Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019
Homework 6 – Multisector Endogenous Growth Models

Problem 1

Consider the following Lucas-Uzawa model with the learning-by-doing effect. The production of the final good (in per worker terms) takes the form:

$$y_t = Ak_t^{\alpha} (u_t h_t)^{1-\alpha}$$

where A is the level of technology, k and h are respectively stocks of physical and human capital per worker, and u is the fraction of human capital used in production of the final good. For simplicity assume that there is no capital depreciation. Final output can be consumed or used to increase the stock of physical capital:

$$\dot{k}_t = y_t - c_t$$

while the accumulation of the human capital takes the form:

$$\dot{h}_t = Bh_t (1 - u_t) + \beta \dot{k}_t$$

where β is the constant parameter that describes the portion of net investment in physical capital that contributes to the accumulation of skills (i.e. the product of β and $\dot{k}$ is the learning by doing effect in the accumulation of human capital that is associated with the process of investment in physical capital). Assume that households maximize their standard CRRA lifetime utility:

$$U = \int_0^{\infty} e^{-\rho t} \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \, \mathrm{d}t$$

(a) In the problem described above what are the control variables and the state variables?

(b) Rewrite the problem by utilizing a new variable $z = h - \beta k$.

(c) Find all the first order conditions characterizing the optimal choice.

(d) Compute the BGP growth rates of physical and human capital, consumption and output as a function of the parameters of the model.

(e) What is the BGP value of u? Is this value smaller or bigger than in the original Lucas-Uzawa model where $\beta = 0$? Why?

Problem 2

Consider an economy where final goods are produced according to the following production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

where L is the constant labor force, M_t is the number of invented types of intermediate goods and x_{it} denotes usage of intermediate good type i in the final goods production. The inventor of type i holds a perpetual patent that gives exclusive, monopolistic rights to produce this type of an intermediate good, with marginal cost of production equal to 1. Assume that the government taxes the monopolists and each of them pays tax T per period.

(a) Solve the profit maximization problem of the final goods producer to obtain the (inverse) demand function for intermediate goods.

(b) Solve the profit maximization problem of the intermediate goods producers (monopolists). Find the optimal price, quantity produced and maximal after-tax profit per period.

(c) Assume that inventing a new type of an intermediate good costs $1/\eta$ units of the final good. Equalize the cost of invention with the discounted after-tax value of profit flows of a new monopolist.

(d) Transform the expression from (c) to obtain the real interest rate. Use the Euler equation $g = (r - \rho) / \sigma$ to obtain the equilibrium growth rate of the economy.

(e) Discuss how the growth rate of the economy depends on the level of taxation T. Should the government aim to reduce the after-tax profits of the monopolists to 0?

Problem 3

Consider a variant of the Romer model where labor is not used in final good production, but is used as the unique input in the intermediate goods production. Assume the final good production function is given by:

$$Y_t = \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Also, suppose that $1/M_t$ units of labor are required to produce one unit of any intermediate good. As in the Romer model, suppose that $\dot{M}_t = \eta M_t L_R$, where L_R is labor allocated to R&D. Define $L_X \equiv L - L_R$ as the amount of labor allocated to production of intermediates.

(a) What is the equilibrium level of intermediate good production?

(b) What is the equilibrium price of a unit of any intermediate good?

(c) Looking at a balanced-growth path equilibrium in which wages and number of intermediate varieties grow at the same rate, compute the maximal profit for intermediate producers.

(d) Write down the research arbitrage condition.

(e) Compute the equilibrium growth rate of the economy.