Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019

1 Dynamic consumption choice

1.1 Two-period optimization

An agent solves the following utility maximization problem:

$$\begin{aligned} \max_{c_1,\, c_2,\, a} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r)\, a \\ & \beta \in [0,1]\,, \text{ given} \\ & y_1, y_2 \geq 0, \text{ given} \\ & r \geq -1, \text{ given} \\ & c_1, c_2 > 0 \end{aligned}$$

where c_1 and c_2 denote agent's consumption in the first and second period, y_1 and y_2 denote agent's income in the first and second period, β is the **discount factor**, and a denotes wealth / net assets (possibly negative) held by the agent at the end of period 1 (at the very beginning of period 2), which yield real rate of interest r.

First construct the lifetime budget constraint:

$$a = \frac{c_2 - y_2}{1+r}$$

$$\begin{aligned} c_1 + \frac{c_2 - y_2}{1+r} &= y_1 \\ c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r} \end{aligned}$$

The lifetime budget constraint says that the present discounted value (PDV) of income (plus initial wealth which is here assumed to be 0) has to equal the PDV of consumption. Rewrite the budget constraint so that one side of the equation equals 0:

$$y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0$$

Set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

where λ is the Lagrange multiplier with the following economic interpretation: by how much would the agent's utility increase if the budget constraint was marginally relaxed, i.e. the agent had marginally higher PDV of income ($\partial U / \partial y_1$).

The Lagrangian involves two choice variables: c_1 and c_2. Therefore, we will need to calculate two first order conditions.

First order conditions (FOCs):

$$\begin{aligned} c_1 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda\,[-1] = 0 && \rightarrow \quad \lambda = \frac{1}{c_1} \\ c_2 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda \left[-\frac{1}{1+r} \right] = 0 && \rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2} \end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta(1+r)c_1$$

An equation linking in optimum consumption in different time periods is called the intertemporal (from Latin: "between time (periods)") condition or the **Euler equation**.

Plug the optimality condition into the budget constraint:

$$\begin{aligned} c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r} \\ c_1 + \frac{\beta(1+r)c_1}{1+r} &= y_1 + \frac{y_2}{1+r} \\ c_1 + \beta c_1 &= y_1 + \frac{y_2}{1+r} \end{aligned}$$

Finally, we can obtain the optimal levels of consumption and assets at the end of the first period:

$$\begin{aligned} c_1 &= \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \\ c_2 &= \frac{\beta}{1+\beta}\left[(1+r)y_1 + y_2\right] \\ a &= y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] = \frac{\beta}{1+\beta}y_1 - \frac{1}{1+\beta}\frac{y_2}{1+r} \end{aligned}$$

The Euler equation can be also obtained in the "Euler way": using calculus of variation. Suppose that the agent has found the optimal solution to the problem. She now considers consuming x units less in the first period so that she can consume $(1+r)x$ units more in the second period. Her utility level is now given by:

$$U = \ln(c_1 - x) + \beta\ln(c_2 + (1+r)x)$$

If the solution was truly optimal, the derivative of the utility function with respect to x, evaluated at $x = 0$, should be equal to 0:

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{1}{c_1 - x}\cdot(-1) + \beta\frac{1}{c_2 + (1+r)x}\cdot(1+r) \\ \frac{\partial U}{\partial x}|_{x=0} &= -\frac{1}{c_1} + \beta(1+r)\frac{1}{c_2} = 0 \quad \rightarrow \quad c_2 = \beta(1+r)c_1 \end{aligned}$$

1.1.1 Comparative statics

Let us take a look at how changes in the parameters of our problem affect the agent's choices.

Changes in β

The higher is β, the more patient is our agent. Intuitively, since the agent with higher β cares more about the future, she would save more in the first period (and thus consume less in the first period) in order to consume more in the second period:

$$\begin{aligned} \frac{\partial c_1}{\partial \beta} &= -\frac{1}{(1+\beta)^2}\left[y_1 + \frac{y_2}{1+r}\right] < 0 \\ \frac{\partial c_2}{\partial \beta} &= \frac{1+\beta-\beta}{(1+\beta)^2}\left[(1+r)y_1 + y_2\right] > 0 \\ \frac{\partial a}{\partial \beta} &= \frac{1+\beta-\beta}{(1+\beta)^2}y_1 - \left[-\frac{1}{(1+\beta)^2}\right]\frac{y_2}{1+r} > 0 \end{aligned}$$

Changes in y_1
An increase in first period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional first period income to second period consumption, assets will also increase:

$$\frac{\partial c_1}{\partial y_1} = \frac{1}{1+\beta} > 0$$
$$\frac{\partial c_2}{\partial y_1} = \frac{\beta}{1+\beta}(1+r) > 0$$
$$\frac{\partial a}{\partial y_1} = \frac{\beta}{1+\beta} > 0$$

Changes in y_2
An increase in second period income increases consumption in both time periods. Since the agent will want to transfer a part of the additional second period income to first period consumption, assets will decrease:

$$\frac{\partial c_1}{\partial y_2} = \frac{1}{1+\beta}\frac{1}{1+r} > 0$$
$$\frac{\partial c_2}{\partial y_2} = \frac{\beta}{1+\beta} > 0$$
$$\frac{\partial a}{\partial y_2} = -\frac{1}{1+\beta}\frac{1}{1+r} < 0$$

Changes in r
An increase in the interest rate generates two effects: substitution effect and income effect. Substitution effect is generated by the change in relative prices of consumption in the first period versus consumption in the second period. It induces the agent to consume more in the second period and consume less in the first period. Income effect is generated by the fact that the PDV of income changes due to the changes in the discounting rate. The sign of the effect depends on whether an agent is a saver or a borrower. An increase in interest rates creates a positive income effect for the saver, increasing her consumption in both periods. Conversely, an increase in interest rates creates a negative income effect for the borrower, decreasing her consumption in both periods.

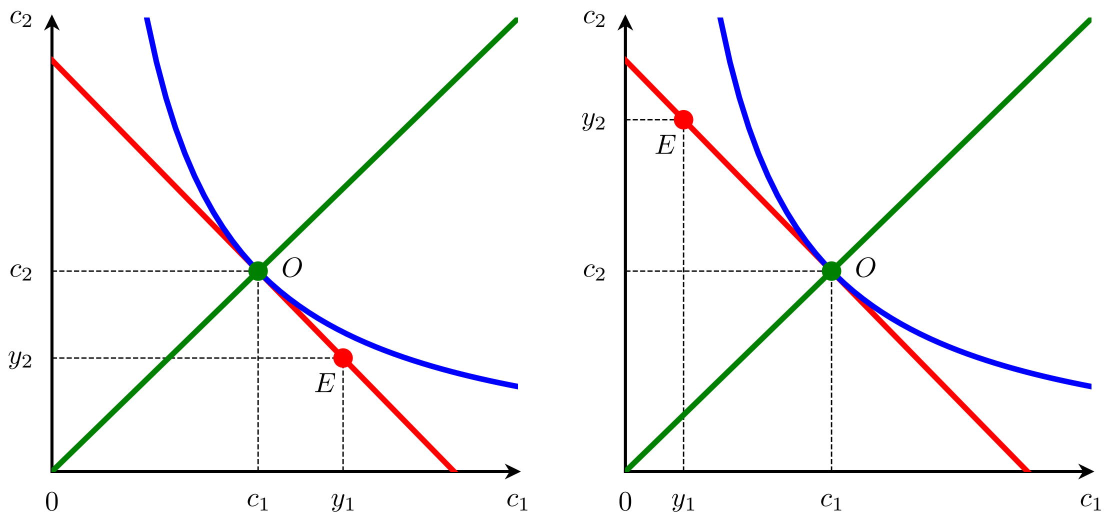
These effects can be summarized by the following table, where pluses and minuses describe the direction of change in a variable for which we can be certain of the direction, regardless of the assumed utility function:

Effects of an	Saver			Borrower		
increase in r	c_1	c_2	a	c_1	c_2	a
Income	+	+	-	-	-	+
Substitution	-	+	+	-	+	+
Net	?	+	?	-	?	+

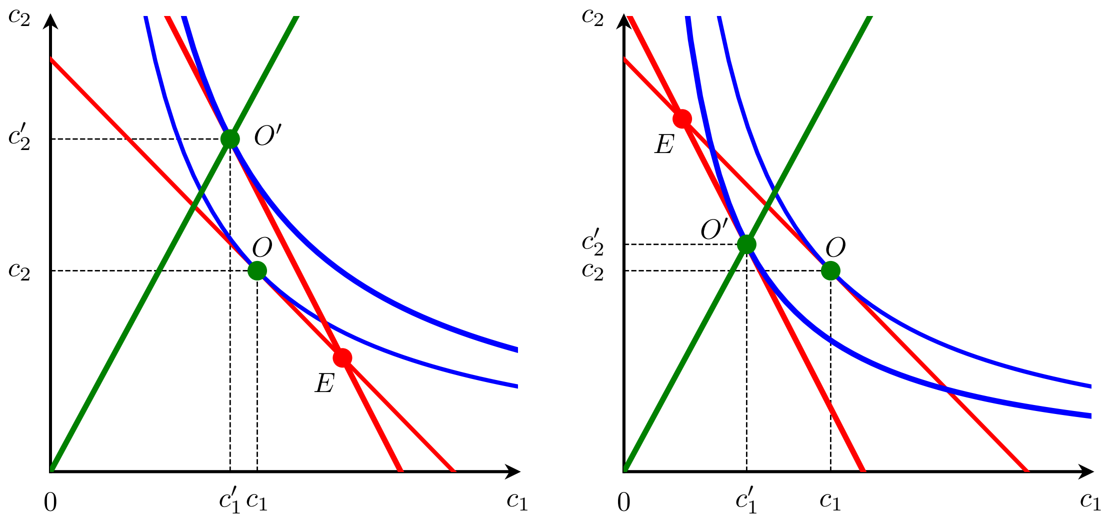
There are however also changes in variables marked with question marks, where the direction of change in a variable depends on the relative strength of substitution and income effects. In the case of a saver, if the income effect is stronger (weaker) than the substitution effect, first period consumption increases (decreases) and assets decrease (increase). In the case of a borrower, if the income effect is stronger (weaker) than the substitution effect, second period consumption decreases (increases).

Saver **Borrower**

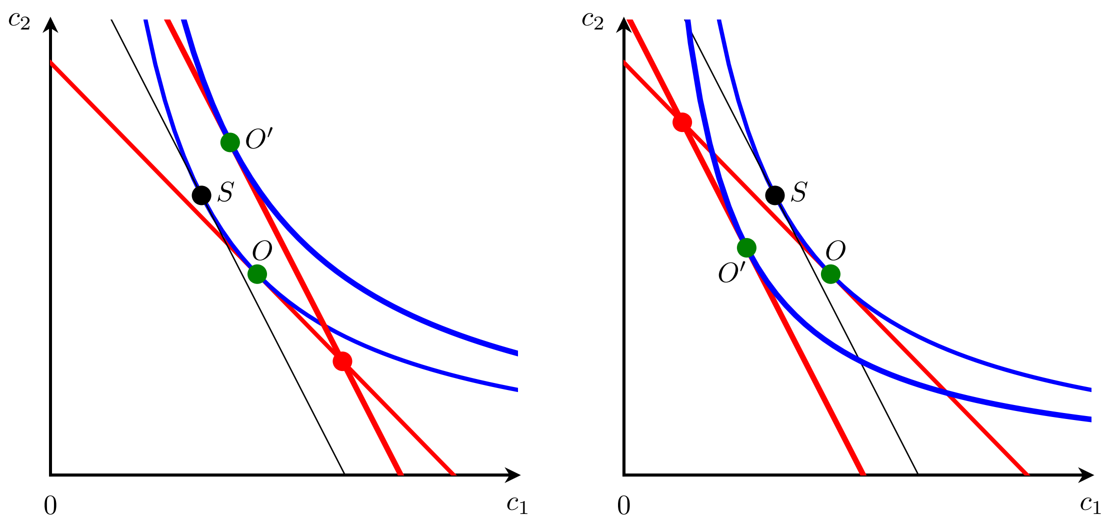
Optimal choice is labeled O. E denotes the initial endowment point.



Reaction to an increase in r. New optimal point is labeled O'.



Decomposition of effects. $O \to S$ is pure substitution effect while $S \to O'$ is pure income effect



The relative strength of the two effects depends on the shape of the indifference curves. For our agent:

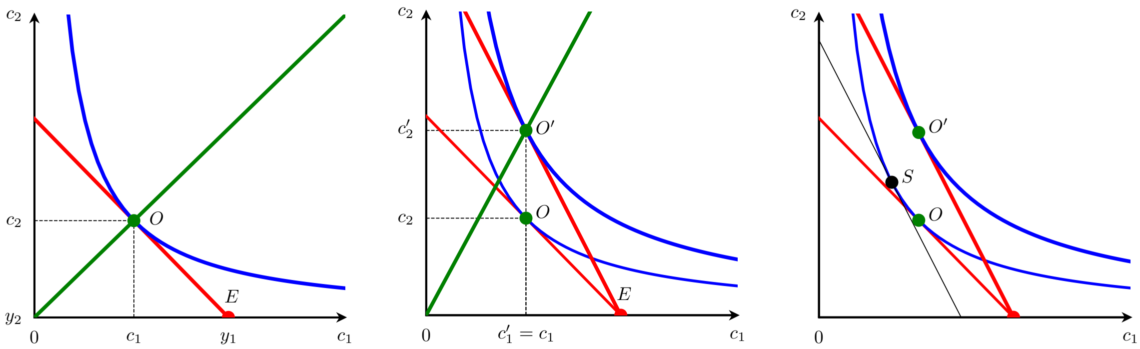
$$\frac{\partial c_1}{\partial r} = \frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] < 0$$

$$\frac{\partial c_2}{\partial r} = \frac{\beta}{1+\beta} y_1 > 0$$

$$\frac{\partial a}{\partial r} = -\frac{1}{1+\beta}\left[-\frac{y_2}{(1+r)^2}\right] > 0$$

In response to the increase in the interest rate, first period consumption decreases (if only $y_2 > 0$), second period consumption increases (if only $y_1 > 0$) and savings increase (if only $y_2 > 0$). The negative relationship between the level of interest rates and consumption is a usual assumption in macroeconomics, and is present in the undergraduate macroeconomic models such as IS-LM and AS-AD.

The case of logarithmic utility and $y_2 = 0$ is a very special case where both first period consumption and savings stay constant while second period consumption increases in response to the increase in the interest rate. The income and substitution effects have exactly equal strength and cancel each other out.



1.2 Inequality constraints: non-borrowing constraint

An agent solves the same problem as before, but now the agent cannot borrow and so $a \geq 0$:

$$\begin{aligned} \max_{c_1, c_2, a} \quad & U = \ln c_1 + \beta \ln c_2 \\ \text{subject to} \quad & c_1 + a = y_1 \\ & c_2 = y_2 + (1+r)\, a \\ & a \geq 0 \end{aligned}$$

Note: mathematical convention requires that any inequality constraint is written as "bigger than" or "bigger or equal than" so that our multiplier has the correct sign for interpretation purposes.

We will consider now two approaches to solve this problem. In the first approach the level of assets at the end of the first period a will be eliminated and substituted with $y_1 - c_1$, similar to what we have done previously. In the second approach a will remain explicit and will be an additional choice variable over which the agent will optimize.

1.2.1 Approach I: a is eliminated from the constraints

Obtain the lifetime budget constraint:

$$c_1 + a = y_1$$
$$c_2 = y_2 + (1+r)\, a$$
$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$y_1 + \frac{y_2}{1+r} - c_1 + \frac{c_2}{1+r} = 0$$

Non-borrowing constraint:

$$a \geq 0 \quad \rightarrow \quad y_1 - c_1 \geq 0$$

Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 + \frac{c_2}{1+r} \right] + \mu \left(y_1 - c_1 \right)$$

The Lagrangian involves two choice variables: c_1 and c_2, with a given implicitly by $y_1 - c_1$. Therefore, we will need to calculate two first order conditions. Additionally, since one of the constraints is an inequality constraint, we will add a **complementary slackness** constraint that says that either $y_1 - c_1 = 0$ and the constraint is binding ($\mu \geq 0$), or $y_1 \geq c_1$ and the constraint is not binding ($\mu = 0$).

First order conditions (FOCs):

$$\begin{aligned} c_1 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda\,[-1] + \mu\,[-1] = 0 \quad \rightarrow \quad \lambda = \frac{1}{c_1} - \mu \\ c_2 \quad &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda \left[-\frac{1}{1+r} \right] = 0 \quad \rightarrow \quad \lambda = \beta\,(1+r)\,\frac{1}{c_2} \\ \text{CS} \quad &: \quad \mu\,(y_1 - c_1) = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad y_1 - c_1 \geq 0 \end{aligned}$$

Because the non-borrowing constraint might not be binding, we have to consider the following two cases:

Case 1: constraint not binding, $\mu = 0$, $y_1 \geq c_1$:
Resulting optimality condition:

$$\frac{1}{c_1} = \beta\,(1+r)\,\frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta\,(1+r)\,c_1$$

Plug the optimality condition into the budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \frac{\beta\,(1+r)\,c_1}{1+r} = y_1 + \frac{y_2}{1+r}$$
$$c_1 + \beta c_1 = y_1 + \frac{y_2}{1+r}$$

$$\begin{aligned} c_1 &= \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right] \\ c_2 &= \frac{\beta}{1+\beta} \left[(1+r)\, y_1 + y_2 \right] \\ a &= y_1 - \frac{1}{1+\beta} \left[y_1 + \frac{y_2}{1+r} \right] = \frac{\beta}{1+\beta} y_1 - \frac{1}{1+\beta} \frac{y_2}{1+r} \end{aligned}$$

We should now check whether $a \geq 0$ (or, equivalently, if $c_1 \leq y_1$). If yes, then this is the optimal solution. Otherwise the optimal solution lies in case 2.

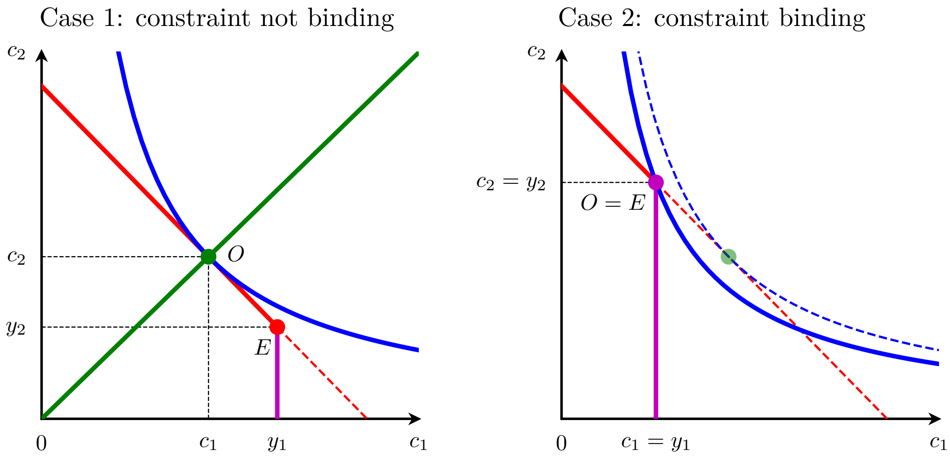
Case 2: constraint binding, $\mu \geq 0$, $y_1 = c_1$:
The solution is now simpler:

$$
\begin{aligned}
c_1 &= y_1 \\
a &= y_1 - c_1 = 0 \\
c_2 &= y_2 + (1+r)\, a = y_2
\end{aligned}
$$

What remains to find is the value of μ:

$$\mu = \frac{1}{c_1} - \lambda = \frac{1}{c_1} - \beta\,(1+r)\,\frac{1}{c_2} = \frac{1}{y_1} - \beta\,(1+r)\,\frac{1}{y_2}$$

If the agent is indeed constrained, then the multiplier μ should be positive. It has the following interpretation: by how much would the utility of an agent improve if we allowed this agent to borrow a marginal amount. In other words, the higher is μ the more utility is “lost” because of the non-borrowing condition. It is evident that μ increases with y_2 and decreases with y_1. Intuitively, the higher is y_2 relative to y_1, the more is the agent induced to decrease assets, which eventually become negative. At this point, however, the non-borrowing constraint kicks in and the agents “suffers”.



1.2.2 Approach II: a remains explicit

Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda_1\,[y_1 - c_1 - a] + \lambda_2\,[y_2 + (1+r)\,a - c_2] + \mu a$$

The Lagrangian involves three choice variables: c_1, c_2 and a. Therefore, we will need to calculate three first order conditions. Additionally, since one of the constraints is an inequality constraint, we will add a complementary slackness (CS) constraint that says that either $a = 0$ and the constraint is binding ($\mu \geq 0$), or $a \geq 0$ and the constraint is not binding ($\mu = 0$).

First order conditions (FOCs):

$$
\begin{aligned}
c_1 &: \quad \frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} + \lambda_1\,[-1] = 0 && \rightarrow \quad \lambda_1 = \frac{1}{c_1} \\
c_2 &: \quad \frac{\partial \mathcal{L}}{\partial c_2} = \beta \cdot \frac{1}{c_2} + \lambda_2\,[-1] = 0 && \rightarrow \quad \lambda_2 = \frac{\beta}{c_2} \\
a &: \quad \frac{\partial \mathcal{L}}{\partial a} = \lambda_1\,[-1] + \lambda_2\,[1+r] + \mu = 0 && \rightarrow \quad \lambda_1 = \lambda_2\,(1+r) + \mu \\
\text{CS} &: \quad \mu a = 0 \quad \text{and} \quad \mu \geq 0 \quad \text{and} \quad a \geq 0
\end{aligned}
$$

Because of the potentially not binding constraint, we have again the following two cases:

Case 1: constraint nonbinding, $\mu = 0$, $a \geq 0$:

$$\begin{aligned}
\lambda_1 &= \frac{1}{c_1} \\
\lambda_2 &= \frac{\beta}{c_2} \\
\lambda_1 &= \lambda_2 (1 + r)
\end{aligned}$$

Resulting optimality condition:

$$\frac{1}{c_1} = \beta (1 + r) \frac{1}{c_2} \quad \rightarrow \quad c_2 = \beta (1 + r) c_1$$

Plug the optimality condition into the budget constraint:

$$\begin{aligned}
c_1 + \frac{c_2}{1 + r} &= y_1 + \frac{y_2}{1 + r} \\
c_1 + \frac{\beta (1 + r) c_1}{1 + r} &= y_1 + \frac{y_2}{1 + r} \\
c_1 (1 + \beta) &= y_1 + \frac{y_2}{1 + r} \\
c_1 &= \frac{1}{1 + \beta} \left[y_1 + \frac{y_2}{1 + r} \right] \\
c_2 &= \frac{\beta}{1 + \beta} \left[(1 + r) y_1 + y_2 \right] \\
a &= \frac{\beta}{1 + \beta} y_1 - \frac{1}{1 + \beta} \frac{y_2}{1 + r}
\end{aligned}$$

We should now check whether $a \geq 0$. If yes, then this is the optimal solution. Otherwise the optimal solution lies in case 2.

Case 2: constraint binding, $\mu \geq 0$, $a = 0$:

The solution is now simpler:

$$\begin{aligned}
a &= 0 \\
c_1 &= y_1 - a = y_1 \\
c_2 &= y_2 + (1 + r) a = y_2
\end{aligned}$$

What remains to find is the value of μ:

$$\begin{aligned}
\mu &= \lambda_1 - \lambda_2 (1 + r) \\
\mu &= \frac{1}{c_1} - \frac{\beta}{c_2} (1 + r) \\
\mu &= \frac{1}{y_1} - \frac{\beta}{y_2} (1 + r)
\end{aligned}$$

If the agent is indeed constrained, then the multiplier μ should be positive.