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# 1 Dynamic consumption choice: multiple periods

## 1.1 Three-period optimization

An agent solves the following utility maximization problem:

$$\begin{aligned}
\max_{c_1, c_2, c_3, a_2, a_3} \quad & U = \ln c_1 + \beta \ln c_2 + \beta^2 \ln c_3 \\
\text{subject to} \quad & c_1 + a_2 = y_1 + (1+r)\, a_1 \\
& c_2 + a_3 = y_2 + (1+r)\, a_2 \\
& c_3 = y_3 + (1+r)\, a_3
\end{aligned}$$

Note that we have introduced a timing convention of agent's assets that seems counterintuitive at first: we denote assets available at the beginning of period $t$ with index $t$, while assets chosen at the end of period $t$ are assigned index $t+1$. The advantage of this notational convention will become obvious later.

First, combine the period-specific budget constraints into a lifetime budget constraint:

$$\begin{aligned}
c_3 &= y_3 + (1+r)\, a_3 \\
a_3 &= \frac{c_3 - y_3}{1+r} \\
c_2 + a_3 &= y_2 + (1+r)\, a_2 \\
a_2 &= \frac{c_2 - y_2}{1+r} + \frac{a_3}{1+r} \\
a_2 &= \frac{c_2 - y_2}{1+r} + \frac{c_3 - y_3}{(1+r)^2} \\
c_1 + a_2 &= y_1 + (1+r)\, a_1 \\
c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} &= (1+r)\, a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2}
\end{aligned}$$

Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \ln c_1 + \beta \ln c_2 + \beta^2 \ln c_3 \\
& + \lambda_1 \left[y_1 + (1+r)\, a_1 - c_1 - a_2\right] \\
& + \lambda_2 \left[y_2 + (1+r)\, a_2 - c_2 - a_3\right] \\
& + \lambda_3 \left[y_3 + (1+r)\, a_3 - c_3\right]
\end{aligned}$$

The Lagrangian involves five choice variables: $c_1$, $c_2$, $c_3$, $a_2$ and $a_3$. Therefore, we will need to calculate five first order conditions.

First order conditions (FOCs):

$$\begin{aligned}
c_1 &: \quad \frac{1}{c_1} - \lambda_1 = 0 && \rightarrow \quad \lambda_1 = \frac{1}{c_1} \\
c_2 &: \quad \frac{\beta}{c_2} - \lambda_2 = 0 && \rightarrow \quad \lambda_2 = \beta \frac{1}{c_2} \\
c_3 &: \quad \frac{\beta^2}{c_3} - \lambda_3 = 0 && \rightarrow \quad \lambda_3 = \beta^2 \frac{1}{c_3} \\
a_2 &: \quad -\lambda_1 + (1+r)\, \lambda_2 = 0 && \rightarrow \quad \lambda_1 = (1+r)\, \lambda_2 \\
a_3 &: \quad -\lambda_2 + (1+r)\, \lambda_3 = 0 && \rightarrow \quad \lambda_2 = (1+r)\, \lambda_3
\end{aligned}$$

Resulting optimality conditions (Euler equations):

$$\begin{aligned}
\lambda_1 &= (1+r)\,\lambda_2 \\
\frac{1}{c_1} &= (1+r)\,\beta\frac{1}{c_2} \\
c_2 &= \beta\,(1+r)\,c_1
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= (1+r)\,\lambda_3 \\
\lambda_1 &= (1+r)^2\,\lambda_3 \\
\frac{1}{c_1} &= (1+r)^2\,\beta^2\frac{1}{c_3} \\
c_3 &= \beta^2\,(1+r)^2\,c_1
\end{aligned}$$

Plug the Euler equations into the lifetime budget constraint:

$$\begin{aligned}
c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} &= (1+r)\,a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \\
c_1 + \frac{\beta\,(1+r)\,c_1}{1+r} + \frac{\beta^2\,(1+r)^2\,c_1}{(1+r)^2} &= (1+r)\,a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \\
c_1 + \beta c_1 + \beta^2 c_1 &= (1+r)\,a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2} \\
c_1\left(1+\beta+\beta^2\right) &= (1+r)\,a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2}
\end{aligned}$$

Optimal levels of consumption:

$$\begin{aligned}
c_1 &= \frac{1}{1+\beta+\beta^2}\left[(1+r)\,a_1 + y_1 + \frac{y_2}{1+r} + \frac{y_3}{(1+r)^2}\right] \\
c_2 &= \frac{\beta}{1+\beta+\beta^2}\left[(1+r)^2\,a_1 + (1+r)\,y_1 + y_2 + \frac{y_3}{1+r}\right] \\
c_3 &= \frac{\beta^2}{1+\beta+\beta^2}\left[(1+r)^3\,a_1 + (1+r)^2\,y_1 + (1+r)\,y_2 + y_3\right]
\end{aligned}$$

Note well how this expression for the consumption in the first period relates to the corresponding expression in the two-period case:

$$c_1 = \frac{1}{1+\beta}\left[(1+r)\,a_1 + y_1 + \frac{y_2}{1+r}\right]$$

The denominator has gained an additional discounting term, $\beta^2$, and the PDV of income includes also income from period 3. Let us now generalize such expressions to the case of an arbitrary number of periods $T$.

## 1.2 Finite time horizon optimization

For notational convenience, let us denote the "first" time period with time subscript 0, the "second" time period with time subscript 1, and so on. We add the condition that the agent's assets at the end of the planning horizon cannot be negative. The agent solves the following utility maximization problem:

$$\begin{aligned}
\max_{\{c_t\}_{t=0}^T, \{a_{t+1}\}_{t=0}^T} \quad & U = \ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2 + \ldots + \beta^T \ln c_T \equiv \sum_{t=0}^{T} \beta^t \ln c_t \\
\text{subject to} \quad & c_t + a_{t+1} = y_t + (1+r)\, a_t \quad \text{for all } t = 0, 1, 2, \ldots, T \\
& a_{T+1} \geq 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

For convenience, we will construct the Lagrangian using a series of per-period budget constraints:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \ln c_t + \sum_{t=0}^{T} \lambda_t \left[y_t + (1+r)\, a_t - c_t - a_{t+1}\right] + \mu a_{T+1}$$

We can expand the Lagrangian for an easier derivation of the first order conditions:

$$\begin{aligned}
\mathcal{L} = & \ldots + \beta^t \ln c_t + \ldots + \lambda_t \left[y_t + (1+r)\, a_t - c_t - a_{t+1}\right] + \lambda_{t+1} \left[y_{t+1} + (1+r)\, a_{t+1} - c_{t+1} - a_{t+2}\right] \\
& + \ldots + \lambda_T \left[y_T + (1+r)\, a_T - c_T - a_{T+1}\right] + \mu a_{T+1}
\end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned}
c_t \quad &: \quad \beta^t \frac{1}{c_t} - \lambda_t = 0 && \rightarrow \quad \lambda_t = \beta^t \frac{1}{c_t} && \text{for all } t = 0, 1, 2, \ldots, T \\
a_{t+1} \quad &: \quad -\lambda_t + \lambda_{t+1}(1+r) = 0 && \rightarrow \quad \lambda_t = (1+r)\, \lambda_{t+1} && \text{for all } t = 0, 1, 2, \ldots, T-1 \\
a_{T+1} \quad &: \quad -\lambda_T + \mu = 0 \\
\text{CS} \quad &: \quad \mu a_{T+1} = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad a_{T+1} \geq 0
\end{aligned}$$

Let us first focus on the complementary slackness condition and the FOC with respect to $a_{T+1}$:

$$\mu = \lambda_T = \beta^T \frac{1}{c_T} > 0$$

The multiplier $\mu$ has to be positive, unless consumption in period $T$ is infinite and the marginal utility of consumption is 0. Therefore, the complementary slackness condition implies that in optimum the agent holds no assets at the end of the planning horizon, which we have been implicitly assuming before:

$$\mu > 0 \quad \text{and} \quad \mu a_{T+1} = 0 \quad \rightarrow \quad a_{T+1} = 0$$

Note that this allows us to restate the original constraint of the no end-of-life debt in a form that is often called a **transversality condition**, and is necessary for solving infinite time horizon problems:

$$\lambda_T a_{T+1} = 0$$

Use the information from the FOCs to get the optimality condition (Euler equation):

$$\beta^t \frac{1}{c_t} = (1+r)\, \beta^{t+1} \frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1} = \beta\,(1+r)\, c_t$$

Since the above Euler equation is true for any two neighboring time periods, we can also rewrite it as:

$$\begin{aligned}
c_{t+1} &= \beta\,(1+r)\, c_t \\
c_{t+2} &= \beta\,(1+r)\, c_{t+1}
\end{aligned}$$

By substitution we can get:

$$c_{t+2} = \left[\beta\,(1+r)\right]^2 c_t$$

And we can relate $t+i$ period consumption (for any $i$) to $t$ period consumption as follows:

$$c_{t+i} = \left[\beta\,(1+r)\right]^i c_t \quad \rightarrow \quad c_t = \left[\beta\,(1+r)\right]^t c_0$$

Let us now construct the lifetime budget constraint. Start from the "first" time period:

$$c_0 + a_1 = y_0 + (1+r)\,a_0$$
$$(1+r)\,a_0 = c_0 - y_0 + a_1$$

Now consider the "second" time period:

$$c_1 + a_2 = y_1 + (1+r)\,a_1$$
$$(1+r)\,a_1 = c_1 - y_1 + a_2$$
$$a_1 = \frac{c_1 - y_1}{1+r} + \frac{a_2}{1+r}$$

Plug the expression for $a_1$ into the "first" period budget constraint:

$$(1+r)\,a_0 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{a_2}{1+r}$$

We could also obtain the expression for $a_2$ from the "third" period budget constraint:

$$a_2 = \frac{c_2 - y_2}{1+r} + \frac{a_3}{1+r}$$

and plug the expression for $a_2$ into the "first" period budget constraint:

$$(1+r)\,a_0 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{c_2 - y_2}{(1+r)^2} + \frac{a_3}{(1+r)^2}$$

If we continue this substitution process, we shall finally arrive at the following lifetime budget constraint:

$$(1+r)\,a_0 = c_0 - y_0 + \frac{c_1 - y_1}{1+r} + \frac{c_2 - y_2}{(1+r)^2} + \ldots + \frac{c_T - y_T}{(1+r)^T} + \frac{a_{T+1}}{(1+r)^T}$$

which we can express compactly by using the summation notation:

$$(1+r)\,a_0 = \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T}$$

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T} = (1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

By deriving and reworking the FOCs, we have obtained the following results:

$$a_{T+1} = 0 \quad \text{and} \quad c_t = [\beta\,(1+r)]^t\,c_0$$

Plug these two pieces of information into the lifetime budget constraint:

$$\sum_{t=0}^{T} \frac{[\beta\,(1+r)]^t\,c_0}{(1+r)^t} = (1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$
$$c_0 \cdot \sum_{t=0}^{T} \beta^t = (1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

We have arrived at the expression for the optimal consumption expenditure in the "first" time period, which is a generalized version of the formulas we have developed for two- and three-period cases:

$$c_0 = \frac{1}{\sum_{t=0}^{T} \beta^t} \left[ (1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \right]$$

We can also apply the formula for a **sum of the finite geometric series** to get:

$$c_0 = \frac{1-\beta}{1-\beta^{T+1}} \left[ (1+r)\,a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \right]$$

## 1.3 Infinite time horizon optimization

The agent solves the following infinite-horizon utility maximization problem:

$$
\begin{aligned}
\max_{\{c_t\}_{t=0}^{\infty},\{a_{t+1}\}_{t=0}^{\infty}} \quad & U=\sum_{t=0}^{\infty}\beta^t \ln c_t \\
\text{subject to} \quad & c_t+a_{t+1}=y_t+(1+r)\,a_t \quad \text{for all } t=0,1,2,\ldots,\infty \\
& \lim_{T\to\infty}\lambda_T a_{T+1}=0 \\
& a_0 \quad \text{given}
\end{aligned}
$$

Again we include the condition that the agents' assets at the "end" cannot be negative. In this problem the planning horizon is infinite and so we need to make use of the transversality condition. The transversality condition eliminates such paths where consumer becomes more and more indebted over time.

Set up the Lagrangian and expand it for an easier derivation of the first order conditions:

$$
\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty}\beta^t \ln c_t+\sum_{t=0}^{\infty}\lambda_t\left[y_t+(1+r)\,a_t-c_t-a_{t+1}\right] \\
&= \ldots+\beta^t \ln c_t+\ldots+\lambda_t\left[y_t+(1+r)\,a_t-c_t-a_{t+1}\right]+\lambda_{t+1}\left[y_{t+1}+(1+r)\,a_{t+1}-c_{t+1}-a_{t+2}\right]+\ldots
\end{aligned}
$$

First order conditions (FOCs):

$$
\begin{aligned}
c_t &: \quad \beta^t\frac{1}{c_t}-\lambda_t=0 && \rightarrow \quad \lambda_t=\beta^t\frac{1}{c_t} && \text{for all } t=0,1,2,\ldots,\infty \\
a_{t+1} &: \quad -\lambda_t+\lambda_{t+1}(1+r)=0 && \rightarrow \quad \lambda_t=(1+r)\,\lambda_{t+1} && \text{for all } t=0,1,2,\ldots,\infty
\end{aligned}
$$

Resulting optimality condition:

$$
\beta^t\frac{1}{c_t}=(1+r)\,\beta^{t+1}\frac{1}{c_{t+1}} \quad \rightarrow \quad c_{t+1}=\beta\,(1+r)\,c_t
$$

And as before, we can generalize the Euler equation to link consumption in any two time periods:

$$
c_{t+i}=\left[\beta\,(1+r)\right]^i c_t \quad \rightarrow \quad c_t=\left[\beta\,(1+r)\right]^t c_0
$$

The process for constructing the lifetime budget constraint is analogous to the finite horizon case. Imagine we have used our substitution procedure for the first $T$ time periods and so we have that:

$$
(1+r)\,a_0=\sum_{t=0}^{T}\frac{c_t-y_t}{(1+r)^t}+\frac{a_{T+1}}{(1+r)^T}
$$

Now consider what happens as $T\to\infty$. We know from the assets FOC that:

$$
\lambda_t=(1+r)\,\lambda_{t+1} \quad \rightarrow \quad \lambda_{t+1}=\frac{\lambda_t}{1+r} \quad \rightarrow \quad \lambda_{t+i}=\frac{\lambda_t}{(1+r)^i} \quad \rightarrow \quad \lambda_T=\frac{\lambda_0}{(1+r)^T}
$$

The transversality condition guarantees that in the limit (since $\lambda_0=\frac{1}{c_0}>0$):

$$
\lim_{T\to\infty}\lambda_T a_{T+1}=\lim_{T\to\infty}\lambda_0\frac{a_{T+1}}{(1+r)^T}=0 \quad \rightarrow \quad \lim_{T\to\infty}\frac{a_{T+1}}{(1+r)^T}=0
$$

Going back to the lifetime budget constraint we then have:

$$
(1+r)\,a_0=\sum_{t=0}^{\infty}\frac{c_t-y_t}{(1+r)^t} \quad \rightarrow \quad \sum_{t=0}^{\infty}\frac{c_t}{(1+r)^t}=(1+r)\,a_0+\sum_{t=0}^{\infty}\frac{y_t}{(1+r)^t}
$$

If we plug in the Euler equation, we get:

$$\sum_{t=0}^{\infty} \frac{[\beta (1+r)]^t c_0}{(1+r)^t} = (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}$$
$$c_0 \cdot \sum_{t=0}^{\infty} \beta^t = (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}$$

$$c_0 = \frac{1}{\sum_{t=0}^{\infty} \beta^t} \left[ (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \right]$$

From the properties of the infinite geometric series we get:

$$c_0 = (1-\beta) \left[ (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \right]$$

Since the planning horizon is infinite, the above expression can be applied to any time period $t$:

$$c_t = (1-\beta) \left[ (1+r) a_t + \sum_{i=0}^{\infty} \frac{y_{t+i}}{(1+r)^i} \right]$$

Note also that this is the limit of the expression found for the finite horizon case, as $T$ goes to infinity:

$$\lim_{T\to\infty} \frac{1-\beta}{1-\beta^{T+1}} \left[ (1+r) a_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} \right] = (1-\beta) \left[ (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \right]$$

### 1.3.1 Temporary vs permanent income changes

Consider now how consumption in time period $t$ would react to an increase in time period $t$ income:

$$\Delta c_t = (1-\beta) \left[ \frac{\Delta y_t}{(1+r)^0} \right] = (1-\beta) \Delta y_t$$

If we consider yearly time intervals, then $\beta \approx 0.96$ and the reaction of consumption is very small.

On the other hand, if income is expected to increase in all future time periods, then we get:

$$\Delta c_t = (1-\beta) \left[ \sum_{i=0}^{\infty} \frac{\Delta \bar{y}_t}{(1+r)^i} \right] = (1-\beta) \left[ \Delta \bar{y}_t \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} \right] = (1-\beta) \frac{1}{1-\frac{1}{1+r}} \Delta \bar{y}_t = (1-\beta) \frac{1+r}{r} \Delta \bar{y}_t$$

Now if we assume that $\beta (1+r) = 1$, then $\Delta c_t = \Delta \bar{y}_t$. This is the **permanent income hypothesis**.

## 1.4 Choice under uncertainty

Consider now the problem where the agent is uncertain about future levels of income and interest rates. Additionally, instead of assuming a specific functional form for the utility function, we consider a generic function $u$, satisfying the standard assumptions that $u'(c) > 0$, $u''(c) < 0$ and $\lim_{c \to 0} u(c) = -\infty$.

$$\begin{aligned}
\max \quad & U = E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\
\text{subject to} \quad & c_t + a_{t+1} = y_t + (1 + r_t) a_t \quad \text{for all } t = 0, 1, \ldots, \infty \\
& \lim_{T \to \infty} \lambda_T a_{T+1} = 0 \\
& a_0 \quad \text{given}
\end{aligned}$$

Set up the (expanded) Lagrangian:

$$\begin{aligned}
\mathcal{L} &= E_0 \left[ \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t) + \lambda_t \left[ y_t + (1 + r_t) a_t - c_t - a_{t+1} \right] \right\} \right] \\
&= \ldots + E_0 \left[ \beta^t u(c_t) + \lambda_t \left[ y_t + (1 + r_t) a_t - c_t - a_{t+1} \right] \right] \\
&\quad + E_0 \left[ \beta^{t+1} u(c_{t+1}) + \lambda_{t+1} \left[ y_{t+1} + (1 + r_{t+1}) a_{t+1} - c_{t+1} - a_{t+2} \right] \right] + \ldots
\end{aligned}$$

First order conditions (FOCs):

$$\begin{aligned}
c_t \quad &: \quad E_0 \left[ \beta^t u'(c_t) - \lambda_t \right] = 0 \quad \to \quad \lambda_t = \beta^t u'(c_t) \quad \text{for all } t = 0, 1, \ldots, \infty \\
a_{t+1} \quad &: \quad E_0 \left[ -\lambda_t \right] + E_0 \left[ \lambda_{t+1} (1 + r_{t+1}) \right] = 0 \quad \to \quad \lambda_t = E_t \left[ \lambda_{t+1} (1 + r_{t+1}) \right] \quad \text{for all } t = 0, 1, \ldots, \infty
\end{aligned}$$

where in the simplifications above we have assumed that in period $t$ the agent has full knowledge about period $t$ variables, but is uncertain about the values of variables in time period $t+1$.

Resulting optimality condition:

$$\beta^t u'(c_t) = E_t \left[ \beta^{t+1} u'(c_{t+1}) (1 + r_{t+1}) \right] \quad \to \quad u'(c_t) = E_t \left[ \beta u'(c_{t+1}) (1 + r_{t+1}) \right]$$

Note well that now we need to be extra careful not to break any expectation operators. Let us rewrite the Euler equation in the following way:

$$1 = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot (1 + r_{t+1}) \right]$$

The above equation can be thought also as an asset pricing equation. In this particular example, the price of a unit of savings is one unit of period $t$ consumption. The payoff from having an asset in $t+1$ period will be $(1 + r_{t+1})$ per unit of savings. The term $\beta \cdot u'(c_{t+1}) / u'(c_t)$ is called the **stochastic discount factor** and measures the relative marginal utility of consumption across periods.

A general form of consumption-based asset equation can be expressed as $p_t = E_t [m_{t+1} \cdot x_{t+1}]$, where $p_t$ denotes asset price, $m_{t+1}$ is the stochastic discount factor and $x_{t+1}$ is the future asset payoff. This equation is the basis of John Cochrane's book **Asset Pricing**, which covers a variety of cases:

| | Price $p_t$ | Payoff $x_{t+1}$ |
|---|---|---|
| Stock | $p_t$ | $p_{t+1} + d_{t+1}$ |
| Return | 1 | $R_{t+1}$ |
| Price-dividend ratio | $\frac{p_t}{d_t}$ | $\left( \frac{p_{t+1}}{d_{t+1}} + 1 \right) \frac{d_{t+1}}{d_t}$ |
| Excess return | 0 | $R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$ |
| Managed portfolio | $z_t$ | $z_t R_{t+1}$ |
| Moment condition | $E[p_t z_t]$ | $x_{t+1} z_t$ |
| One-period bond | $p_t$ | 1 |
| Risk-free rate | 1 | $R^f$ |
| Option | $C$ | $\max \{S_T - K, 0\}$ |