Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019
Homework 7 - Business Cycle Models - Solutions

Problem 1

Consider the following model with no labor disutility ($h_t = 1$). Households maximize expected utility subject to the standard budget constraint:

$$\begin{aligned} \max \quad & U_t = E_t\left[\sum_{i=0}^{\infty}\beta^i \ln c_{t+i}\right] \\ \text{subject to} \quad & c_t + k_{t+1} = w_t h_t + (1+r_t)\,k_t + div_t \end{aligned}$$

Firms maximize profits subject to the Cobb-Douglas production function:

$$\begin{aligned} \max \quad & div_t = y_t - w_t h_t - (r_t+\delta)\,k_t \\ \text{subject to} \quad & y_t = z_t k_t^{\alpha} h_t^{1-\alpha} \end{aligned}$$

where $\delta = 1$ (capital deprecates fully). The technology constant z evolves according to the process:

$$z_t = (1-\rho_z) + \rho_z z_{t-1} + \varepsilon_{z,t}$$

(a) Derive the first order conditions of the households.

$$\mathcal{L} = \sum_{i=0}^{\infty}\beta^i \cdot E_t\left[\ln c_{t+i} + \lambda_{t+i}\left(w_{t+i}h_{t+i} + (1+r_{t+i})\,k_{t+i} + div_{t+i} - c_{t+i} - k_{t+i+1}\right)\right]$$

First order conditions:

$$\begin{aligned} c_t \quad &: \quad \beta^0 \cdot E_t\left[\frac{1}{c_t} - \lambda_t\right] = 0 \quad \rightarrow \quad \lambda_t = \frac{1}{c_t} \\ k_{t+1} \quad &: \quad \beta^0 \cdot E_t\left[-\lambda_t\right] + \beta^1 \cdot E_t\left[\lambda_{t+1}(1+r_{t+1})\right] = 0 \quad \rightarrow \quad \lambda_t = \beta E_t\left[\lambda_{t+1}(1+r_{t+1})\right] \end{aligned}$$

Euler equation:

$$\frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right]$$

(b) Derive the first order conditions of the firm.

$$\max \quad div_t = z_t k_t^{\alpha} h_t^{1-\alpha} - w_t h_t - (r_t+\delta)\,k_t$$

First order conditions:

$$\begin{aligned} k_t \quad &: \quad \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - (r_t+\delta) = 0 \quad \rightarrow \quad r_t = \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - \delta \\ h_t \quad &: \quad (1-\alpha)\,z_t k_t^{\alpha} h_t^{-\alpha} - w_t = 0 \quad \rightarrow \quad w_t = (1-\alpha)\,z_t k_t^{\alpha} h_t^{-\alpha} \end{aligned}$$

Use assumptions that $h_t = 1$ and $\delta = 1$:

$$\begin{aligned} y_t &= z_t k_t^{\alpha} \\ r_t &= \alpha z_t k_t^{\alpha-1} - 1 \\ w_t &= (1-\alpha)\,z_t k_t^{\alpha} \end{aligned}$$

(c) Find the steady state of the system.
Let's plug in the firm dividends into the household's budget constraint:

$$\begin{aligned} c_t + k_{t+1} &= w_t h_t + (1+r_t)\,k_t + (y_t - w_t h_t - (r_t+\delta)\,k_t) \\ c_t + k_{t+1} &= y_t + (1-\delta)\,k_t \\ c_t + k_{t+1} &= y_t \end{aligned}$$

We have the following system of equations:

$$\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1}) \right] \\
y_t &= c_t + k_{t+1} \\
y_t &= z_t k_t^{\alpha} \\
r_t &= \alpha z_t k_t^{\alpha-1} - 1 \\
w_t &= (1-\alpha) z_t k_t^{\alpha} \\
z_t &= (1-\rho_z) + \rho_z z_{t-1} + \varepsilon_{z,t}
\end{aligned}$$

Now rewrite the system under the steady state:

$$\begin{aligned}
\frac{1}{c} &= \beta \left[\frac{1}{c} (1 + r) \right] \\
y &= c + k \\
y &= z k^{\alpha} \\
r &= \alpha z k^{\alpha-1} - 1 \\
w &= (1-\alpha) z k^{\alpha} \\
z &= (1-\rho_z) + \rho_z z
\end{aligned}$$

And we conclude that:

$$\begin{aligned}
z &= 1 \\
r &= \frac{1}{\beta} - 1 \\
k^{\alpha-1} &= \frac{1+r}{\alpha} \quad \rightarrow \quad k = \left(\frac{\alpha}{1+r} \right)^{1/(1-\alpha)} = (\alpha\beta)^{1/(1-\alpha)} \\
y &= k^{\alpha} \\
w &= (1-\alpha) k^{\alpha} \\
c &= y - k = k^{\alpha} - k
\end{aligned}$$

(d) Assuming that household behavior can be expressed as $c_t = (1-s) y_t$ where s is a constant, find the value of s as a function of model parameters.
Start with the Euler equation:

$$\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1}) \right] \\
\frac{1}{(1-s) y_t} &= \beta E_t \left[\frac{1}{(1-s) y_{t+1}} \cdot \alpha z_{t+1} k_{t+1}^{\alpha-1} \right] \\
\frac{1}{y_t} &= \beta E_t \left[\frac{1}{z_{t+1} k_{t+1}^{\alpha}} \cdot \alpha z_{t+1} k_{t+1}^{\alpha-1} \right] \\
\frac{1}{y_t} &= \alpha\beta E_t \left[\frac{1}{k_{t+1}} \right]
\end{aligned}$$

Now we can express k_{t+1} as a share of current output y_t:

$$k_{t+1} = \alpha\beta \cdot y_t$$

Since output is split between current consumption and future capital, we have that:

$$\begin{aligned}
c_t + k_{t+1} &= y_t \\
c_t &= y_t - k_{t+1} = y_t - \alpha\beta y_t = (1-\alpha\beta) y_t
\end{aligned}$$

This verifies our assumption that households save a constant fraction $s = \alpha\beta$ of their income.

(e) Find the expression for k_{t+1} as a function of variables at time t.

$$k_{t+1} = \alpha\beta \cdot y_t = \alpha\beta \cdot z_t k_t^{\alpha}$$

Problem 2

Suppose that you have the following simplified log-linearized (around 0 inflation steady state) New Keynesian model. The two main non-policy equations of the model can be written:

$$x_t = E_t x_{t+1} - (i_t - E_t \pi_{t+1})$$
$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}$$

where x is output gap, i is the nominal interest rate, π is inflation rate, β is the households' discount rate and $\kappa > 0$ is a constant that depends on model parameters. The central bank obeys a strict inflation targeting rule. In particular, let π_t^* be an exogenous inflation target. The central bank will adjust i_t so that $\pi_t = \pi_t^*$ is consistent with these equations holding. Assume that the inflation target follows an exogenous AR(1) process:

$$\pi_t^* = \rho_\pi \pi_{t-1}^* + \varepsilon_{\pi,t}, \quad \rho_\pi \in (0,1)$$

(a) Derive an analytic expression for i_t as a function of π_t^*.
To do that, we need to exploit our knowledge that $\pi_t = \pi_t^*$ and so $E_t \pi_{t+1} = E_t \pi_{t+1}^* = E_t \left[\rho_\pi \pi_t^* + \varepsilon_{\pi,t+1}\right] = \rho_\pi \pi_t^* = \rho_\pi \pi_t$. Then:

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta E_t \pi_{t+1} \\
\pi_t &= \kappa x_t + \beta \rho \pi_t \\
x_t &= \frac{(1-\beta\rho)}{\kappa} \pi_t \\
E_t x_{t+1} &= \frac{(1-\beta\rho)}{\kappa} E_t \pi_{t+1} = \frac{(1-\beta\rho)}{\kappa} \rho \pi_t
\end{aligned}$$

And:

$$\begin{aligned}
x_t &= E_t x_{t+1} - (i_t - E_t \pi_{t+1}) \\
\frac{(1-\beta\rho)}{\kappa} \pi_t &= \frac{(1-\beta\rho)}{\kappa} \rho \pi_t - (i_t - \rho \pi_t) \\
i_t &= \frac{(1-\beta\rho)}{\kappa} (\rho - 1) \pi_t^* + \rho \pi_t^*
\end{aligned}$$

(b) Suppose that ρ is 0. In which direction must the central bank adjust i_t in order to achieve an increase in π_t^*?
For $\rho = 0$ the expression for nominal interest rate becomes:

$$i_t = -\frac{1}{\kappa} \pi_t^*$$

This means that in order to achieve an increase in the inflation rate, the central bank has to lower the nominal interest rate. This logic underpins standard monetary policymaking.

(c) If ρ is sufficiently close to 1, is it possible that i_t must increase in order to implement an increase in π_t^*? How does this answer depend on the value of κ?
For $\rho \approx 1$ the expression for nominal interest rate becomes:

$$i_t = \pi_t^*$$

and is independent of κ. It now says that in order to implement an increase in the inflation target, the central bank needs to increase the nominal interest rate.

(d) Provide intuition behind the difference in results in (b) and (c).
The difference between results in (b) and (c) results from differing ρ. When $\rho = 0$, we are not actually varying the inflation target, but rather desire to control the inflation over the business cycle. When $\rho \approx 1$, we want to permanently change the inflation target. So the case $\rho = 0$ represents the short run, in which changes in nominal interest rate and inflation rate are negatively correlated. The second case represents the long-run, where nominal interest rate and inflation are positively correlated as the Fisher equation "wants" to return the real interest rate to its natural level:

$$1 + r = \frac{1+i}{1+\pi}$$

Problem 3

In Lecture 17 Slides we considered a model where permanent changes to marginal productivity of labor reduced the unemployment rate. This would imply that with trend productivity growth unemployment would disappear over time. Let's modify our model so that it is consistent with stationary unemployment in face of trend productivity growth. Suppose that the flow cost of a vacancy κ and the imputed value of free time b are functions of the wage rate w (instead being exogenous). In particular, assume that $\kappa_t = \kappa_0 w_t$ and $b_t = b_0 w_t$.

(a) Determine the formula for job creation and wage setting along the balanced growth path (steady state).
The new formulas are:

$$w = mpn - (r+s)\frac{\kappa_0 w}{q(\theta)}$$
$$w = \gamma b_0 w + (1-\gamma)(mpn + \kappa_0 w \theta)$$

(b) How do θ and wages along the balanced growth path react to productivity changes?
First, rewrite both equations such that terms involving w is on the left hand side, and everything else on the other:

$$w\left[1 + (r+s)\frac{\kappa_0}{q(\theta)}\right] = mpn$$
$$w\left[1 - \gamma b_0 - (1-\gamma)\kappa_0\theta\right] = (1-\gamma)\, mpn$$

Now use the mpn from the first equation and plug into the second:

$$w\left[1 - \gamma b_0 - (1-\gamma)\kappa_0\theta\right] = (1-\gamma)\, w\left[1 + (r+s)\frac{\kappa_0}{q(\theta)}\right]$$

In the above equation w gets cancelled out. The remainder is an equation in θ, and the solution does not depend on the marginal productivity of labor mpn. This implies that an increase in the labor productivity has no impact on the labor market tightness θ. On the other hand, wages will increase proportionally with the increase in productivity.

(c) Does a continuous growth of productivity lead to a decrease in the long run unemployment rate?
No, under these assumptions an increase in productivity does not lead to the decrease in the unemployment rate.