a "workhorse" of modern macroeconomics and forms the basis of the majority of modern works in economic growth and business cycles literature.

In the Overlapping Generations model the agents lived finite lives and were "selfish", in the sense that they cared only about their own utility and left no resources after death. However in the real world we see that at least some individuals leave sizeable bequests and support their children financially throughout their lives. This implies that parents care for their children's "utility". In the Ramsey model it is assumed that parents care about their children as much as they care about themselves. Moreover, both the labor and capital income are pooled within the household and split equally across its members. Therefore, each household is in fact a "dynasty" of individuals with an infinite planning horizon, and the welfare function of a household is constructed as follows:

$$U = \sum_{t=0}^{\infty} \beta^t N_t u(c_t)$$

where β is a discount factor, N_t is the number of people alive at period t, and $u(\cdot)$ is an instantaneous utility function (felicity function).

Usually this felicity function is represented by a Constant Relative Risk Aversion (CRRA) function:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

where $\sigma > 0$ regulates the degree of risk aversion.

Firms' problem

The problem of the firms will be the same for both simplified and general cases. The firms want to hire optimal quantities of labor and capital to maximize their profits:

$$\max \quad \Pi_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} - w_t L_t - r_t^k K_t$$

First order conditions:

$$\frac{\partial \Pi_t}{\partial L_t} = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} - w_t = 0$$
$$\frac{\partial \Pi_t}{\partial K_t} = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - r_t^k = 0$$

Simplify the above expressions:

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha}$$
$$r_t^k = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1}$$

The above equations determine both the wage w and the capital rental rate r^k as functions of capital (per effective labor). The real interest rate is equal to the capital rental rate, less the capital depreciation rate:

$$r_t = r_t^k - \delta$$

1.1 Simplified model

We will first consider a Ramsey model in discrete time, with simplifying assumptions that yield an analytical solution, and then consider a general case. For now, we will assume for simplicity that there is no population and technology growth, and without loss of generality we can normalize $N = A = 1$, so that capital per effective labor and capital per worker are equivalent. We will also assume total depreciation of capital ($\delta = 1$) and that the instantaneous utility function is logarithmic ($\sigma = 1$).

Households' problem

The representative "dynasty" solves the following problem:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{subject to} \quad & c_t + a_{t+1} = w_t + (1 + r_t)\, a_t \\ & \lim_{t\to\infty} \lambda_t a_{t+1} = 0 \end{aligned}$$

Write down the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1}\right] \\ &= \ldots + \beta^t \ln c_t + \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1}\right] \\ &\quad \ldots + \lambda_{t+1} \left[w_{t+1} + (1 + r_{t+1})\, a_{t+1} - c_{t+1} - a_{t+2}\right] + \ldots \end{aligned}$$

In each period t we can choose current consumption c_t and future level of assets a_{t+1}.

First order conditions:

$$\begin{aligned} c_t \quad &: \quad \beta^t \frac{1}{c_t} - \lambda_t = 0 & \rightarrow \quad & \lambda_t = \beta^t \frac{1}{c_t} \\ a_{t+1} \quad &: \quad -\lambda_t + \lambda_{t+1}\,(1 + r_{t+1}) = 0 & \rightarrow \quad & \lambda_t = (1 + r_{t+1})\, \lambda_{t+1} \end{aligned}$$

We obtain the familiar Euler equation:

$$\beta^t \frac{1}{c_t} = (1 + r_{t+1})\, \beta^{t+1} \frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta\,(1 + r_{t+1})\, c_t$$

General Equilibrium

Since there is no government and no international trade, domestic capital is the only asset which can be in positive net supply:

$$a_t = k_t \quad \text{for all } t$$

And the budget constraint of the households can be rewritten as:

$$\begin{aligned} c_t + a_{t+1} &= w_t + (1 + r_t)\, a_t \\ c_t + k_{t+1} &= w_t + (1 + r_t)\, k_t \end{aligned}$$

We can also use the expressions for prices (recall that $A = 1$ and $\delta = 1$):

$$\begin{aligned} w_t &= (1 - \alpha)\, A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = (1 - \alpha)\, k_t^{\alpha} \\ r_t^k &= \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha - 1} = \alpha k_t^{\alpha - 1} \\ r_t &= r_t^k - \delta = \alpha k_t^{\alpha - 1} - 1 \end{aligned}$$

And plug them into the budget constraint:

$$\begin{aligned} c_t + k_{t+1} &= (1 - \alpha)\, k_t^{\alpha} + \left(1 + \alpha k_t^{\alpha - 1} - 1\right) k_t \\ c_t + k_{t+1} &= (1 - \alpha)\, k_t^{\alpha} + \alpha k_t^{\alpha} \\ c_t + k_{t+1} &= k_t^{\alpha} \end{aligned}$$

Closed-form solution

Finally, let us guess-and-verify that this economy behaves as a Solow-Swan economy, and the households save a constant fraction s of their income:

$$k_{t+1} = sy_t = sk_t^{\alpha}$$
$$c_t = (1-s)\,y_t = (1-s)\,k_t^{\alpha}$$

Use the Euler equation and the equation for the interest rate:

$$c_{t+1} = \beta\,(1+r_{t+1})\,c_t$$
$$c_{t+1} = \beta\alpha k_{t+1}^{\alpha-1} c_t$$
$$(1-s)\,k_{t+1}^{\alpha} = \beta\alpha k_{t+1}^{\alpha-1}\,(1-s)\,k_t^{\alpha}$$
$$k_{t+1} = \beta\alpha k_t^{\alpha}$$

Indeed, the saving rate is a constant:

$$s = \alpha\beta$$

Note that $\partial s/\partial\beta > 0$ and that $s \leq \alpha = s_{GR}$, with equality when $\beta = 1$ (households care just as much about the future and the present). This implies that the Ramsey economy is never dynamically inefficient.

Under our assumptions, we can obtain an analytical, closed form solution of the Ramsey model:

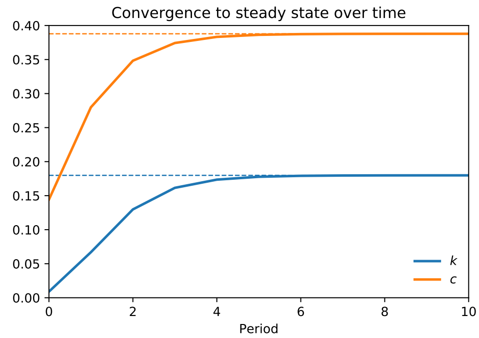
$$c_t = (1-\alpha\beta)\,k_t^{\alpha}$$
$$k_{t+1} = \alpha\beta k_t^{\alpha}$$

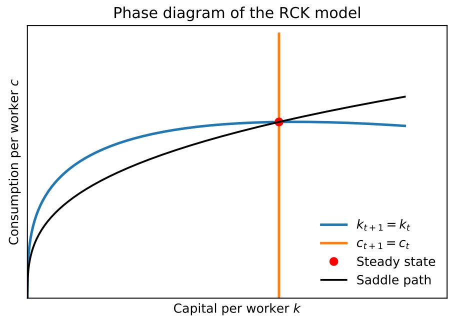
Finally, let us find the steady state of the model:

$$k^* = \alpha\beta\,(k^*)^{\alpha}$$
$$(k^*)^{1-\alpha} = \alpha\beta$$
$$k^* = (\alpha\beta)^{1/(1-\alpha)}$$
$$c^* = (1-\alpha\beta)\,(\alpha\beta)^{\alpha/(1-\alpha)}$$

The first plot below illustrates the process of convergence to the steady state over time. Note that since $\delta = 1$, the convergence is very fast.

The second plot is a new type of plot, called a phase diagram, and will illustrate the convergence to the steady state in the (k, c) space.





The phase diagram shows a number of conditions. The first one, $k_{t+1} = k_t$, describes a set of combinations of k and c for which capital per worker does not change over time. We can find them via the budget/resource constraint:

$$c_t + k_{t+1} = k_t^{\alpha}$$
$$c_t = k_t^{\alpha} - k_{t+1}$$

If $k_{t+1} = k_t = k$:

$$c = k^\alpha - k$$

The second condition, $c_{t+1} = c_t$, describes a set of combinations of k and c for which consumption per worker does not change over time. We can find them via the Euler equation:

$$\begin{aligned}
c_{t+1} &= \beta\left(1 + r_{t+1}\right) c_t \\
1 &= \beta\left(1 + r_{t+1}\right) \\
r_{t+1} &= \frac{1}{\beta} - 1 \\
\alpha k_{t+1}^{\alpha-1} - 1 &= \frac{1}{\beta} - 1 \\
k_{t+1}^{\alpha-1} &= \frac{1}{\alpha\beta} \\
k_{t+1} &= (\alpha\beta)^{1/(1-\alpha)} = k^*
\end{aligned}$$

That means that when the capital per worker is at its steady state level, $c_{t+1} = c_t$ for all c.

Obviously, the intersection of $k_{t+1} = k_t$ and $c_{t+1} = c_t$ is by definition the steady state of the model.

Finally, the "saddle path" is our model's solution. Importantly, there is only one path that leads from any possible level of capital per worker k to the steady state. This guarantees that the model has a unique solution.

1.2 Full model (without technological progress) in discrete time

Households' problem

The representative "dynasty" solves the following problem:

$$\begin{aligned}
\max \quad & U = \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\
\text{subject to} \quad & c_t + (1+n)\, a_{t+1} = w_t + (1 + r_t)\, a_t
\end{aligned}$$

Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - (1+n)\, a_{t+1}\right] \\
&= \ldots + \left(\frac{1+n}{1+\rho}\right)^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left[w_t + (1+r_t)\, a_t - c_t - (1+n)\, a_{t+1}\right] \\
&\quad \ldots + \lambda_{t+1}\left[w_{t+1} + (1+r_{t+1})\, a_{t+1} - c_{t+1} - (1+n)\, a_{t+2}\right] + \ldots
\end{aligned}$$

First order conditions:

$$\begin{aligned}
c_t \quad &: \quad \left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma} - \lambda_t = 0 \qquad \rightarrow \quad \lambda_t = \left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma} \\
a_{t+1} \quad &: \quad -(1+n)\,\lambda_t + \lambda_{t+1}\left(1 + r_{t+1}\right) = 0 \quad \rightarrow \quad \lambda_t = \frac{1+r_{t+1}}{1+n}\lambda_{t+1}
\end{aligned}$$

Derive the Euler equation

$$\begin{aligned}
\left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma} &= \frac{1+r_{t+1}}{1+n}\left(\frac{1+n}{1+\rho}\right)^{t+1} c_{t+1}^{-\sigma} \\
\left(\frac{c_{t+1}}{c_t}\right)^\sigma &= \frac{1+r_{t+1}}{1+\rho} \\
c_{t+1} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\sigma} c_t
\end{aligned}$$

General Equilibrium

Again, since there is no government and no international trade, domestic capital is the only asset which can be in positive net supply:

$$a_t = k_t \quad \text{for all } t$$

And the budget constraint of the households can be rewritten as:

$$\begin{aligned} c_t + (1+n)\, a_{t+1} &= w_t + (1+r_t)\, a_t \\ c_t + (1+n)\, k_{t+1} &= w_t + (1+r_t)\, k_t \end{aligned}$$

We can also use the expressions for prices:

$$\begin{aligned} w_t &= (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha)\, k_t^{\alpha} \\ r_t^k &= \alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \alpha k_t^{\alpha-1} \\ r_t &= r_t^k - \delta = \alpha k_t^{\alpha-1} - \delta \end{aligned}$$

And plug them into the budget constraint:

$$\begin{aligned} c_t + (1+n)\, k_{t+1} &= (1-\alpha)\, k_t^{\alpha} + \left(1 + \alpha k_t^{\alpha-1} - \delta\right) k_t \\ c_t + (1+n)\, k_{t+1} &= (1-\alpha)\, k_t^{\alpha} + \alpha k_t^{\alpha} + (1-\delta)\, k_t \\ (1+n)\, k_{t+1} &= k_t^{\alpha} - c_t + (1-\delta)\, k_t \\ k_{t+1} &= \frac{k_t^{\alpha} - c_t + (1-\delta)\, k_t}{1+n} \end{aligned}$$

The $k_t^{\alpha} - c_t$ part represents gross investment, and is the analogue of sk_t^{α} in the Solow-Swan and OLG models.

Plug in the interest rate into the Euler equation:

$$\begin{aligned} c_{t+1} &= \left(\frac{1 + r_{t+1}}{1+\rho}\right)^{1/\sigma} c_t \\ c_{t+1} &= \left(\frac{1 + \alpha k_{t+1}^{\alpha-1} - \delta}{1+\rho}\right)^{1/\sigma} c_t \end{aligned}$$

Steady state

If $c_{t+1} = c_t = c$, then:

$$\begin{aligned} c &= \left(\frac{1 + \alpha k^{\alpha-1} - \delta}{1+\rho}\right)^{1/\sigma} c \\ 1 + \rho &= 1 + \alpha k^{\alpha-1} - \delta \\ \rho + \delta &= \alpha k^{\alpha-1} \\ k^{1-\alpha} &= \frac{\alpha}{\rho+\delta} \\ k^* &= \left(\frac{\alpha}{\rho+\delta}\right)^{1/(1-\alpha)} \end{aligned}$$

The above gets us the level of capital per worker in the steady state.

If $k_{t+1} = k_t = k$, then:

$$(1+n)\,k = k^{\alpha} - c + (1-\delta)\,k$$
$$c = k^{\alpha} - (\delta + n)\,k$$

And the steady state level of consumption per worker is given by:

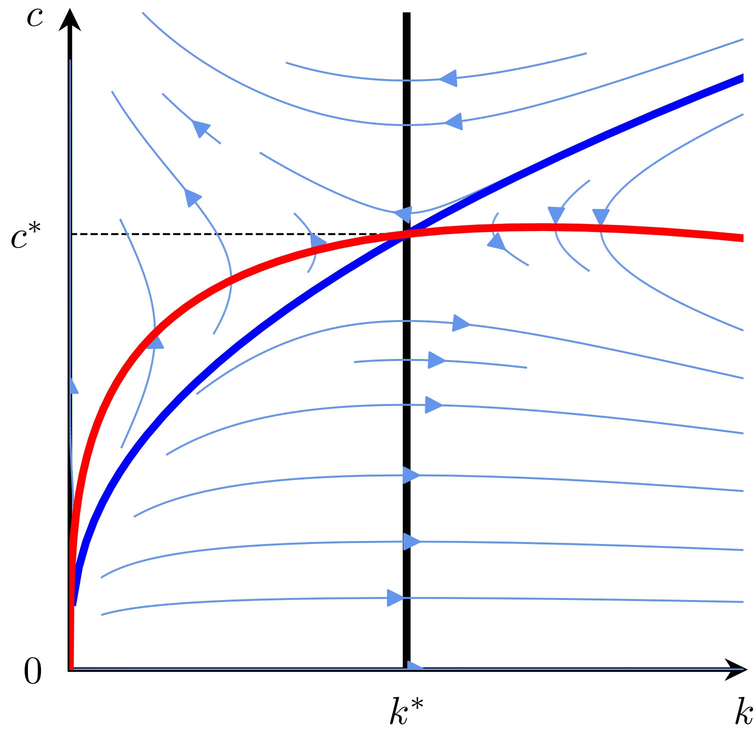
$$c^* = (k^*)^{\alpha} - (\delta + n)\,k^*$$

Transition dynamics (saddle path)

Along the transition to the steady state, the model variables have to obey the following conditions:

$$k_{t+1} = \frac{k_t^{\alpha} - c_t + (1-\delta)\,k_t}{1+n}$$
$$c_{t+1} = \left(\frac{1 + \alpha k_{t+1}^{\alpha-1} - \delta}{1+\rho}\right)^{1/\sigma} c_t$$

Solving the Ramsey model is equivalent to finding, for some initial level of capital per worker k_0, such c_0 that the above forward equations bring k and c to the steady state.



Since the general formulation of the model yields no closed-form solution, the saddle path can be found by using one of many numerical solution procedures. A very simple numerical procedure is the shooting algorithm.