Business cycle facts
Real Business Cycle (RBC) model

Applied Macroeconomics

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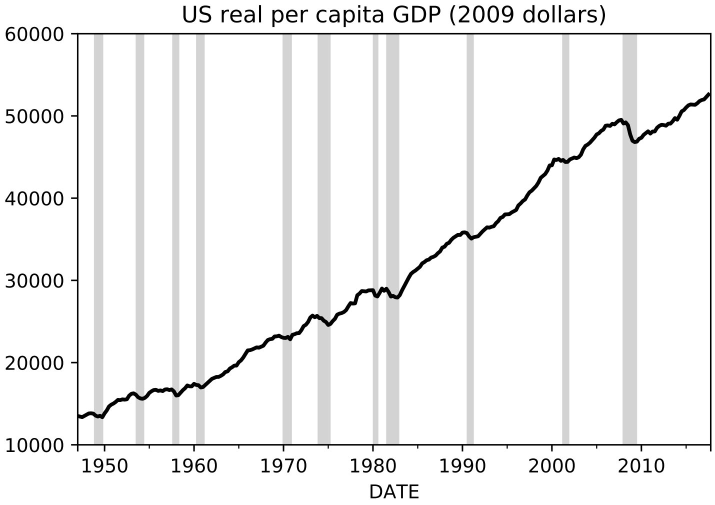
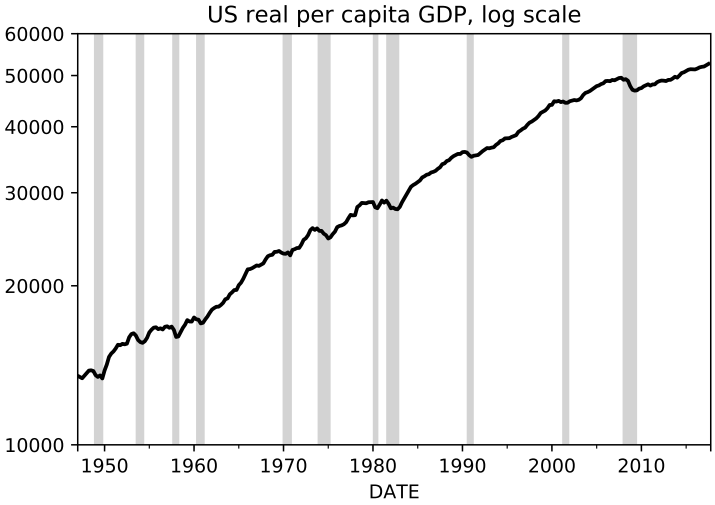
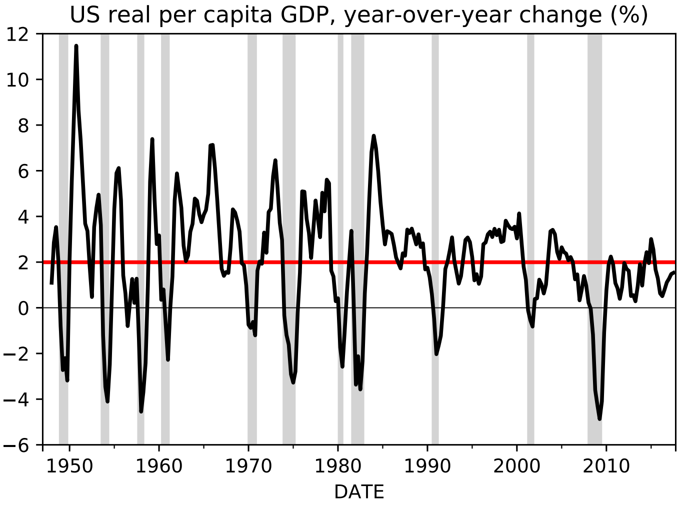
Time series properties of US GDP



Time series properties of US GDP



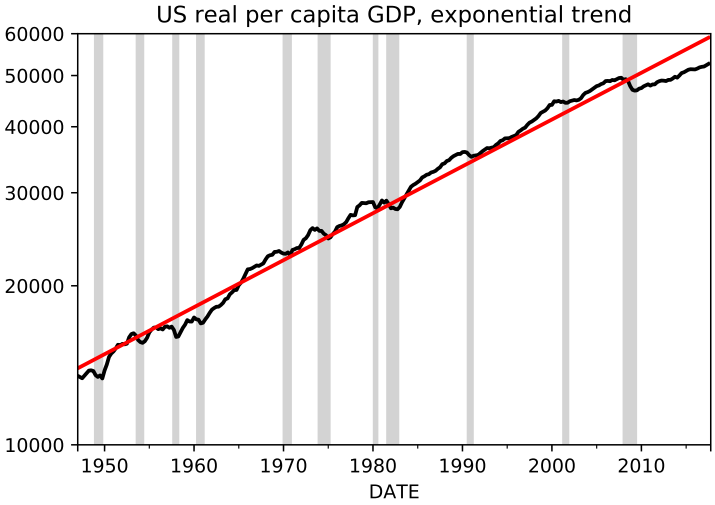
Time series properties of US GDP



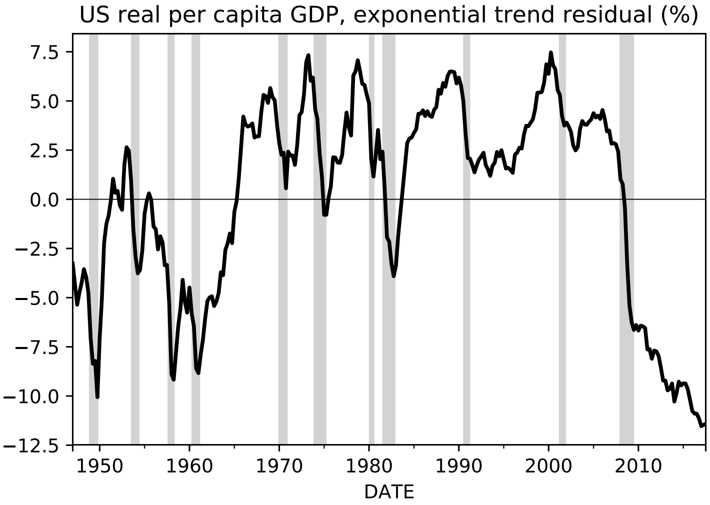
Time series properties of US GDP

- Between 1947 and 2017 per capita US GDP grew on average at around 2% annually
- There is substantial variation in GDP growth rate over time
- Recessions and expansions differ in size, length and frequency
- We would like to separate the trend (growth theory) from cycle (business cycle theory)

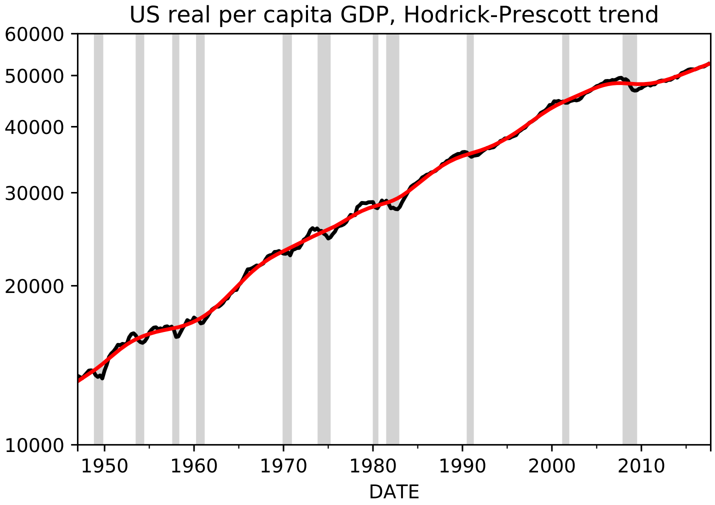
Trend vs cycle: exponential trend



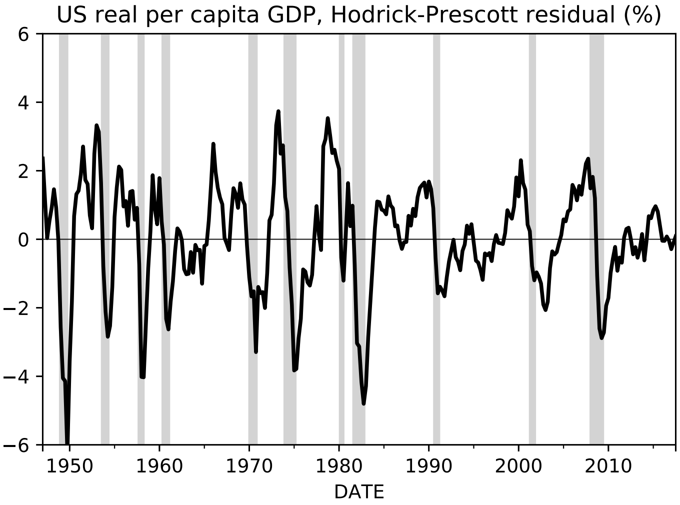
Trend vs cycle: exponential trend



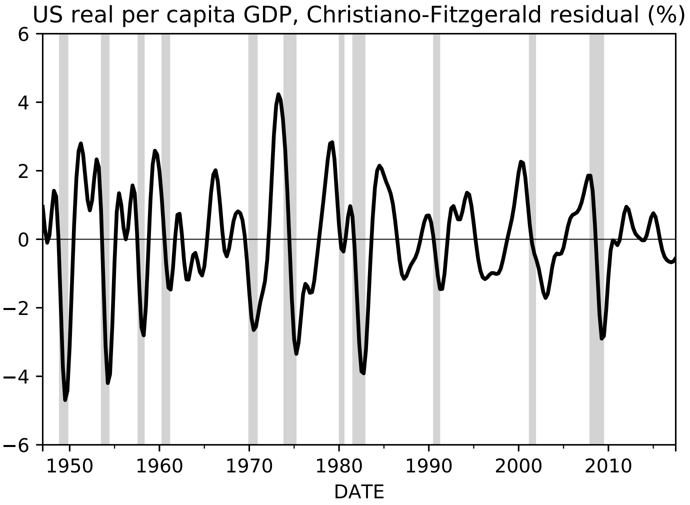
Trend vs cycle: Hodrick-Prescott filter



Trend vs cycle: Hodrick-Prescott filter



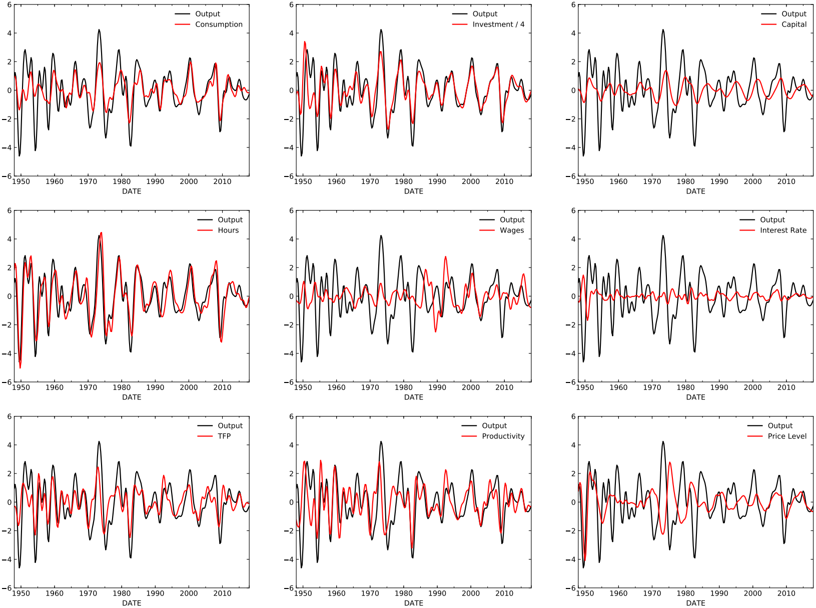
Trend vs cycle: Christiano-Fitzgerald filter



Trend vs cycle

- Most often used filter is the Hodrick-Prescott filter
- Christiano-Fitzgerald filter exhibits similar dynamics,
  but the cyclical component is "smooth" – better for visualization

Business cycle facts: USA 1948Q1-2018Q1



Business cycle facts: USA 1948Q1-2018Q1

- Consumption is coincident, procyclical and less volatile than output
- Investment is coincident, procyclical and more volatile than output
- Price level can be procyclical or countercyclical
- Productivity and TFP are both procyclical and leading output
- Hours are just as volatile as output with a 1-2 quarters lag
- Real wage is procyclical when price level is countercyclical
  and countercyclical when price level is procyclical
- Capital stock is procyclical, mildly volatile and lags output
- Real interest rates are acyclical and the least volatile
  There are potentially large errors in this measurement of r

Business cycle facts: USA 1948Q1-2018Q1

		Std. Dev.	Rel. S. D.	Corr. w. y	Autocorr.
Output	y	1.60	1.00	1.00	0.85
Consumption	c	0.86	0.54	0.76	0.83
Investment	i	4.54	2.83	0.79	0.87
Capital	k	0.57	0.36	0.36	0.97
Hours	h	1.60	1.00	0.81	0.90
Wages	w	0.84	0.52	0.10	0.65
Interest rate	r	0.39	0.25	-0.01	0.40
TFP	z	1.00	0.62	0.67	0.71
Productivity	$\frac{y}{h}$	1.30	0.81	0.51	0.65
Price level	P	0.89	0.55	-0.15	0.91

DSGE models

- Dynamic Stochastic General Equilibrium (DSGE) models aim to replicate business cycle behavior of real-world economies
  - Dynamic: forward-looking behavior of agents
  - Stochastic: the economy is subject to shocks
  - GE: what happens in one market influences other markets
- We can generate quantitative predictions on short-term movements of macroeconomic variables and compare them with the data
- We use those models to
  - Simulate counterfactual scenarios
  - Explain past developments (historical decomposition)
  - Construct forecasts (conditional and uncoditional)
  - Perform policy experiments
- Very active research on the frontier, but well established methods

Method

- All DSGE models are microfounded
- Usual setup
  - Households maximize utility subject to budget constraint
  - Firms maximize profits subject to technology
  - Markets clear
- Derive first order conditions for optimum
- Solve the system
- Check for stability
- Set parameters (calibration or estimation)
- Evaluate model's empirical performance
- Use the model to perform analyses of your choice

Dynare

- Set of instructions for Matlab (commercial) and GNU Octave (open source)
- Open source software under active development
- Solves, estimates, simulates DSGE models
- Good documentation, many example models available, active user forum with answers from developers

Basic Real Business Cycle model

- Ramsey model with endogenous labor supply and stochastic “technology” shocks
- Closed economy with no government
- Perfect competition
- Single final good with price normalized to 1 – all other prices are real
- Two groups of representative agents
  - Households
  - Firms
- Rational expectations – agents make no systematic forecast errors
- Despite simplicity and “unrealistic” assumptions, surprisingly good empirical performance

Households' problem

A representative household solves expected utility maximization problem

$$\begin{aligned} \max \quad & U_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\log c_t + \phi \log \left(1 - h_t \right) \right) \right] \\ \text{subject to} \quad & a_{t+1} + c_t = \left(1 + r_t \right) a_t + w_t h_t + div_t \end{aligned}$$

where

β	discount factor
c	per capita consumption
ϕ	relative preference for leisure
h	per capita hours (as fraction of total available time)
a	per capita assets (physical capital)
r	real interest rate
w	real wage per hour
div	per capita dividends

Households' solution I

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t E_0 \left[\log c_t + \phi \log\left(1 - h_t\right)\right]$$
$$+ \sum_{t=0}^{\infty} \beta^t E_0 \left[\lambda_t \left[\left(1 + r_t\right) a_t + w_t h_t + div_t - a_{t+1} - c_t\right]\right]$$

First Order Conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t E_0 \left[\frac{1}{c_t}\right] - \beta^t E_0 \left[\lambda_t\right] = 0 \quad \longrightarrow \quad \lambda_t = \frac{1}{c_t}$$
$$\frac{\partial \mathcal{L}}{\partial h_t} = \beta^t \cdot E_0 \left[-\frac{\phi}{1 - h_t}\right] + \beta^t E_0 \left[\lambda_t w_t\right] = 0 \quad \longrightarrow \quad \lambda_t = \frac{\phi}{w_t \left(1 - h_t\right)}$$
$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = -E_0 \left[\lambda_t\right] + \beta E_0 \left[\lambda_{t+1} \left(1 + r_{t+1}\right)\right] = 0$$
$$\longrightarrow \quad \lambda_t = \beta E_t \left[\lambda_{t+1} \left(1 + r_{t+1}\right)\right]$$

Households' solution II

First Order Conditions

$$
\begin{aligned}
c_t &: \quad \lambda_t = \frac{1}{c_t} \\
h_t &: \quad \lambda_t = \frac{\phi}{w_t\left(1-h_t\right)} \\
a_{t+1} &: \quad \lambda_t = \beta E_t\left[\lambda_{t+1}\left(1+r_{t+1}\right)\right]
\end{aligned}
$$

Resulting

$$
\begin{aligned}
\text{Intertemporal condition } (c+a) &: \quad \frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}\left(1+r_{t+1}\right)\right] \\
\text{Intratemporal condition } (c+h) &: \quad \frac{1}{c_t} = \frac{\phi}{w_t\left(1-h_t\right)} \\
&\longrightarrow \quad h_t = 1-\phi\frac{c_t}{w_t}
\end{aligned}
$$

Firms' problem

A representative firm solves profit (dividend) maximization problem

$$
\begin{aligned}
\max \quad & div_t = y_t - r_t^k k_t - w_t h_t \\
\text{subject to} \quad & y_t = z_t k_t^{\alpha} h_t^{1-\alpha} \\
& r_t^k = r_t + \delta
\end{aligned}
$$

where

div	per capita dividends
y	per capita output
r^k	capital rental rate
k	per capita physical capital stock
w	real wage per hour
h	per capita hours (as fraction of total available time)
z	stochastic total factor productivity (TFP) level
α	physical capital share in output
r	real interest rate
δ	physical capital depreciation rate

Firms' solution

Rewritten problem

$$\max \quad div_t = z_t k_t^{\alpha} h_t^{1-\alpha} - (r_t + \delta) k_t - w_t h_t$$

First Order Conditions

$$\frac{\partial div_t}{\partial k_t} = \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - (r_t + \delta) = 0 \quad \longrightarrow \quad r_t = \alpha z_t k_t^{\alpha-1} h_t^{1-\alpha} - \delta$$

$$\frac{\partial div_t}{\partial h_t} = (1-\alpha) z_t k_t^{\alpha} h_t^{-\alpha} - w_t = 0 \quad \longrightarrow \quad w_t = (1-\alpha) z_t k_t^{\alpha} h_t^{-\alpha}$$

Alternative expressions for factor prices

$$r_t = \alpha \frac{y_t}{k_t} - \delta$$

$$w_t = (1-\alpha) \frac{y_t}{h_t}$$

Due to perfect competition economic profits equal zero

$$div_t = y_t - r_t^k k_t - w_t h_t = y_t - \alpha \frac{y_t}{k_t} \cdot k_t - (1-\alpha) \frac{y_t}{h_t} \cdot h_t = 0$$

General equilibrium

Capital market clears

$$a_t = k_t$$

Households' budget constraint can be written as resource constraint

$$\begin{aligned}
a_{t+1} + c_t &= (1 + r_t)\, a_t + w_t h_t + div_t \\
k_{t+1} + c_t &= \left(1 + \alpha \frac{y_t}{k_t} - \delta\right) k_t + (1 - \alpha) \frac{y_t}{h_t} \cdot h_t + 0 \\
k_{t+1} + c_t &= \alpha y_t + (1 - \delta)\, k_t + (1 - \alpha)\, y_t \\
k_{t+1} + c_t &= y_t + (1 - \delta)\, k_t
\end{aligned}$$

If we define investment

$$i_t = k_{t+1} - (1 - \delta)\, k_t$$

We can rewrite the resource constraint as the GDP accounting equation

$$y_t = c_t + i_t$$

Stochastic total factor productivity

TFP evolves according to an AR(1) process (in logs)

$$\log z_t = \rho_z \log z_{t-1} + \varepsilon_t$$

where $\rho_z < 1$ regulates shock persistence and ε is zero-mean white noise

It is often assumed that $\varepsilon \sim \mathcal{N}\left(0, \sigma_z^2\right)$

In the absence of shocks $\log z \to 0$ and $z \to 1$

In Dynare one can define shock variance in the following manner

```
shocks;
var epsilon = sigma_z^2;
end;
```

Full set of equilibrium conditions

System of 8 equations and 8 unknowns: $\{y, c, i, k, h, w, r, z\}$

$$
\begin{aligned}
\text{Euler equation} &: \quad \frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} (1 + r_{t+1}) \right] && (1) \\
\text{Consumption-hours choice} &: \quad h_t = 1 - \phi \frac{c_t}{w_t} && (2) \\
\text{Production function} &: \quad y_t = z_t k_t^{\alpha} h_t^{1-\alpha} && (3) \\
\text{Real interest rate} &: \quad r_t = \alpha \frac{y_t}{k_t} - \delta && (4) \\
\text{Real hourly wage} &: \quad w_t = (1 - \alpha) \frac{y_t}{h_t} && (5) \\
\text{Investment} &: \quad i_t = k_{t+1} - (1 - \delta) k_t && (6) \\
\text{Output accounting} &: \quad y_t = c_t + i_t && (7) \\
\text{TFP AR(1) process} &: \quad \log z_t = \rho_z \log z_{t-1} + \varepsilon_t && (8)
\end{aligned}
$$

The first equation can also be written as $1 = \beta E_t \left[\frac{c_t}{c_{t+1}} (1 + r_{t+1}) \right]$
but not as $E_t [c_{t+1}] = \beta E_t [c_t (1 + r_{t+1})]$

Equilibrium conditions in Dynare

$$
\begin{aligned}
1 &= \beta E_t \left[c_t / c_{t+1} \left(1 + r_{t+1} \right) \right] \\
h_t &= 1 - \phi c_t / w_t \\
y_t &= z_t k_t^{\alpha} h_t^{1-\alpha} \\
r_t &= \alpha y_t / k_t - \delta \\
w_t &= \left(1 - \alpha \right) y_t / h_t \\
i_t &= k_{t+1} - \left(1 - \delta \right) k_t \\
y_t &= c_t + i_t \\
\log z_t &= \rho \log z_{t-1} + \varepsilon_t
\end{aligned}
$$

```
model;
1 = beta * c/c(+1) * (1+r(+1));
h = 1 - phi * c/w;
y = z * k^alpha * h^(1-alpha);
r = alpha * y/k - delta;
w = (1-alpha) * y/h;
i = k(+1) - (1-delta)*k;
y = c + i;
log(z) = rho*log(z(-1)) + epsilon;
end;
```

Note that we did not have to specify the expectation operator in eq. 1
Dynare automatically applies E on both sides of equations
Important rule: **never break the expectation operator**

Steady state: closed form solution

Start with the Euler equation

$$\frac{1}{c} = \beta \frac{1}{c}\left(1 + r\right) \quad \longrightarrow \quad r = \frac{1}{\beta} - 1$$

From the interest rate equation obtain the k/h ratio

$$r = \alpha k^{\alpha-1} h^{1-\alpha} - \delta \quad \longrightarrow \quad \frac{k}{h} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

From the production function obtain the y/h ratio and use it to get wage

$$y = k^{\alpha} h^{1-\alpha} \quad \longrightarrow \quad \frac{y}{h} = \left(\frac{k}{h}\right)^{\alpha} \quad \text{and} \quad w = \left(1-\alpha\right)\frac{y}{h}$$

From investment and output accounting equations obtain the c/h ratio

$$i = \delta k \quad \longrightarrow \quad y = c + \delta k \quad \longrightarrow \quad \frac{c}{h} = \frac{y}{h} - \delta \frac{k}{h}$$

Get h from the consumption-hours choice. The rest follows from h

$$h = 1 - \phi \frac{c}{w} \quad \longrightarrow \quad 1 = \frac{1}{h} - \phi \frac{c}{h}\frac{1}{w} \quad \longrightarrow \quad h = 1/\left[1 + \phi \frac{c}{h}\frac{1}{w}\right]$$

Steady state in Dynare

$$
\begin{aligned}
z &= 1 \\
r &= 1/\beta - 1 \\
k/h &= \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} \\
y/h &= (k/h)^{\alpha} \\
w &= (1-\alpha)\, y/h \\
c/h &= y/h - \delta k/h \\
h &= 1/\left[1 + \phi c/h/w\right] \\
\\
k &= k/h \cdot h \\
c &= c/h \cdot h \\
y &= y/h \cdot h \\
i &= y - c
\end{aligned}
$$

```
steady_state_model;
z = 1;
r = 1/beta - 1;
k_h = (alpha/(r+delta))^(1/(1-alpha));
y_h = k_h^alpha;
w = (1-alpha) * y_h;
c_h = y_h - delta * k_h;
h = 1 / (1 + phi * c_h / w);

k = k_h * h;
c = c_h * h;
y = y_h * h;
i = y - c;
end;
```

Transition dynamics

- Our model is a system of non-linear difference equations
- There exist no closed form solutions for the transitional dynamics except for few very special (and uninteresting) cases
- We can solve quite easily an approximated version of the system
  - (log-)linearize by hand
  - let Dynare compute n-th order Taylor expansion
- Solving the DSGE model involves transforming the forward looking system into a VAR (backward looking) system
  - Many good methods: Blanchard-Kahn, Klein, Sims, etc.
- Dynare does that for you: `stoch_simul;`
- This is possible thanks to the Rational Expectations assumption

Dynare .mod file structure

- Preamble
  - endogenous variables: `var y c i k h w r z;`
  - exogenous variables: `varexo epsilon;`
  - parameters:
    `parameters alpha beta delta phi rho_z sigma_z;`
- Model equations: `model; ... end;`
- Steady state equations or initial values
  - steady state: `steady_state_model; ... end;`
  - initial values: `initval; ... end;`
- Shocks: `shocks; ... end;`
- Computation

Dynare computation commands

- `steady;` solves the steady state of the model and displays it
- `check;` checks if the system has one unique solution
- `stoch_simul;` does a number of operations
  - performs *n*-th order approximation of the model
  - computes transition dynamics functions
  - calculates various statistics of model variables
    - mean and variance
    - correlation with other variables
    - autocorrelation
    - shock variance decomposition
  - plots impulse response functions
- `estimation;` estimates model parameters

Parameters

- We need to specify parameter values
  - tell Dynare the parameter values, e.g. `alpha = 0.33;`
- There is a variety of approaches on how to obtain those values
- Two most widely used are
  - **Calibration** – picking parameter values to fit certain long-run (average) features of data. For example, we might want to pick the parameters so that the model's investment share in GDP matches the average share in the data
  - **Estimation** – Dynare allows us to easily run a Bayesian estimation procedure on real data. It still needs as an input prior estimates of parameter values and their confidence intervals, which makes the calibration exercise very useful
- Most models in recent papers are estimated
- Today's toy model is calibrated

Parameter values

The following parameter values are standard in the literature

	Value justification	Mean	Conf. int.
α	Capital income share of GDP	0.33	± 0.05
β	From average real interest rate	0.99	± 0.005
δ	From investment share of GDP	0.025	± 0.05
ϕ	Work for 1/3 of time endowment	1.75	± 0.05
ρ_z	Coefficient in TFP AR(1) regression	0.97	± 0.02
σ_z	Error term in TFP AR(1) regression	0.007	± 0.005

I am going to use $\rho_z = 0.9622$ from our estimation
and $\sigma_z = 0.00853$ to match the standard deviation of output in the data

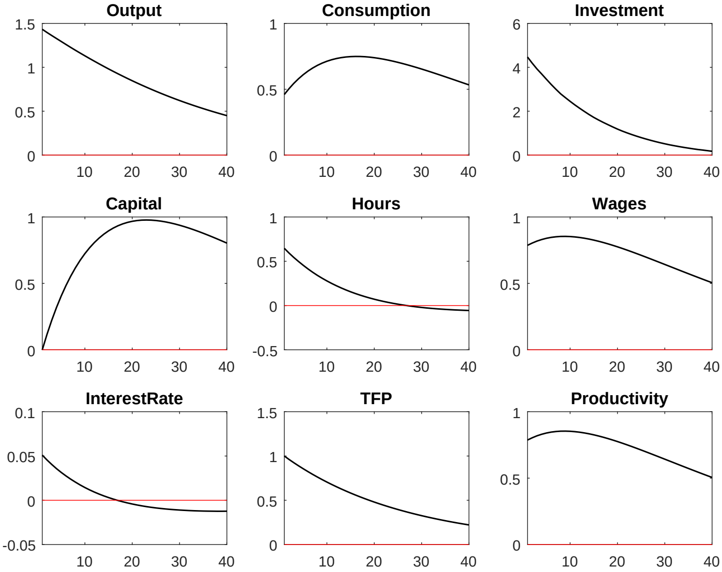
Dynare conventions

- **Notation**: always end all lines with a semicolon ;
  there are only a few exceptions to this rule
- **Timing**: Dynare uses end of period convention for stock variables
  If you have equation like $k_{t+1} = i_t + (1 - \delta)\, k_t$, you can either
  - Write it as `k(+1) = i + (1-delta)*k;`
    and in the preamble declare `predetermined_variables k;`
  - Write it as `k = i + (1-delta)*k(-1);`
    and everywhere else replace `k` with `k(-1)`
- The reason for that is that Dynare treats variables without lead or lag as variables that can change in the current time period
- In this case the level of k_t was decided in period $t - 1$!

Model evaluation

- Usually we match the behavior of model variables to real-world variables at quarterly (sometimes monthly, rarely annual) frequency
- To compare models with data we use
  - Moment matching
  - Impulse response functions matching
- Today we will use moment matching

Model impulse response functions

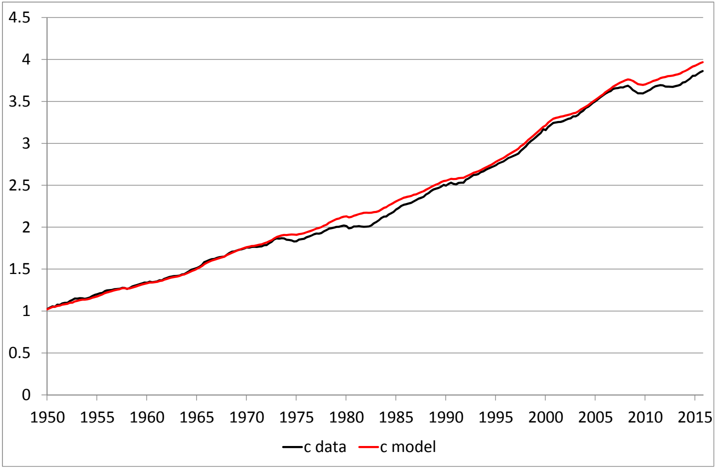


Percent deviations from steady state values (for r percentage points)

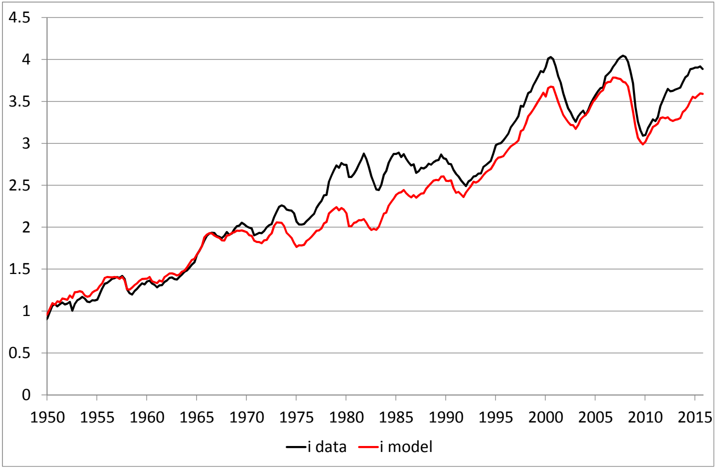
Model vs data comparison

		Std. Dev.		Corr. w. y		Autocorr.	
		Data	Model	Data	Model	Data	Model
Output	y	1.60	1.60	1.00	1.00	0.85	0.72
Consumption	c	0.86	0.57	0.76	0.92	0.83	0.80
Investment	i	4.54	5.14	0.79	0.99	0.87	0.71
Capital	k	0.57	0.46	0.36	0.08	0.97	0.96
Hours	h	**1.60**	**0.73**	0.81	0.98	0.90	0.71
Wage	w	0.84	0.73	**0.10**	**0.99**	0.65	0.75
Interest rate	r	0.39	0.06	**-0.01**	**0.96**	0.40	0.71
TFP	z	1.00	1.15	**0.67**	**1.00**	0.71	0.72
Productivity	$\frac{y}{h}$	1.30	0.95	**0.51**	**0.99**	0.65	0.75

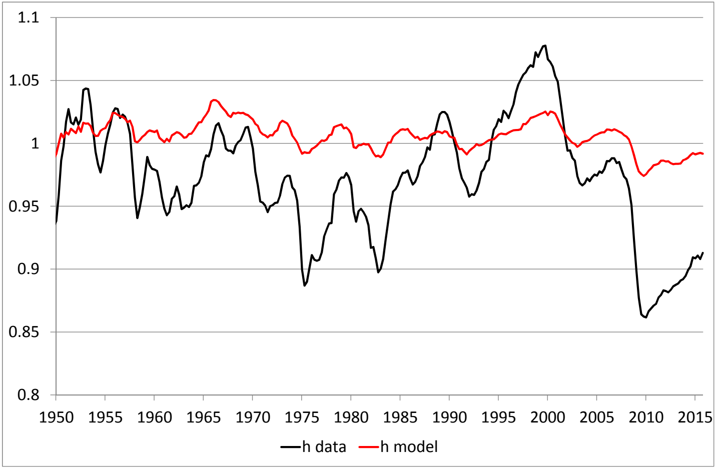
Model vs data comparison: consumption



Model vs data comparison: investment



Model vs data comparison: hours



Model vs data comparison

- Model performance is quite good – it was a big surprise in the 1980s!
- There are some problems with it though
  - In the data, hours are just as volatile as output
  - In the model, hours are less than half as volatile as output
  - In the data, real wage can be either pro- or countercyclical
  - In the model, real wage is strongly procyclical
  - In the data TFP and productivity are mildly correlated with output
  - In the model both are 1:1 correlated with output
- These results suggest that
  - We need to focus more on labor market
    – should improve behavior of hours and real wage
  - Need some room for nominal variables
  - More shocks than just TFP are needed
- This is what we are going to do over the next lectures