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1 Investment

1.1 Neoclassical investment theory

Households

Assume that the economy is populated by N identical, infinitely lived households that each supply one unit of labor and solve the following problem:

$$\begin{aligned} \max \quad & U=\sum_{t=0}^{\infty} \beta^t \cdot u\left(c_t\right) \\ \text{subject to} \quad & c_t+b_{t+1}+p_t s_{t+1}=w_t+(1+r) b_t+\left(d_t+p_t\right) s_t \quad \text{for all } t=0,1,\ldots,\infty \\ & b_0,\ s_0 \quad \text{given} \end{aligned}$$

where b denotes corporate bonds that pay real interest rate r, w denotes the real wage that the household receives for supplying labor, p is the price of one share of a firm, s is the number of shares owned by the household and d is the dividend per share paid by the firm. We assume that the dividends are paid at the end of a time period and trade in firm shares takes place immediately after.

Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \cdot u\left(c_t\right)+\sum_{t=0}^{\infty} \lambda_t\left[w_t+(1+r) b_t+\left(d_t+p_t\right) s_t-c_t-b_{t+1}-p_t s_{t+1}\right] \\ &= \ldots+\beta^t \cdot u\left(c_t\right)+\lambda_t\left[w_t+(1+r) b_t+\left(d_t+p_t\right) s_t-c_t-b_{t+1}-p_t s_{t+1}\right] \\ &\quad +\lambda_{t+1}\left[w_{t+1}+(1+r) b_{t+1}+\left(d_{t+1}+p_{t+1}\right) s_{t+1}-c_{t+1}-b_{t+2}-p_t s_{t+2}\right]+\ldots \end{aligned}$$

First order conditions:

$$\begin{aligned} c_t &: \quad \beta^t u'\left(c_t\right)-\lambda_t=0 & \rightarrow \quad & \lambda_t=\beta^t u'\left(c_t\right) \\ b_{t+1} &: \quad -\lambda_t+\lambda_{t+1}(1+r)=0 & \rightarrow \quad & \lambda_t=\lambda_{t+1}(1+r) \\ s_{t+1} &: \quad -\lambda_t p_t+\lambda_{t+1}\left(d_{t+1}+p_{t+1}\right)=0 & \rightarrow \quad & \lambda_t p_t=\lambda_{t+1}\left(d_{t+1}+p_{t+1}\right) \end{aligned}$$

Combining the FOCs for consumption and bonds, we get the usual Euler equation:

$$u'\left(c_t\right)=\beta(1+r) u'\left(c_{t+1}\right)$$

Combining the FOCs for bonds and shares, we get the fundamental pricing equation:

$$p_t=\frac{d_{t+1}+p_{t+1}}{1+r}$$

Denote with S the entire stock of firm shares. Then, total dividend is given by $D=d \cdot S$ and total market value of the firm is given by $V=p \cdot S$. The fundamental pricing equation can be rewritten as:

$$V_t=p_t S=\frac{d_{t+1} S+p_{t+1} S}{1+r}=\frac{D_{t+1}+V_{t+1}}{1+r}$$

By iterating the above formula ad infinitum one concludes that the fundamental value of the firm can be expressed as the PDV sum of the future dividend flows:

$$V_t=\sum_{i=1}^{\infty} \frac{D_{t+i}}{(1+r)^i}$$

provided that the value of the firm does not increase faster than the PDV discount factor:

$$\lim_{i \rightarrow \infty} \frac{V_{t+i}}{(1+r)^i}=0$$

Firm

A single, representative firm converts inputs (capital K and labor L) into output Y according to the neoclassical production function (where A is a measure of productivity/technology):

$$Y_t = A_t F\left(K_t, L_t\right)$$

We assume that the firm buys investment goods to increase its capital stock. Thus their dividend flow can be expressed as:

$$D_t = A_t F\left(K_t, L_t\right) - w_t L_t - I_t + B_{t+1} - \left(1+r\right) B_t$$

where I is investment, and B are the corporate bonds issued by the firm. The capital stock of the firm follows the capital accumulation equation (where δ denotes the depreciation rate):

$$K_{t+1} = \left(1-\delta\right) K_t + I_t$$

We assume that firm managers want to maximize the current dividend flow and value of the firm, which is consistent with shareholders' preferences. Since the current value of the firm is the PDV sum of future dividend flows, the objective function is then the NPV sum of current and future profit flows:

$$\begin{aligned}
\max \quad & \left(D_t + V_t\right) = \sum_{i=0}^{\infty} \frac{1}{\left(1+r\right)^i} \left[A_{t+i} F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - I_{t+i} + B_{t+i+1} - \left(1+r\right) B_{t+i}\right] \\
\text{subject to} \quad & K_{t+i+1} = \left(1-\delta\right) K_{t+i} + I_{t+i} \quad \text{for all } i = 0, 1, \ldots, \infty \\
& K_t \quad \text{given}
\end{aligned}$$

Lagrangian (note that the Lagrange multiplier q is already discounted):

$$\begin{aligned}
\mathcal{L} &= \sum_{i=0}^{\infty} \frac{1}{\left(1+r\right)^i} \begin{bmatrix} A_{t+i} F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - I_{t+i} + B_{t+i+1} - \left(1+r\right) B_{t+i} \\ + q_{t+i}\left[\left(1-\delta\right) K_{t+i} + I_{t+i} - K_{t+i+1}\right] \end{bmatrix} \\
&= A_t F\left(K_t, L_t\right) - w_t L_t - I_t + B_{t+1} - \left(1+r\right) B_t + q_t\left[\left(1-\delta\right) K_t + I_t - K_{t+1}\right] \\
&+ \frac{1}{\left(1+r\right)} \left[A_{t+1} F\left(K_{t+1}, L_{t+1}\right) - w_{t+1} L_{t+1} - I_{t+1} + B_{t+2} - \left(1+r\right) B_{t+1}\right] \\
&+ \frac{1}{\left(1+r\right)} q_{t+1}\left[\left(1-\delta\right) K_{t+1} + I_{t+1} - K_{t+2}\right] + \ldots
\end{aligned}$$

First order conditions:

$$\begin{aligned}
L_t &: \quad A_t F_L\left(K_t, L_t\right) - w_t = 0 && \rightarrow \quad w_t = A_t F_L\left(K_t, L_t\right) \\
I_t^n &: \quad -1 + q_t = 0 && \rightarrow \quad q_t = 1 \\
B_{t+1} &: \quad 1 + \frac{1}{1+r}\left[-\left(1+r\right)\right] = 0 && \rightarrow \quad 1 = \frac{1+r}{1+r} \\
K_{t+1} &: \quad -q_t + \frac{1}{1+r}\left[A_{t+1} F_K\left(K_{t+1}, L_{t+1}\right) + \left(1-\delta\right) q_{t+1}\right] = 0 \\
& \quad \hookrightarrow \quad q_t = \frac{1}{1+r}\left[A_{t+1} F_K\left(K_{t+1}, L_{t+1}\right) + \left(1-\delta\right) q_{t+1}\right]
\end{aligned}$$

where F_L and F_K denote the derivatives of the production function with respect to employment and capital, respectively.

The FOC for employment says that in optimum the firm should equate the marginal product of labor with the per-worker wages. The FOC for net investment says that the Lagrange multiplier q is always equal to 1, which can be interpreted that the cost of a marginal unit of investment is exactly equal to the NPV sum of additional future profits generated by this extra investment. The FOC for firm bonds is satisfied always, independently from the level of B. That means that any amount of firm debt is consistent with profit-maximizing behavior (i.e. leverage does not matter) and the firm can finance investment equally well either through debt or through retained earnings, because the internal and external cost of capital are equal. This is a version of the **Modigliani-Miller (1958) theorem**.

Since $q_t = q_{t+1} = 1$, we can rewrite the FOC for capital as:[1]

$$r + \delta = A_{t+1} F_K \left(K_{t+1}, L_{t+1}\right)$$

which says that the marginal product of capital has to equal its acquisition cost (interest rate) plus depreciation cost.

If we assume for simplicity full employment and constant population ($L_t = L_{t+1} = N$), the desired capital stock can be expressed as a function of the interest rate and the technology variable, $K^*_{t+1} = K\left(A_{t+1}, r\right)$ and net investment equals $I^n_t = K^*_{t+1}\left(A_{t+1}, r\right) - K_t$. Thus, if the technology improves, net investment will be positive. Moreover, if interest rates increase (decrease), investment decreases (increases).

The simple neoclassical setup has a problem, in that it predicts very large (infinite in the continuous time version of the model) fluctuations of investment in response to change in interest rates. To amend this, we turn to the q theory of investment.

1.2 Capital adjustment costs: q theory of investment

Previously we have assumed that the only cost related with new capital is its acquisition cost. Here we will assume that there is an adjustment cost, which will be proportional to the investment/capital stock ratio. For simplicity, we will assume that the depreciation rate is 0:

$$D_t = A_t F\left(K_t, L_t\right) - w_t L_t - I_t \left(1 + \frac{\chi}{2}\frac{I_t}{K_t}\right) + B_{t+1} - (1+r) B_t$$

Lagrangian:

$$\begin{aligned}
\mathcal{L} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} & \left[A_{t+i} F\left(K_{t+i}, L_{t+i}\right) - w_{t+i} L_{t+i} - I_{t+i}\left(1 + \frac{\chi}{2}\frac{I_{t+i}}{K_{t+i}}\right) + B_{t+i+1} - (1+r) B_{t+i} \right. \\
& \left. + q_{t+i}\left[K_{t+i} + I_{t+i} - K_{t+i+1}\right] \right] \\
= {} & A_t F\left(K_t, L_t\right) - w_t L_t - I_t - \frac{\chi}{2}\frac{I_t^2}{K_t} + B_{t+1} - (1+r) B_t + q_t\left[K_t + I_t - K_{t+1}\right] \\
& + \frac{1}{(1+r)}\left[A_{t+1} F\left(K_{t+1}, L_{t+1}\right) - w_{t+1} L_{t+1} - I_{t+1} - \frac{\chi}{2}\frac{I_{t+1}^2}{K_{t+1}} + B_{t+2} - (1+r) B_{t+1}\right] \\
& + \frac{1}{(1+r)} q_{t+1}\left[K_{t+1} + I_{t+1} - K_{t+2}\right] + \dots
\end{aligned}$$

First order conditions:

$$\begin{aligned}
L_t &: \quad A_t F_L\left(K_t, L_t\right) - w_t = 0 \\
I_t &: \quad -1 - \chi\frac{I_t}{K_t} + q_t = 0 \\
B_{t+1} &: \quad 1 + \frac{1}{1+r}\left[-(1+r)\right] = 0 \\
K_{t+1} &: \quad -q_t + \frac{1}{1+r}\left[A_{t+1} F_K\left(K_{t+1}, L_{t+1}\right) + \frac{\chi}{2}\left(\frac{I_{t+1}}{K_{t+1}}\right)^2 + q_{t+1}\right] = 0
\end{aligned}$$

Since the FOCs for employment and bonds are the same as previously, we can focus solely on the changes in FOCs for investment and capital stock.

[1] If we allowed for the price of capital goods p^K to differ from the price of final goods, we would arrive at the following formula, which takes into account the possibility of change in price of capital goods over time:

$$A \cdot F_K\left(K, L\right) = p^K\left(r + \delta - \Delta p^K / p^K\right)$$

We can see now that q is not always equal to 1 but is related to investment/capital ratio. We can express q as the PDV sum of future marginal products of capital net of depreciation and gains from increased capital stock which makes future investment less costly:

$$q_t = \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} \left[A_{t+i} F_K \left(K_{t+i}, L_{t+i} \right) + \frac{\chi}{2} \left(\frac{I_{t+i}}{K_{t+i}} \right)^2 \right]$$

Since the system is now dynamic, as decisions in the current period will affect the future (unlike in the neoclassical example), we will be interested in two objects: the steady state of the system and its dynamics. To do that, we shall find the expressions for $\Delta K_{t+1} = K_{t+1} - K_t$ and $\Delta q_{t+1} = q_{t+1} - q_t$. For simplicity, assume $A_t = A_{t+1} = A$ and $L_t = L_{t+1} = L$. Summarizing our knowledge thus far, we have:

$$\begin{aligned} q_t &= 1 + \chi \frac{I_t}{K_t} \\ K_{t+1} &= K_t + I_t \\ q_t &= \frac{1}{1+r} \left[A F_K \left(K_{t+1}, L \right) + \frac{\chi}{2} \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 + q_{t+1} \right] \end{aligned}$$

We can rewrite the above conditions in the difference equations form:

$$\begin{aligned} q_t &= 1 + \chi \frac{I_t}{K_t} \\ \Delta K_{t+1} &= I_t \\ \Delta q_t &= r q_t - A F_K \left(K_{t+1}, L \right) + \frac{\chi}{2} \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 \end{aligned}$$

To simplify the problem further, we will rewrite the above equations in continuous time, which allows us to "forget" about the time subscripts:

$$\begin{aligned} q &= 1 + \chi \frac{I}{K} \\ \dot{K} &= I \\ \dot{q} &= r q - \left[A F_K \left(K, L \right) + \frac{\chi}{2} \left(\frac{I}{K} \right)^2 \right] \end{aligned}$$

And get rid of I by substituting:

$$I = \frac{q-1}{\chi} K$$

Then:

$$\begin{aligned} \dot{K} &= \frac{q-1}{\chi} K \\ \dot{q} &= r q - \left[A F_K \left(K, L \right) + \frac{\chi}{2} \left(\frac{q-1}{\chi} \right)^2 \right] \end{aligned}$$

The steady state of the system is found by setting $\dot{K}$ and $\dot{q}$ to 0:

$$\begin{aligned} &\dot{K} = 0 \quad \rightarrow \quad \frac{q-1}{\chi} K = 0 \quad \rightarrow \quad q^* = 1 \\ &\dot{q} = 0 \quad \rightarrow \quad r q = A F_K \left(K, L \right) + \frac{(q-1)^2}{2\chi} \quad \underbrace{\rightarrow}_{q=q^*=1} \quad r = A F_K \left(K, L \right) \end{aligned}$$

The steady state replicates the neoclassical case.

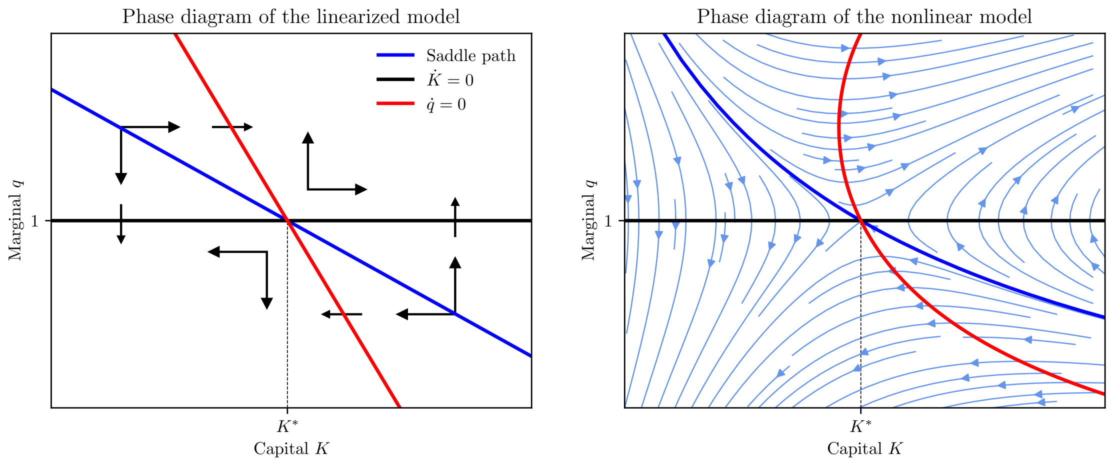
Now we turn to the dynamics of the system. Since the system is nonlinear and in general has no analytical solution, we will examine its linear approximation around the steady state by using a first-order Taylor expansion of a function of two variables (q and K):

$$\dot{K} \approx \frac{1}{\chi} K^* \left(q - q^*\right) + \frac{q^* - 1}{\chi} \left(K - K^*\right) = \frac{1}{\chi} K^* \left(q - 1\right)$$

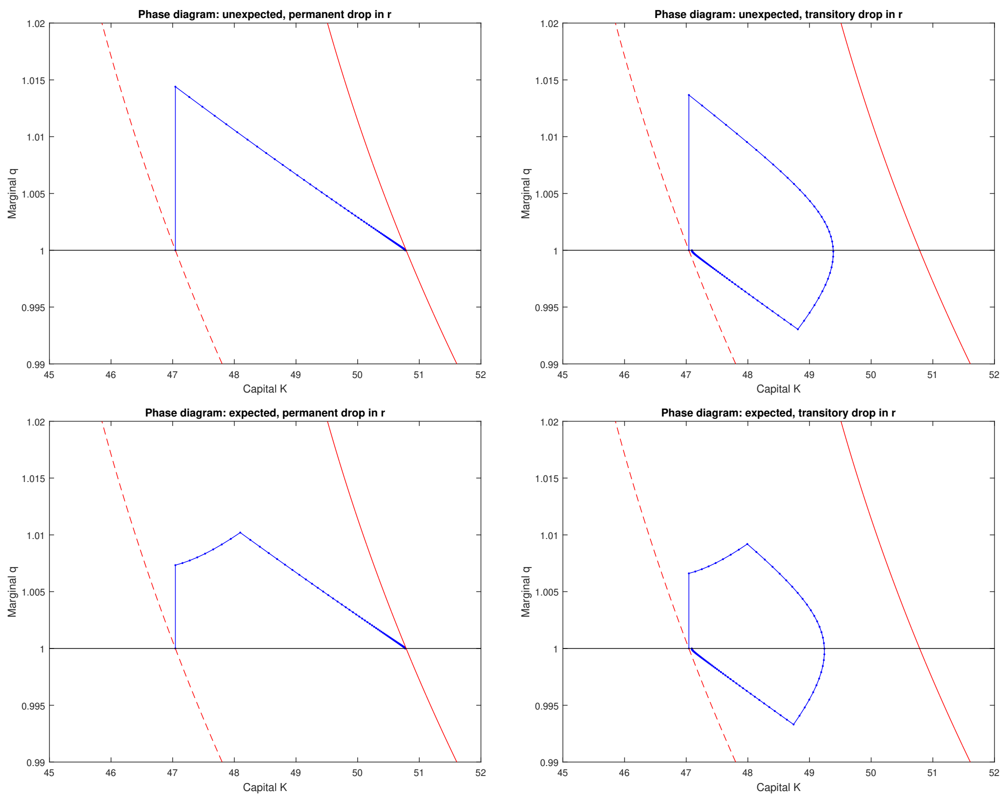
$$\dot{q} \approx \left(r - \frac{q^* - 1}{\chi}\right) \left(q - q^*\right) - AF_{KK} \left(K^*, L\right) \left(K - K^*\right) = r \left(q - 1\right) - AF^*_{KK} \left(K - K^*\right)$$

The $\dot{q}$ function is downward sloping in the (K, q) space (at least in the neighborhood of the steady state), because the neoclassical production function has the property that $F_{KK} < 0$.

We can summarize our considerations on a phase diagram:



We can also analyze the dynamic effects of changes in the interest rate r:



1.3 Tobin's (1969) q and the stock market value

From the point of view of the firm's manager, since q_t is the shadow price of capital installed at the end of period t in the firm, a natural approach to value the firm is $V_t = q_t K_{t+1}$. From the outsiders standpoint, q is unobservable. But we have some expectations about future dividend flows, reflected in the stock-market valuation of the firm. It turns out that we can get useful information from the stock market value. Let us start with the expression for q_t:

$$\begin{aligned}
q_t &= \frac{1}{1+r}\left[A_{t+1}F_K\left(K_{t+1},L_{t+1}\right)+\frac{\chi}{2}\left(\frac{I_{t+1}}{K_{t+1}}\right)^2+q_{t+1}\right] \quad | \quad \cdot K_{t+1}=K_{t+2}-I_{t+1} \\
q_tK_{t+1} &= \frac{1}{1+r}\left[A_{t+1}F_K\left(K_{t+1},L_{t+1}\right)K_{t+1} = +\frac{\chi}{2}\frac{I_{t+1}^2}{K_{t+1}}+q_{t+1}\left(K_{t+2}-I_{t+1}\right)\right] \\
q_tK_{t+1} &= \frac{1}{1+r}\left[A_{t+1}F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}+\frac{\chi}{2}\frac{I_{t+1}^2}{K_{t+1}}+q_{t+1}K_{t+2}-I_{t+1}\left(1+\chi\frac{I_{t+1}}{K_{t+1}}\right)\right] \\
q_tK_{t+1} &= \frac{1}{1+r}\left[A_{t+1}F_K\left(K_{t+1},L_{t+1}\right)K_{t+1}-I_{t+1}\left(1+\frac{\chi}{2}\frac{I_{t+1}}{K_{t+1}}\right)+q_{t+1}K_{t+2}\right] \\
q_tK_{t+1} &= \sum_{i=1}^{\infty}\frac{1}{(1+r)^i}\left[A_{t+i}F_K\left(K_{t+i},L_{t+i}\right)K_{t+i}-I_{t+i}\left(1+\frac{\chi}{2}\frac{I_{t+i}}{K_{t+i}}\right)\right]
\end{aligned}$$

Recall that:

$$V_t = \sum_{i=1}^{\infty}\frac{1}{(1+r)^i}\left[A_{t+i}F_K\left(K_{t+i},L_{t+i}\right)-w_{t+i}L_{t+i}-I_{t+i}\left(1+\frac{\chi}{2}\frac{I_{t+i}}{K_{t+i}}\right)+B_{t+i+1}-(1+r)B_{t+i}\right]$$

We can without loss of generality ignore B and focus on the case where the firm never issues bonds (recall the Modigliani-Miller theorem). We will also make use of the properties of neoclassical production functions and of the FOC for employment:

$$AF(K,L) = AF_K(K,L)\cdot K + AF_L(K,L)\cdot L \quad \rightarrow \quad AF_K(K,L)\cdot K = AF(K,L) - wL$$

Then we can write:

$$q_tK_{t+1} = \sum_{i=1}^{\infty}\frac{1}{(1+r)^i}\left[A_{t+i}F_K\left(K_{t+i},L_{t+i}\right)-w_{t+i}L_{t+i}-I_{t+i}\left(1+\frac{\chi}{2}\frac{I_{t+i}}{K_{t+i}}\right)\right]$$

So it would appear that we could extract q_t using the formula: $q_t = V_t/K_{t+1}$. Is that true?

1.4 Hayashi's (1982) theorem

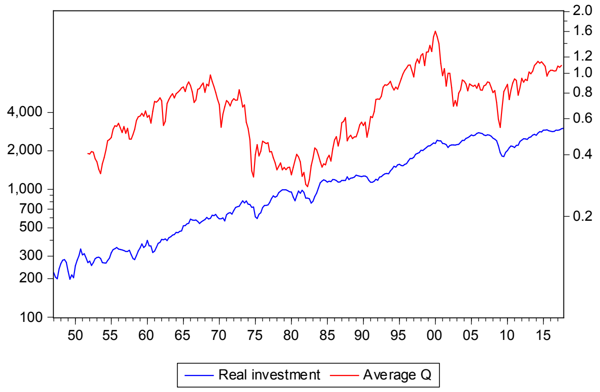
In general, V_t/K_{t+1} is the average Q, not marginal q. However, under certain (very restrictive) assumptions those two concepts coincide:

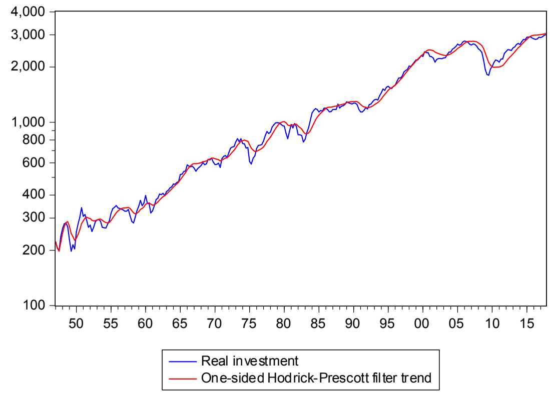
1. Production function and total adjustment cost function exhibit constant returns to scale.
2. Capital goods are homogeneous.
3. Stock market is efficient (uses fundamental pricing).

If the above assumptions are satisfied, the firm should invest whenever $V/K > 1$ and disinvest when $V/K < 1$.

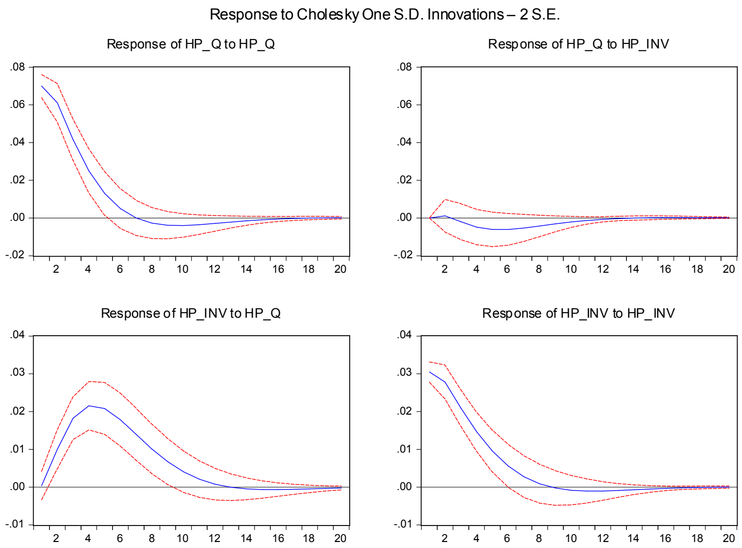
1.5 Using average Q to forecast real aggregate investment

Even if the assumptions of the Hayashi's theorem are not satisfied, average Q does provide information that can be used in forecasting investment. The following graph shows the relevant time series for the US at quarterly frequency[2]. As can be easily seen, real investment is increasing over time and average Q is usually far from 1. However, one can notice that significant drops in Q usually translate to drops in investment a few quarters later. To perform formal analysis, I first detrend (the logarithms of) both time series using the one-sided **Hodrick-Prescott (1997) filter**. This filter can be thought of as a special moving average filter that isolates deviations from trend at business cycle frequency:





Next I set up a simple VAR model on HP-deviations from trend of investment and Q. Using a variety of lag selection criteria I decide to include two lags. Then I estimate the VAR model and produce the Impulse Response Functions plot, seen on the right. The HP-deviations of investment and average Q exhibit significant autocorrelation. While the shocks to investment do not translate to the changes in Q, shocks to Q indeed translate to changes in investment, influencing it over the horizon of up to 3 years, and having the strongest impact for 4-6 quarters after the initial shock to Q.



[2]US real investment data can be downloaded from **https://fred.stlouisfed.org/series/GPDIC1**. Average Q is approximated by the ratio between **nonfinancial corporate business equities** and **net worth**.

Finally, I use my VAR model to produce one quarter ahead forecasts for HP-deviations of investment, which I combine with the previously isolated HP trend. Basing on the data up to 2017Q3, I forecasted that US investment in 2017Q4 would be equal to 3009 billions of 2009 dollars, with the standard deviation of forecast error of 3%. The actual 2017Q4 real investment was equal to 3002 billions of 2009 dollars and the actual forecast error was smaller than 0.3%.