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1 Solow-Swan model

1.1 Introduction

Proposed independently by **Robert Solow** and **Trevor Swan** in 1956. The Solow-Swan model is used as a departure point for the entire field of economic growth.

Key assumptions:

- One sector economy. The economy produces a single good that can be either consumed or invested.
- Output is produced using capital and labor according to a neoclassical production function.
- Markets for the final good and factors of production are perfectly competitive.
- Complete information, no externalities.
- Two types of agents: firms and households.
- Households own capital and rent it to the firms.
- Households save a fixed fraction of their income.
- No government, closed economy.

Three key equations of the Solow-Swan model:

- Capital accumulation equation (accounting identity): $\dot{K}_t = I_t - \delta K_t$
- Neoclassical production function (functional restriction): $Y_t = F(K_t, L_t, A_t)$
- Savings rate is independent of income level (behavioral assumption): $S_t = sY_t$

A consequence of the (No government, closed economy) assumption is that private savings are equal to private investment. This is an accounting identity and says nothing about any causal relationships between savings and investment.

Start from the national accounting identity and assume no government ($G = 0$) and no international trade ($NX = 0$):

$$Y = C + I + \underbrace{G}_{0} + \underbrace{NX}_{0} \quad \rightarrow \quad Y = C + I$$

Then consider the disposition of households' disposable income Y^d between consumption and savings. If there is no government, taxes T are 0 and disposable income is equal to the economy's output.

$$Y^d = C + S \quad \text{and} \quad Y^d = Y - \underbrace{T}_{0} \quad \rightarrow \quad Y = C + S$$

Together those two accounting exercises yield the result that savings are equal to investment:

$$Y = C + I \quad \text{and} \quad Y = C + S \quad \rightarrow \quad S = I$$

1.2 Production function

A production function describes how capital K and labor L are transformed into output Y using technology A, and is written in its general form as $Y_t = F(K_t, L_t, A_t)$. A neoclassical production function satisfies the following properties:

- Continuous and at least twice differentiable.
- Constant returns to scale in K and L. Increasing both capital and labor inputs by a certain proportion translates to an increase of output produced by that same proportion:
$$F(\lambda \cdot K,\, \lambda \cdot L,\, A) = \lambda \cdot F(K, L, A) = \lambda \cdot Y \quad \text{for all } \lambda > 0 \tag{1}$$
- Positive but diminishing marginal products of K and L:
$$\begin{aligned} \frac{\partial F(K,L,A)}{\partial K} &\equiv F_K(K,L,A) > 0 \quad \text{and} \quad \frac{\partial F(K,L,A)}{\partial L} \equiv F_L(K,L,A) > 0 \\ \frac{\partial^2 F(K,L,A)}{\partial K^2} &\equiv F_{KK}(K,L,A) < 0 \quad \text{and} \quad \frac{\partial^2 F(K,L,A)}{\partial L^2} \equiv F_{LL}(K,L,A) < 0 \end{aligned} \tag{2}$$
- Often (but not always) we also impose the following **Inada**'s conditions:
$$\begin{aligned} \lim_{K\to 0} F_K(K,L,A) &= \infty \quad \text{and} \quad \lim_{L\to 0} F_L(K,L,A) = \infty \\ \lim_{K\to \infty} F_K(K,L,A) &= 0 \quad \text{and} \quad \lim_{L\to \infty} F_L(K,L,A) = 0 \end{aligned} \tag{3}$$

The assumption of constant returns to scale has important implications. For example, it makes the number of firms indeterminate, as the economy behaves identically in the case of one firm that employs all resources and in the case of M firms, each employing a $1/M$ fraction of all resources. Therefore we often assume that the firms' sector is represented by a single representative firm.

Definition: function homogenous of degree m

Let $X \in \mathbb{N}$. A function $f : \mathbb{R}^{X+2} \to \mathbb{R}$ is homogenous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if
$$f(\lambda \cdot x,\, \lambda \cdot y,\, z) = \lambda^m \cdot f(x,y,z) \quad \text{for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^X$$
According to the above definition, a neoclassical production function is homogenous of degree one in capital K and labor L.

Theorem: Euler's homogeneous function theorem

Suppose that $f : \mathbb{R}^{X+2} \to \mathbb{R}$ is differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by f_x and f_y, and is homogeneous of degree m in x and y. Then:
$$m\gamma^{m-1} \cdot f(x,y,z) = f_x(x,y,z) \cdot x + f_y(x,y,z) \cdot y \quad \text{for all } x \in \mathbb{R},\, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^X$$
Moreover, $f_x(x,y,z)$ and $f_x(x,y,z)$ are themselves homogeneous of degree $m-1$ in x and y.

Proof

Differentiate the definition for function homogenous of degree m with respect to λ:
$$m\lambda^{m-1} \cdot (x,y,z) = f_x(\lambda \cdot x,\, \lambda \cdot y,\, z) \cdot x + f_y(\lambda \cdot x,\, \lambda \cdot y,\, z) \cdot y$$
Setting $\lambda = 1$ proves the first result of the theorem. To obtain the second result, differentiate the definition for function homogenous of degree m with respect to x:
$$\begin{aligned} f_x(\lambda \cdot x,\, \lambda \cdot y,\, z) \cdot \lambda &= \lambda^m \cdot f_x(x,y,z) \\ f_x(\lambda \cdot x,\, \lambda \cdot y,\, z) &= \lambda^{m-1} \cdot f_x(x,y,z) \end{aligned}$$
In our context, the theorem translates to the following results:
$$F(K,L,A) = F_K(K,L,A) \cdot K + F_L(K,L,A) \cdot L \tag{4}$$
and
$$F_K(\lambda \cdot K,\, \lambda \cdot L,\, A) = F_K(K,L,A) \quad \text{and} \quad F_L(\lambda \cdot K,\, \lambda \cdot L,\, A) = F_L(K,L,A) \tag{5}$$

1.3 Balanced growth path

Consider the fundamental equation of a one-sector closed economy:

$$\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t \tag{6}$$

It turns out that this equation generates a lot of interesting results.

Definition: balanced growth path (BGP)

A balanced growth path is a path $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ along which the quantities Y_t, K_t and C_t are positive and grow at constant rates, which we denote g_Y, g_K and g_C, respectively.

Proposition: equivalence of balanced growth and constancy of key ratios

Let $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t and $I_t = Y_t - C_t$ are positive for all $t \geq 0$. Then, given the capital accumulation equation 6, the following holds:

1. If there is balanced growth, then $g_Y = g_K = g_C = g_I$ and the ratios K/Y, C/Y and I/Y are constant.
2. If K/Y and C/Y are constant, then Y, K, C and I all grow at the same constant rate, i.e. there is not only balanced growth, but balanced growth where $g_Y = g_K = g_C = g_I$.

Note that (1) means that any economy on a balanced growth path has to generate constant key ratios. (2) ensures that if we observe constant ratios, the economy is on a balanced growth path.

Proof

1. Consider an economy on a balanced growth path. Then, by definition, g_Y, g_K and g_C are constant. If we use equation 6, we get the result that $g_I = g_K$:

$$\begin{aligned} \dot{K}_t &= I_t - \delta K_t \quad | \quad : K_t \\ g_K = \frac{\dot{K}_t}{K_t} &= \frac{I_t}{K_t} - \delta \\ \frac{I_t}{K_t} &= g_K + \delta \end{aligned}$$

Since the right hand side of the above equation is constant, then the I/K ratio is constant and that means that the rates of growth of I and K have to be identical.

Focus now on the national accounting relationship:

$$Y_t = C_t + I_t \quad \rightarrow \quad \frac{\mathrm{d}Y_t}{\mathrm{d}t} = \frac{\mathrm{d}C_t}{\mathrm{d}t} + \frac{\mathrm{d}I_t}{\mathrm{d}t} \quad \rightarrow \quad \dot{Y}_t = \dot{C}_t + \dot{I}_t$$

$$\begin{aligned} g_Y &= \frac{\dot{Y}_t}{Y_t} = \frac{\dot{C}_t}{Y_t} + \frac{\dot{I}_t}{Y_t} = \frac{\dot{C}_t}{C_t}\frac{C_t}{Y_t} + \frac{\dot{I}_t}{I_t}\frac{I_t}{Y_t} = g_C\frac{C_t}{Y_t} + g_I\frac{Y_t - C_t}{Y_t} = \frac{C_t}{Y_t}(g_C - g_I) + g_I \\ g_Y &= \frac{C_t}{Y_t}(g_C - g_K) + g_K \end{aligned}$$

We have now two possibilities: either $g_C = g_K$ (and in consequence $g_Y = g_K$) or $g_C \neq g_K$. Let us assume that $g_C \neq g_K$. Then we can rewrite the above expression as:

$$\frac{C_t}{Y_t} = \frac{g_Y - g_K}{g_C - g_K}$$

The right hand side contains only growth rates, which are constant by the definition of the BGP. That means that the C/Y ratio is constant as well, and $g_C = g_Y$. But this implies that the right hand side is equal to 1 and $C_t = Y_t$, which contradicts our assumption that $I_t = Y_t - C_t$ is positive.

That means that our assumption $g_C \neq g_K$ was wrong and so $g_C = g_K$. This also implies that $g_Y = g_K = g_I = g_C$.

2. Suppose that K/Y and C/Y are constant. Then $g_Y = g_K = g_C$ and as a consequence ($I/Y = 1 - C/Y$) $g_Y = g_I = g_K$. Now we need to show that the growth rates are also constant. To that end we use equation 6:

$$\dot{K}_t = I_t - \delta K_t \quad | \quad : K_t$$
$$g_K = \frac{I_t}{K_t} - \delta$$

Since $g_I = g_K$ then I/K is constant and in consequence g_K and other growth rates are constant.

1.4 Technological progress

There are at least three possibilities in how technological progress might manifest itself in our production function. Let $\tilde{F}$ denote the "true" production function and $\tilde{A}$ the "true" form of technological progress:

- Hicks-neutral: technology acts as a multiplicative constant: $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = A_t \cdot F\left(K_t, L_t\right)$
- Solow-neutral: technology increases productivity of capital: $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = F\left(A_t \cdot K_t, L_t\right)$
- Harrod-neutral: technology increases productivity of labor: $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = F\left(K_t, A_t \cdot L_t\right)$

Obviously the technological progress might manifest itself as a combination of the above possibilities: $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = A_{H,t} \cdot F\left(A_{K,t} \cdot K_t, A_{L,t} \cdot L_t\right)$

Despite all of the above forms appearing ex ante plausible, balanced growth requires that the "true" production function has a representation of the Harrod-neutral form.

Theorem: Uzawa's balanced growth path theorem

Let $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$, where $0 < C_t < Y_t$ for all $t \geq 0$ be a path satisfying the capital accumulation equation 6. Suppose also that $\tilde{F}\left(K_t, L_t, \tilde{A}_t\right)$ exhibits constant returns to scale in K and L and worker population grows at a rate n such that $L_t = L_0 \cdot \exp(nt)$.

Then a necessary condition for this path to be a balanced growth path is that:

$$Y_t = \tilde{F}\left(K_t, L_t, \tilde{A}_t\right) = F\left(K_t, A_t \cdot L_t\right) \tag{7}$$

where $A_t = \exp(gt)$ $(g_A = g)$ with $g \equiv g_Y - n$.

Proof

Suppose the path $\{Y_t, K_t, C_t\}_{t=0}^{\infty}$ is a balanced growth path. Then g_Y and g_K are equal and constant. As a consequence, we can write $Y_t = Y_0 \cdot \exp(g_K t)$ and $K_t = K_0 \cdot \exp(g_K t)$. We then have:

$$Y_t \cdot \exp(-g_Y t) = Y_0 = \tilde{F}\left(K_0, L_0, \tilde{A}_0\right) = \tilde{F}\left(K_t \cdot \exp(-g_K t), L_t \cdot \exp(-nt), \tilde{A}_0\right)$$

Because $g_Y = g_K$ and the function $\tilde{F}$ satifies CRS property, we can rewrite the above equation as:

$$Y_t = \tilde{F}\left(K_t \cdot \exp(-g_K t) \cdot \exp(g_Y t), L_t \cdot \exp(-nt) \cdot \exp(g_Y t), \tilde{A}_0\right) = \tilde{F}\left(K_t, L_t \cdot \exp((g_Y - n)t), \tilde{A}_0\right)$$

Note that in the above formulation $\tilde{A}$ is reduced to a constant $\tilde{A}_0$. Therefore, there exists a function $F(K, L)$ such that:

$$Y_t = F\left(K_t, L_t \cdot \exp((g_Y - n)t)\right)$$

which we can rewrite as

$$Y_t = F\left(K_t, A_t \cdot L_t\right)$$

where $A_t = \exp((g_Y - n)t)$ and $g = g_Y - n$ is the rate of "rewritten" technology growth.

Note that it does not mean that literally all technological progress directly increases labor productivity, just that the "true" production function can be rewritten in the required functional form.

1.5 Solution of the Solow-Swan model

Assume that the number of workers L_t, all employed, grows at the same rate as the total population N_t at rate n ($\dot{L}_t/L_t = \dot{N}_t/N_t = n$) and technology A_t improves at rate g ($\dot{A}_t/A_t = g$).

Introduce the following notation:

- capital per worker $k_t = K_t/L_t$
- output per worker $y_t = Y_t/L_t$
- consumption per worker $c_t = C_t/L_t$
- capital per effective units of labor $\hat{k}_t = K_t/\left(A_t L_t\right)$
- output per effective units of labor $\hat{y}_t = Y_t/\left(A_t L_t\right)$
- consumption per effective units of labor $\hat{c}_t = C_t/\left(A_t L_t\right)$

Now recall the set of our three key equations and combine them into a single one:

$$\begin{aligned} \dot{K}_t &= I_t - \delta K_t \\ I_t &= S_t = sY_t \quad \rightarrow \quad \dot{K}_t = sF\left(K_t, A_t L_t\right) - \delta K_t \\ Y_t &= F\left(K_t, A_t L_t\right) \end{aligned}$$

Because both the population and technology change, the aggregate K (and other aggregate variables) will exhibit a trend. The variables that do not change along the balanced growth path are variables per effective units of labor. Therefore, we need to rewrite the above equation in per effective labor terms.

$$\begin{aligned} \dot{K}_t &= sF\left(K_t, A_t L_t\right) - \delta K_t \quad \mid \quad : \left(A_t L_t\right) \\ \frac{\dot{K}_t}{A_t L_t} &= s\frac{F\left(K_t, A_t L_t\right)}{A_t L_t} - \delta\frac{K_t}{A_t L_t} \end{aligned} \tag{8}$$

To find out what $\dot{K}_t/\left(A_t L_t\right)$ is equal to, consider the time derivative of $\hat{k}_t$:

$$\begin{aligned} \dot{\hat{k}}_t &= \frac{\mathrm{d}\hat{k}_t}{\mathrm{d}t} = \frac{\mathrm{d}\left(K_t/\left(A_t L_t\right)\right)}{\mathrm{d}t} = \frac{\frac{\mathrm{d}K_t}{\mathrm{d}t} \cdot \left(A_t L_t\right) - K_t \cdot \frac{\mathrm{d}\left(A_t L_t\right)}{\mathrm{d}t}}{\left(A_t L_t\right)^2} \\ \dot{\hat{k}}_t &= \frac{\dot{K}_t}{A_t L_t} \cdot \frac{A_t L_t}{A_t L_t} - \frac{K_t}{A_t L_t} \cdot \frac{\frac{\mathrm{d}A_t}{\mathrm{d}t} \cdot L_t + A_t \cdot \frac{\mathrm{d}L_t}{\mathrm{d}t}}{A_t L_t} = \frac{\dot{K}_t}{A_t L_t} - \hat{k}_t \cdot \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{L}_t}{L_t}\right) \\ \dot{\hat{k}}_t &= \frac{\dot{K}_t}{A_t L_t} - \hat{k}_t \cdot (g + n) \end{aligned} \tag{9}$$

To find out what $F\left(K_t, A_t L_t\right)/\left(A_t L_t\right)$ is, use the CRS property (equation 1) of the F function:

$$F\left(\lambda \cdot K_t, \lambda \cdot A_t L_t\right) = \lambda \cdot F\left(K_t, A_t L_t\right)$$

Now pick $\lambda = 1/\left(A_t L_t\right)$:

$$\frac{F\left(K_t, A_t L_t\right)}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, \frac{A_t L_t}{A_t L_t}\right) = F(\hat{k}_t, 1) \equiv f(\hat{k}_t) \tag{10}$$

where for notational convenience we introduce a function $f\left(\cdot\right) = F\left(\cdot, 1\right)$.

We can now return to equation 8 and plug in the results from equations 9 and 10:

$$\begin{aligned} \dot{\hat{k}}_t &= \frac{\dot{K}_t}{A_t L_t} - \left(n + g\right)\hat{k}_t = sf(\hat{k}_t) - \delta\hat{k}_t - \left(n + g\right)\hat{k}_t \\ \dot{\hat{k}}_t &= sf(\hat{k}_t) - \left(\delta + n + g\right)\hat{k}_t \end{aligned} \tag{11}$$

The above equation is sometimes called the fundamental equation of the Solow-Swan model.

1.5.1 Behavior of factor prices

In the Solow-Swan economy, factor prices (that is, wages and the interest rate) are determined by the relative availability of capital and labor. To see this relationship more clearly, let us solve the problem of profit maximization by the representative firm:

$$\max \quad F\left(K_t, A_t L_t\right) - w_t L_t - \left(r_t + \delta\right) K_t$$

It is here more convenient to rewrite the problem in the "per effective labor" form:

$$\max \quad A_t L_t \left[f(\hat{k}_t) - \hat{w}_t - \left(r_t + \delta\right) \hat{k}_t \right]$$

where $\hat{w}_t \equiv w_t / A_t$. First order conditions:

$$\begin{aligned}
\hat{k}_t \quad &: \quad A_t L_t \left[f'(\hat{k}_t) - (r_t + \delta) \right] = 0 && \rightarrow \quad r_t = f'(\hat{k}_t) - \delta \\
L_t \quad &: \quad A_t \left[f(\hat{k}_t) - \hat{w}_t - (r_t + \delta)\, \hat{k}_t \right] = 0 && \rightarrow \quad \hat{w}_t = f(\hat{k}_t) - (r_t + \delta)\, \hat{k}_t = f(\hat{k}_t) - \hat{k}_t \cdot f'(\hat{k}_t)
\end{aligned}$$

The interest rate is decreasing in capital per effective labor, while the "effective" wage $\hat{w}$ is increasing in capital per effective labor.[1] Along the balanced growth path, interest rate is predicted to be constant, while the real wage is predicted to grow at the rate of technological progress.

1.5.2 Steady state (balanced growth path)

We can now find the steady state of the fundamental equation 11 which will determine the balanced growth path level of $\hat{k}^*$. Set $\dot{\hat{k}}_t = 0$:

$$\begin{aligned}
0 &= s f(\hat{k}^*) - (\delta + n + g)\, \hat{k}^* \\
\frac{\hat{k}^*}{f(\hat{k}^*)} &= \frac{s}{\delta + n + g}
\end{aligned} \tag{12}$$

We can show that the left hand side of the above equation increases with $\hat{k}^*$:

$$\frac{\mathrm{d}[\hat{k}/f(\hat{k})]}{\mathrm{d}\hat{k}} = \frac{f(\hat{k}) - \hat{k} \cdot f'(\hat{k})}{[f(\hat{k})]^2} = \frac{\hat{w}(\hat{k})}{[f(\hat{k})]^2} > 0$$

And so we can conclude that the steady state level of $\hat{k}$ increases with savings rate s, and decreases with depreciation rate δ, population growth n and technology growth g.

Example (Cobb-Douglas production function)

Let $Y_t = K_t^\alpha \left(A_t L_t\right)^{1-\alpha}$. Then:

$$f(\hat{k}_t) = \frac{K_t^\alpha \left(A_t L_t\right)^{1-\alpha}}{A_t L_t} = \frac{K_t^\alpha \left(A_t L_t\right)^{1-\alpha}}{\left(A_t L_t\right)^\alpha \cdot \left(A_t L_t\right)^{1-\alpha}} = \left(\frac{K_t}{A_t L_t}\right)^\alpha = \hat{k}_t^\alpha$$

$$\frac{\hat{k}^*}{(\hat{k}^*)^\alpha} = \frac{s}{\delta + n + g} \quad \rightarrow \quad \hat{k}^* = \left(\frac{s}{\delta + n + g}\right)^{1/(1-\alpha)}$$

$$\hat{y}^* = f(\hat{k}^*) = \left(\frac{s}{\delta + n + g}\right)^{\alpha/(1-\alpha)}$$

$$\hat{c}^* = (1 - s)\, \hat{y}^* = (1 - s) \left(\frac{s}{\delta + n + g}\right)^{\alpha/(1-\alpha)}$$

[1] To see this, take the derivative of the effective wage with respect to capital per effective labor:

$$\partial \hat{w} / \partial \hat{k} = f'(\hat{k}) - \left[f'(\hat{k}) + \hat{k} \cdot f''(\hat{k}) \right] = -\hat{k} \cdot f''(\hat{k}) > 0$$

The above expression is positive, since the production function has the property of diminishing marginal products (eq. 2).

Rates of growth of variables in the steady state (along the BGP)

By definition, variables per effective labor do not change along the BGP, i.e. $g^*_{\hat{k}} = g^*_{\hat{y}} = g^*_{\hat{c}} = 0$. Using our definitions of variables per effective labor and per worker we can easily obtain the rates of growth of variables per worker (for example consumption):

$$\begin{aligned} c_t &= A_t \cdot \hat{c}_t \quad \Big| \quad \frac{\mathrm{d}}{\mathrm{d}t} \\ \dot{c}_t &= \dot{A}_t \cdot \hat{c}_t + A_t \cdot \dot{\hat{c}}_t \quad \Big| \quad : c_t \\ g_c = \frac{\dot{c}_t}{c_t} &= \frac{\dot{A}_t \cdot \hat{c}_t}{A_t \cdot \hat{c}_t} + \frac{A_t \cdot \dot{\hat{c}}_t}{A_t \cdot \hat{c}_t} = g_A + g_{\hat{c}} \\ g^*_c &= g_A + g^*_{\hat{c}} = g \end{aligned}$$

Along the BGP all per worker variables will grow at a rate equal to the rate of technological progress. This is a remarkable result as it implies that in the (very) long run the only factor responsible for the improvement of our lives is the rate of improvement in labor productivity. So if we want to make future generations better off we should focus on inventing new and/or better ways of production.

The BGP rate of growth of aggregate variables is also easy to find:

$$\begin{aligned} C_t &= c_t \cdot L_t \\ g_C &= g_c + g_L \\ g^*_C &= g + n \end{aligned}$$

1.5.3 Transition dynamics

Outside the balanced growth path the economy is in the process of convergence to the BGP. That means that if, say, $\hat{k}_t < \hat{k}^*$ then $g_{\hat{k}} > 0$. As a consequence, both per worker and aggregate variables will also grow faster than g and $g + n$, respectively. We describe this as the "catching-up process", as the economies that are further away from their steady states grow faster than the ones that are closer to theirs.

The general formula for the rate of growth of capital per effective worker can be obtained from the fundamental equation 11:

$$\begin{aligned} \dot{\hat{k}}_t &= sf(\hat{k}) - (\delta + n + g)\,\hat{k}_t \\ g_{\hat{k}} = \frac{\dot{\hat{k}}_t}{\hat{k}} &= s\frac{f(\hat{k})}{\hat{k}_t} - (\delta + n + g) \end{aligned} \tag{13}$$

Recall that we have shown that $\hat{k}/f(\hat{k})$ increases with $\hat{k}$, which means that $f(\hat{k})/\hat{k}$ has to decrease with $\hat{k}$. Again, that means that if $\hat{k}_t < \hat{k}^*$ then $g_{\hat{k}} > 0$ (and if $\hat{k}_t > \hat{k}^*$ then $g_{\hat{k}} < 0$). We would also like to assign some numerical values for the rates of growth outside the BGP.

Example (Cobb-Douglas production function)

Let $f(\hat{k}) = \hat{k}_t^{\alpha}$. Then the rate of growth of capital per effective labor is given by:

$$g_{\hat{k}} = s\hat{k}_t^{\alpha-1} - (\delta + n + g)$$

But usually we are much more interested in the rate of growth of output per worker. After some algebra the formula one can obtain the following formula relating the growth rate of output per worker to the distance of the economy to its steady state:

$$g_y \approx g - (1 - \alpha)\,(\delta + n + g)\,(\ln y_t - \ln y^*_t)$$

The above equation is very often used in empirical research.