Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019
Homework 5 – One Sector Growth Models

Problem 1

Consider a Ramsey-Cass-Koopmans economy where for simplicity we assume $g = 0$ and $A = 1$. The representative households solve the following utility maximization problem:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} dt \\ \text{subject to} \quad & \dot{a}_t = (r_t - n)\, a_t + w_t - c_t + v_t \end{aligned}$$

where v is the lump-sum transfer from the government to households.

The representative firm solves the following profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_t = (1 - \tau^y)\, Y_t - (r_t + \delta)\, K_t - w_t L_t \\ \text{subject to} \quad & Y_t = K_t^\alpha L_t^{1-\alpha} \end{aligned}$$

where τ^y is the firm revenue tax (equivalent to taxing all households' income regardless of its source).

(a) Derive the first order conditions of the household.

(b) Recast the problem of the firm in per worker terms. Derive the first order conditions of the firm.

(c) Write down the government budget constraint. Using the assumptions of closed economy and balanced government budget, find the conditions for general equilibrium in this economy.

(d) Find the steady state level of capital per worker k^* and consumption per worker c^* in this economy. Discuss how they depend on the tax rate τ_y.

Problem 2

Consider an perfectly competitive economy where individual price taking firms face the following production function:

$$Y_{it} = A_t K_{it}^\alpha L_{it}^{1-\alpha}$$

Assume that publicly available technology depends on the average level of capital per worker k:

$$A_t = \left(\frac{\sum_i K_{it}}{\sum_i L_{it}} \right)^\eta = k_t^\eta$$

where η represents a learning-by-doing externality.

The aggregate final goods production is a sum of individual firms' outputs:

$$Y_t = \sum_i Y_{it}$$

Consumers solve the following utility maximization problem:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} dt \\ \text{subject to} \quad & \dot{a}_t = (r_t - n)\, a_t + w_t - c_t \end{aligned}$$

(a) Find the first order conditions characterizing the optimal choice of the consumer.

(b) Find the first order conditions characterizing the optimal behavior of the firm assuming that there is a constant rate of capital depreciation δ.

(c) Describe the general equilibrium in this economy using (a) and (b).

(d) Draw a phase diagram in the (k, c) space; will the long run equilibrium in this economy be stable if $\alpha + \eta < 1$? What about if $\alpha + \eta = 1$?

(e) Assuming that the initial level of capital in this economy is below its steady-state value describe the behavior of k, c, y and the growth rate of per capita income over time in the two above mentioned cases.

Problem 3

Suppose the economy's production function depends positively ($p'(\cdot) > 0$) on the ratio of government expenditures to GDP, denoted with $\omega \equiv G/Y$:

$$Y_t = AK_t \cdot p(\omega)$$

Assume no population growth for simplicity. Then the problem of the households can be stated using aggregate variables:

$$\begin{aligned} \max \quad & U = \int_0^\infty e^{-\rho t} \frac{C_t^{1-\sigma} - 1}{1 - \sigma} \, \mathrm{d}t \\ \text{subject to} \quad & \dot{K}_t = rK_t - C_t \end{aligned}$$

Assume that there is a firm revenue tax τ^y and the representative firm solves the following profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_t = (1 - \tau^y) Y_t - (r + \delta) K_t \\ \text{subject to} \quad & Y_t = AK_t \cdot p(\omega) \end{aligned}$$

(a) Find the first order conditions characterizing the optimal choice of the consumer.

(b) Find the first order conditions characterizing the optimal behavior of the firm.

(c) Describe the general equilibrium in this economy using (a) and (b).

(d) Solve the social planner's problem using the following resource constraint:

$$\dot{K}_t = AK_t \cdot p(G_t/Y_t) - \delta K_t - C_t - G_t \quad \rightarrow \quad \dot{K}_t = (1 - \omega) AK_t \cdot p(\omega) - \delta K_t - C_t$$

Note that ω can be chosen by the social planner.

(e) Under which conditions there is equivalence between the decentralized equilibrium from (c) and the social planner's equilibrium from (d)?