Marcin Bielecki, Advanced Macroeconomics IE, Spring 2019
Homework 1 – Intertemporal Optimization

Problem 1

Consider the following two-period utility maximization problem. This utility function belongs to the CRRA (Constant Relative Risk Aversion) class of functions which can be thought of as generalized logarithmic functions. An agent lives for two periods and in both receives some positive income y. Solve for the optimal consumption values c.

$$\begin{aligned} \max_{c_t, c_{t+1}, a_{t+1}} \quad & U = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & c_t + a_{t+1} = y_t \\ & c_{t+1} = y_{t+1} + (1+r)\, a_{t+1} \end{aligned}$$

where $\sigma \geq 0$,[1] $\beta \in [0,1]$ and $r \geq -1$.

(a) Rewrite the budget constraints into a single lifetime budget constraint and set up the Lagrangian.

(b) Obtain the first order conditions for c_t and c_{t+1}. Express c_{t+1} as a function of c_t.

(c) Using the lifetime budget constraint obtain the formulas for optimal c_t and c_{t+1}.

(d) Set $\sigma = 1$ and verify that the formulas for optimal c_t and c_{t+1} are identical to the ones we obtained in class for the utility function $U = \ln c_t + \beta \ln c_{t+1}$.

(e) Return to expressions obtained in (c). Assume now that $y_{t+1} = 0$. How does c_t react when interest rate r increases? How does it depend on σ? How does σ impact the relative strength of income and substitution effects?

Problem 2

In a two period model suppose the government provides good g to a household that must be paid for with taxes τ levied on this household. Assume that the household receives this government good only in the first period of their lives. The government must collect taxes in the first period to finance the provision of the government good. The household chooses the optimal amount of consumption in each period taking as given the government good. The household solves thus the following utility maximization problem:

$$\begin{aligned} \max_{c_t, c_{t+1}} \quad & U = \ln c_t + \gamma \ln g_t + \beta \ln c_{t+1} \\ \text{subject to} \quad & c_t + \frac{c_{t+1}}{1+r} = y_t - \tau_t \end{aligned}$$

where $\gamma \geq 0$.

(a) Find the optimal values of c_t and c_{t+1}.

(b) Substitute the optimal values you obtained in (a) back into the utility function to obtain an expression for utility that depends on g_t and τ_t.

(c) Now have the government choose the optimal level of τ_t that maximizes the utility expression in (b) subject to the constraint that $g_t = \tau_t$ (balanced budget).

(d) Given the solution to τ_t obtained in (c) substitute it back to the expressions for c_t and c_{t+1} in (a).

(e) If $\beta = 1$ and $r = 0$, what is the relationship between optimal c_t and c_{t+1}? Can you provide intuition for this result?

[1] For $\sigma = 1$ the CRRA function becomes logarithmic: $U = \ln c_t + \beta \ln c_{t+1}$. This can be easily proven by using the L'Hôpital's rule to compute the following limit: $\lim_{\sigma \to 1} \frac{c^{1-\sigma}-1}{1-\sigma}$.

Problem 3

In this problem we will show how changes in taxation can change consumption of an agent if she is borrowing-constrained. The agent's income in the first period is 1/3 of her income in the second period. Assume that $\beta(1+r) = 1$ and consider the following utility maximization problem:

$$\begin{aligned} \max_{c_t, c_{t+1}, a_{t+1}} \quad & U = \ln c_t + \beta \ln c_{t+1} \\ \text{subject to} \quad & c_t + a_{t+1} = y/3 \\ & c_{t+1} = y + (1+r)\, a_{t+1} \end{aligned}$$

(a) Using the Lagrangian method find optimal c_t, c_{t+1} and a_{t+1}. Are savings positive or negative?

(b) Assume now that the agent cannot borrow and faces an additional non-borrowing constraint: $a_{t+1} \geq 0$. Using the Lagrangian method find optimal c_t, c_{t+1} and a_{t+1}.

(c) Show graphically in the (c_t, c_{t+1}) space the problem of the agent and especially show that the agent would be on a higher indifference curve were she allowed to borrow.

(d) Suppose that the government arranges a transfer v to this agent by issuing bonds. In the future, the government will tax the agent to be able to buy back the bonds. The new constraints of the agent are:

$$\begin{aligned} & c_t + a_{t+1} = y/3 + v \\ & c_{t+1} = y + (1+r)\, a_{t+1} - (1+r)\, v \\ & a_{t+1} \geq 0 \end{aligned}$$

What is the impact of the government transfer on the agent's first period consumption?

(e) Show graphically in the (c_t, c_{t+1}) space the effect of the transfer scheme from (d).

(f) Suppose that the transfer is large enough so that the agent's savings become positive. What would be the impact of even bigger transfers on first period consumption?