Marcin Bielecki, Applied Macroeconomics: Endogenous Growth[1]

1 Expanding product variety

Based on **Romer (1990)** *Endogenous Technological Change*.

1.1 Households

Assume for simplicity constant population L. Utility maximization problem:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & a_{t+1} = w_t + (1 + r_t)\, a_t - c_t \end{aligned}$$

Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1} \right] \\ &= \ldots + \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \ldots + \lambda_t \left[w_t + (1 + r_t)\, a_t - c_t - a_{t+1} \right] + \lambda_{t+1} \left[w_{t+1} + (1 + r_{t+1})\, a_{t+1} - c_{t+1} - a_{t+2} \right] + \ldots \end{aligned}$$

FOCs:

$$\begin{aligned} c_t &: \quad \beta^t c_t^{-\sigma} - \lambda_t = 0 \\ a_{t+1} &: \quad -\lambda_t + \lambda_{t+1} (1 + r_{t+1}) = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda_t &= \beta^t c_t^{-\sigma} \\ \lambda_t &= \lambda_{t+1} (1 + r_{t+1}) \end{aligned}$$

Euler equation:

$$\begin{aligned} \beta^t c_t^{-\sigma} &= \beta^{t+1} c_{t+1}^{-\sigma} (1 + r_{t+1}) \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \beta (1 + r_{t+1}) \\ \frac{c_{t+1}}{c_t} &= \left(\frac{1 + r_{t+1}}{1 + \rho} \right)^{\frac{1}{\sigma}} \end{aligned}$$

where we used the relationship between the discount factor and the discount rate: $\beta \equiv \frac{1}{1+\rho}$.

Approximate (this results in expressions that are accurate in continuous time)[2]:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \left(\frac{(1 + r_{t+1})(1 - \rho)}{(1 + \rho)(1 - \rho)} \right)^{\frac{1}{\sigma}} = \left(\frac{1 + r_{t+1} - \rho - \rho r_{t+1}}{1 - \rho^2} \right)^{\frac{1}{\sigma}} \approx (1 + r_{t+1} - \rho)^{\frac{1}{\sigma}} \approx 1 + \frac{r_{t+1} - \rho}{\sigma} \\ \frac{\Delta c_{t+1}}{c_t} &\approx \frac{r_{t+1} - \rho}{\sigma} \end{aligned}$$

Growth rate of consumption per capita along the Balanced Growth Path:

$$g_c = \frac{r - \rho}{\sigma}$$

[1] This set of lecture notes is based on chapters 3–5 and 7 from **Aghion and Howitt (2009)** *The Economics of Growth*.
[2] At the end of the first line we used first-order Taylor expansion around $x_0 = 0$: $(1 + x)^a = 1^a + a \cdot 1^{a-1} \cdot (x - 0) + \mathcal{O}(2)$.

1.2 Producers

Two types of goods:

- homogenous final goods Y_t produced by perfectly competitive, representative firm
- differentiated intermediate goods (machines) x_i produced by M_t monopolists

1.2.1 Final goods

Production function:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha}$$

Profit maximization problem:

$$\max_{L, x_{it}} \quad L^{1-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M_t} p_{it} x_{it}$$

FOCs:

$$\begin{aligned} L \quad &: \quad w_t = (1-\alpha) L^{-\alpha} \sum_{i=1}^{M_t} x_{it}^{\alpha} = (1-\alpha) Y_t / L \\ x_{it} \quad &: \quad p_{it} = \alpha L^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Demand for intermediate good of type i:

$$x_{it} = L \left(\frac{\alpha}{p_{it}} \right)^{\frac{1}{1-\alpha}}$$

1.2.2 Intermediate goods

One unit of intermediate good is produced from one unit of final good.

Profit maximization problem:

$$\begin{aligned} \max_{x_{it}, p_{it}} \quad & \Pi_{it} = p_{it} x_{it} - x_{it} \\ \text{subject to} \quad & p_{it} = \alpha L^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Simplify:

$$\max_{x_{it}} \quad \Pi_{it} = \alpha L^{1-\alpha} x_{it}^{\alpha} - x_{it}$$

FOC:

$$x_{it} \quad : \quad \alpha^2 L^{1-\alpha} x_{it}^{\alpha-1} - 1 = 0$$

Optimal level of production:

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} L$$

The level of production of all intermediate goods will be the same and constant over time. We can drop subscripts i and t.

Optimal price:

$$p = \alpha L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} L \right)^{\alpha-1} = \alpha \cdot \alpha^{-2} = \frac{1}{\alpha}$$

Profit:

$$\Pi = (p-1) x = \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{2}{1-\alpha}} L$$

1.3 Research and Development

Developing a new type of an intermediate good requires sacrificing $1/\eta$ units of final good. Parameter η measures the productivity of the R&D sector. Let R_t denote the amount of resources devoted to R&D. Then the number of varieties will increase by:

$$\Delta M_{t+1} = \eta R_t$$

Assume that the research sector is perfectly competitive. Then the cost of invention $1/\eta$ will have to be equal to the discounted value of profit flows of a monopolist:

$$\frac{1}{\eta} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}\Pi = \Pi \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} = \Pi \frac{1}{1-\frac{1}{1+r}} = \Pi \frac{1+r}{r} = \Pi \frac{(1+r)(1-r)}{r(1-r)} = \Pi \frac{1-r^2}{r-r^2} \approx \frac{\Pi}{r}$$

And the interest rate will have to satisfy:

$$r = \eta \Pi$$

1.4 General Equilibrium

We can now plug the above interest rate into the Euler equation:

$$g_c = \frac{\eta \Pi - \rho}{\sigma} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

The last formality is to show that the rate of growth of consumption, output and product variety are identical. Start with the output accounting identity:

$$\begin{aligned} Y_t &= Lc_t + M_t x + R_t \\ Lc_t &= Y_t - M_t x - R_t \end{aligned}$$

If we are able to show that the RHS grows at the rate of product variety growth, then consumption will also grow at that rate. Consider final goods output:

$$Y_t = L^{1-\alpha} \sum_{i=1}^{M_t} x_i^{\alpha} = L^{1-\alpha} M_t \left(\alpha^{\frac{2}{1-\alpha}} L\right)^{\alpha} = M_t \alpha^{\frac{2\alpha}{1-\alpha}} L \quad \longrightarrow \quad g_Y = g_M$$

Now take a look at the R&D sector:

$$\Delta M_{t+1} = \eta R_t \longrightarrow g_M = \frac{\Delta M_{t+1}}{M_t} = \eta \frac{R_t}{M_t}$$

This implies that if we are on the Balanced Growth Path and variables grow at constant rates, then:

$$g_R = g_M$$

because otherwise g_M would not be constant. Since both Y_t and R_t grow at the same rate as M_t, then consumption c_t also grows at that rate and the rate of growth of the economy is equal to:

$$g = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma}$$

Growth increases with the productivity of R&D as measured by the parameter η and with the size of the economy as measured by labor supply L, and decreases with the rate of time preference ρ and degree of risk aversion σ.

The prediction that g should increase with L was first seen as a virtue of the model, suggesting that larger countries or larger free-trade zones should grow faster. However, **Jones (1995)** pointed out that this prediction is counterfactual, to the extent that the number of researchers has substantially increased in the United States over the period since 1950, whereas the growth rate has remained on average at slightly below 2 percent over the same period.

1.5 Socially optimal rate of growth

Set up the social planner's problem. For simplicity, normalize $L = 1$:

$$\begin{aligned} \max \quad & U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\ \text{subject to} \quad & M_t x_t^{\alpha} = c_t + M_t x_t + R_t \\ & M_{t+1} = M_t + \eta R_t \end{aligned}$$

Lagrangian:

$$\mathcal{L}^{sp} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left[M_t x_t^{\alpha} - c_t - M_t x_t - R_t \right] + \sum_{t=0}^{\infty} \mu_t \left[M_t + \eta R_t - M_{t+1} \right]$$

FOCs:

$$\begin{aligned} c_t \quad &: \quad \beta^t c_t^{-\sigma} - \lambda_t = 0 \\ x_t \quad &: \quad \lambda_t \left[M_t \alpha x_t^{\alpha-1} - M_t \right] = 0 \\ R_t \quad &: \quad -\lambda_t + \mu_t \eta = 0 \\ M_{t+1} \quad &: \quad -\mu_t + \lambda_{t+1} \left(x_{t+1}^{\alpha} - x_{t+1} \right) + \mu_{t+1} = 0 \end{aligned}$$

Simplify and rewrite:

$$\begin{aligned} \lambda_t &= \beta^t c_t^{-\sigma} \\ x_t &= \alpha^{\frac{1}{1-\alpha}} \\ \mu_t &= \frac{\lambda_t}{\eta} \\ \mu_t &= \lambda_{t+1} \left(x_{t+1}^{\alpha} - x_{t+1} \right) + \mu_{t+1} \end{aligned}$$

Euler:

$$\begin{aligned} \frac{\lambda_t}{\eta} &= \lambda_{t+1} x_{t+1} \left(x_{t+1}^{\alpha-1} - 1 \right) + \frac{\lambda_{t+1}}{\eta} \\ \beta^t c_t^{-\sigma} &= \beta^{t+1} c_{t+1}^{-\sigma} \left[\eta \alpha^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) + 1 \right] \\ \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} &= \beta \left[1 + \eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} \right] \\ \frac{c_{t+1}}{c_t} &= \left(\frac{1 + \eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}}}{1+\rho} \right)^{\frac{1}{\sigma}} \end{aligned}$$

Obtain rate of growth:

$$\frac{c_{t+1}}{c_t} \approx \left(1 + \eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} - \rho \right)^{\frac{1}{\sigma}} \approx 1 + \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} - \rho}{\sigma} = 1 + g^{sp}$$

Compare with the growth rate in the decentralized economy:

$$g^{sp} = \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} - \rho}{\sigma} > \frac{\eta \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{2}{1-\alpha}} - \rho}{\sigma} = g$$

Because intermediate goods producers do not internalize their contribution to product diversity and because researchers do not internalize research spillovers, the growth rate in the decentralized economy is always lower than is socially optimal[3].

[3] Note however that **Benassy (1998)** shows that with a slight modification of the production function the decentralized economy's rate of growth might exceed socially optimal rate of growth.

1.6 Labor as R&D Input

The original **Romer (1990)** model supposed that labor was the only R&D input. Suppose now that labor can be either used to produce final goods (L_Y) or to perform R&D activities (L_R):

$$L = L_Y + L_R$$

We can transfer some results from the previous sections, we just need to be careful in replacing L with L_Y.

Output is given by:

$$Y_t = L_Y^{1-\alpha} M_t x_t^{\alpha}$$

The optimal level of intermediate goods production equals:

$$x_t = \alpha^{\frac{2}{1-\alpha}} L_Y$$

And the profit flow of a monopolist is given by:

$$\Pi = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L_Y$$

Wages equal:

$$w_t = (1-\alpha) Y_t / L_Y = (1-\alpha) L_Y^{-\alpha} M_t x_t^{\alpha} = (1-\alpha) L_Y^{-\alpha} M_t \left(\alpha^{\frac{2}{1-\alpha}} L_Y\right)^{\alpha} = (1-\alpha) \alpha^{\frac{2\alpha}{1-\alpha}} M_t$$

Now the number of varieties M_t grows at a rate that depends upon the amount of labor devoted to research:

$$\Delta M_{t+1} = \eta M_t L_R$$

This equation reflects the existence of spillovers in research activities; that is, all researchers can make use of the accumulated knowledge M_t embodied in existing designs.

Research arbitrage condition now implies:

$$\frac{w_t}{\eta M_t} = \frac{\Pi}{r}$$

$$r = \frac{\eta M_t \Pi}{w_t} = \frac{\eta M_t \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L_Y}{(1-\alpha) \alpha^{\frac{2\alpha}{1-\alpha}} M_t} = \eta L_Y \alpha^{-1} \alpha^{\frac{2}{1-\alpha}} \alpha^{-\frac{2\alpha}{1-\alpha}} = \alpha \eta L_Y$$

Now we need to find the actual division of labor between production and research. Let's look at research:

$$g_M = \frac{\Delta M_{t+1}}{M_t} = \eta L_R = \eta (L - L_Y)$$

Then:

$$L_Y = L - \frac{g_M}{\eta} \quad \longrightarrow \quad r = \alpha (\eta L - g_M)$$

Plug into the Euler equation:

$$g_c = \frac{r-\rho}{\sigma} = \frac{\alpha(\eta L - g_M) - \rho}{\sigma}$$

By assuming the two rates of growth are equal we get:

$$g = \frac{\alpha \eta L - \rho}{\alpha + \sigma}$$

And the proof that the two rates of growth are indeed equal is trivial since now output is split only between consumption and intermediate goods:

$$Y_t = Lc_t + M_t x_t \quad \longrightarrow \quad Lc_t = L_Y^{1-\alpha} M_t x_t^{\alpha} + M_t x_t$$

It is now obvious that both consumption and the number of varieties grow at exactly the same rates.

2 Increasing product quality (Schumpeterian growth)

Based on **Aghion and Howitt (1992)** *A Model of Growth Through Creative Destruction*.

Assume constant population L for simplicity. Analyze the case of a single intermediate good first.

2.1 Producers

2.1.1 Final goods

Production function:

$$Y_t = (A_t L)^{1-\alpha} x_t^{\alpha}$$

where A is the level of quality/productivity of the intermediate good.

Profit maximization problem:

$$\max_{x_t} \quad (A_t L)^{1-\alpha} x_t^{\alpha} - w_t L - p_t x_t$$

FOC:

$$x_t \quad : \quad p_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1}$$

2.1.2 Intermediate goods

One unit of intermediate good is produced from one unit of final good.

Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_t = p_t x_t - x_t \\ \text{subject to} \quad & p_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1} \end{aligned}$$

Simplify:

$$\max \quad \Pi_t = \alpha (A_t L)^{1-\alpha} x_t^{\alpha} - x_t$$

FOC:

$$x_t \quad : \quad \alpha \cdot \alpha (A_t L)^{1-\alpha} x_t^{\alpha-1} - 1 = 0$$

Optimal level of production:

$$x_t = \alpha^{\frac{2}{1-\alpha}} A_t L$$

Optimal price:

$$p_t = \alpha (A_t L)^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_t L\right)^{\alpha-1} = \alpha \cdot \alpha^{-2} = \frac{1}{\alpha}$$

Profit:

$$\Pi_t = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A_t L \equiv \pi A_t L$$

Value of the firm:

$$V_t = \sum_{i=0}^{\infty} \left(\frac{1-z}{1+r}\right)^i \Pi_t \approx \frac{\Pi_t}{r+z}$$

where z is the probability of being replaced by a successful innovator. Note here that I assume that both the real interest rate r and the innovation probability z are constant in equilibrium, which is indeed the case.

Final goods output:

$$Y_t = (A_t L)^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_t L\right)^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}} A_t L$$

Note that output grows at a rate of growth of productivity A.

2.2 Research and Development

A successful innovator replaces the monopolist and increases productivity of the intermediate good by $\gamma > 1$:

$$A_{t+1} = \gamma A_t$$

Success probability z depends on the amount of devoted R&D resources R, adjusted by target productivity level A^*, reflecting the notion that as technology advances it becomes harder to improve upon:

$$z_t = \eta \frac{R_t}{A_t^*} \quad \text{where} \quad A_t^* \equiv \gamma A_t$$

with parameter η reflecting the productivity of the R&D sector.

If successful, the innovator will gain ownership of a firm of value:

$$V_t^* = \frac{\pi A_t^* L}{r + z}$$

The expected net benefit of R&D activity is:

$$z_t \cdot V_t^* - R_t = \eta \frac{R_t}{A_t^*} \cdot \frac{\pi A_t^* L}{r + z} - R_t = \eta R_t \cdot \frac{\pi L}{r + z} - R_t$$

Innovators choose the amount of R&D resources R_t to maximize expected net benefits of R&D:

$$R_t \quad : \quad \frac{\eta \pi L}{r + z} - 1 = 0 \quad \longrightarrow \quad \eta \pi L = r + z$$

2.3 General Equilibrium

Solving the standard utility maximization problem of the consumer results in the Euler equation:

$$g_c = \frac{r - \rho}{\sigma}$$

The expected rate of productivity growth driven by innovation is given by:

$$E\left[g_A\right] = E\left[\frac{A_{t+1}}{A_t} - 1\right] = \frac{z\gamma A_t + (1 - z) A_t}{A_t} - 1 = z_t (\gamma - 1)$$

By assuming that along the Balanced Growth Path rates of growth of consumption and productivity are equal, as is indeed the case, we get the following system of three equations linking real interest rate r, innovative success probability z and rate of growth of the economy g:

$$\begin{cases} \sigma g = r - \rho & \text{Euler equation} \\ r = \eta \pi L - z & \text{Optimal R\&D intensity} \\ z = g / (\gamma - 1) & \text{Expected growth rate} \end{cases}$$

Solving the system:

$$\sigma g = \eta \pi L - z - \rho$$
$$\sigma g = \eta \pi L - g / (\gamma - 1) - \rho$$

$$g = \frac{\eta \pi L - \rho}{\sigma + (\gamma - 1)^{-1}} = \frac{\eta \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} L - \rho}{\sigma + (\gamma - 1)^{-1}}$$

$$z = \frac{\eta \pi L - \rho}{1 + \sigma (\gamma - 1)} \quad \text{and} \quad r = \frac{\rho + \sigma (\gamma - 1) \eta \pi L}{1 + \sigma (\gamma - 1)}$$

Growth increases with the productivity of R&D η, the size of innovative step γ and with the size of the economy as measured by labor supply L, and decreases with the rate of time preference ρ and degree of risk aversion σ.

2.4 Eliminating Scale Effects

Consider now the case of M distinct intermediate goods, produced by their respective monopolists.

Final goods

Production function:

$$Y_t = \left(\frac{L}{M_t}\right)^{1-\alpha} \sum_{i=1}^{M_t} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

This production function is the same as the one assumed in the expanding product variety model, except that (1) each product has its own unique productivity parameter A_{it} instead of having $A_{it} = 1$ for all products, and (2) we assume that what matters is not the absolute input L of labor but the input per product L/M[4].

Now we have to specify the process by which product variety increases. The simplest scheme is to suppose that each person has a probability ψ of inventing a new intermediate product, with no expenditure at all on research. Suppose also that the exogenous fraction ε of products disappears each period. The number of intermediate products will stabilize at a level proportional to population:

$$M_{t+1} = (1-\varepsilon) M_t + \psi L \quad \longrightarrow \quad \varepsilon M = \psi L \quad \longrightarrow \quad \frac{M}{L} = \frac{\psi}{\varepsilon} \quad \longrightarrow \quad \frac{L}{M} = \frac{\varepsilon}{\psi} \equiv \ell$$

where ℓ denotes workers per product line. Assume we have already reached the Balanced Growth Path.

Profit maximization problem:

$$\max_{x_{it}} \quad \ell^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M} p_{it} x_{it}$$

FOC:

$$x_{it} \quad : \quad p_{it} = \alpha \ell^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1}$$

Intermediate goods

Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_{it} = p_{it} x_{it} - x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Simplify:

$$\max \quad \Pi_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha} - x_{it}$$

FOC:

$$x_{it} \quad : \quad \alpha^2 \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1} - 1 = 0$$

Optimal level of production:

$$x_{it} = \alpha^{\frac{2}{1-\alpha}} A_{it} \ell$$

Optimal price:

$$p_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A_{it}\ell\right)^{\alpha-1} = \alpha \cdot \alpha^{-2} = \frac{1}{\alpha}$$

[4] The production function is a special case of the one that Benassy (1998) showed does not necessarily yield a positive productivity effect of product variety.

Profit:

$$\Pi_{it} = \left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}A_{it}\ell \equiv \pi A_{it}\ell$$

Value of the firm:

$$V_{it} = \frac{\Pi_{it}}{r+z}$$

Final goods output:

$$Y_t = \left(\frac{L}{M}\right)^{1-\alpha}\sum_{i=1}^{M}A_{it}^{1-\alpha}\left(\alpha^{\frac{2}{1-\alpha}}A_{it}\frac{L}{M}\right)^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}}L\left(\frac{1}{M}\sum_{i=1}^{M}A_{it}\right) = \alpha^{\frac{2\alpha}{1-\alpha}}A_tL$$

where the aggregate productivity is just the unweighted numerical average of all the individual productivity parameters:

$$A_t \equiv \frac{1}{M}\sum_{i=1}^{M}A_{it}$$

Research and Development

As before, a successful innovation will increase the productivity of an intermediate good by γ:

$$A_{i,t+1} = \gamma A_{it}$$

and the success probability depends on target productivity adjusted R&D resources:

$$z_{it} = \eta\frac{R_{it}}{A_{it}^{*}}$$

The expected net benefit of R&D activity is:

$$\eta\frac{R_{it}}{A_{it}^{*}}\cdot\frac{\pi A_{it}^{*}\ell}{r+z} - R_{it}$$

FOC:

$$R_{it} \quad : \quad \frac{\eta\pi\ell}{r+z} - 1 = 0 \quad \longrightarrow \quad \eta\pi\ell = r+z$$

Note that in equlibrium the probability of a successful innovation will be the same for all intermediates.

General Equilibrium

Because the number of intermediate good types is large, the growth rate of the economy will be “smooth”:

$$A_{t+1} = \frac{1}{M}\sum_{i=1}^{M}\left[z\gamma A_{it} + (1-z)A_{it}\right] = \left[z(\gamma-1)+1\right]\frac{1}{M}\sum_{i=1}^{M}A_{it} = \left[z(\gamma-1)+1\right]A_t \quad \longrightarrow \quad g_A = z(\gamma-1)$$

The growth rate of the economy is given by the expression similar to one for the single-good case, however this time population size plays no role:

$$g = \frac{\eta\pi\ell-\rho}{\sigma+(\gamma-1)^{-1}} = \frac{\eta\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}\frac{\varepsilon}{\psi}-\rho}{\sigma+(\gamma-1)^{-1}}$$

3 Innovation and capital accumulation

Suppose now that intermediate goods are produced using capital, which can accumulate over time.

Final goods

Production function:

$$Y_t = \left(\frac{L}{M}\right)^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha} = \ell^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha}$$

where as before $\ell \equiv L/M$ denotes workers per product line along the Balanced Growth Path.

Profit maximization problem:

$$\max_{x_{it}} \quad \ell^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha} - w_t L - \sum_{i=1}^{M} p_{it} x_{it}$$

FOC:

$$x_{it} \quad : \quad p_{it} = \alpha \ell^{1-\alpha} A_{it}^{1-\alpha} x_{it}^{\alpha-1}$$

Intermediate goods

Now we assume that in order to produce one unit of an intermediate good, its producer needs to rent one unit of capital at capital rental rate r_t^k. Profit maximization problem:

$$\begin{aligned} \max \quad & \Pi_{it} = p_{it} x_{it} - r_t^k x_{it} \\ \text{subject to} \quad & p_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1} \end{aligned}$$

Simplify:

$$\max \quad \Pi_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha} - r_t^k x_{it}$$

FOC:

$$x_{it} \quad : \quad \alpha^2 \left(A_{it}\ell\right)^{1-\alpha} x_{it}^{\alpha-1} - r_t^k = 0$$

Optimal level of production:

$$x_{it} = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell$$

The rental rate is determined in the market for capital, where the supply is the historically predetermined capital stock K_t and the demand is the sum of all intermediate goods demands:

$$K_t = \sum_{i=1}^{M} x_{it} = \sum_{i=1}^{M} \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} A_{it}\ell = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} L \cdot \frac{1}{M} \sum_{i=1}^{M} A_{it} = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} A_t L$$

where $A_t \equiv \frac{1}{M} \sum_{i=1}^{M} A_{it}$ is the average productivity parameter.

Denote with $\hat{k}$ the level of capital per effective labor:

$$\hat{k}_t \equiv \frac{K_t}{A_t L}$$

Rental rate of capital is then given by:

$$K_t = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} A_t L \quad \longrightarrow \quad \hat{k}_t = \left(\alpha^2 / r_t^k\right)^{\frac{1}{1-\alpha}} \quad \longrightarrow \quad r_t^k = \alpha^2 \hat{k}_t^{\alpha-1}$$

The optimal level of production can be expressed also as:

$$x_{it} = \left(\frac{\alpha^2}{\alpha^2 \hat{k}_t^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} A_{it}\ell = A_{it}\ell\hat{k}_t$$

Profit:

$$\Pi_{it} = \alpha \left(A_{it}\ell\right)^{1-\alpha} \left(A_{it}\ell\hat{k}_t\right)^{\alpha} - \alpha^2 \hat{k}_t^{\alpha-1} \cdot A_{it}\ell\hat{k}_t = \alpha\left(1-\alpha\right)\hat{k}_t^{\alpha} \cdot A_{it}\ell \equiv \pi\left(\hat{k}_t\right) A_{it}\ell$$

Profits increase with capital per effective labor, because an increase in $\hat{k}_t$ reduces the monopolist's per-unit cost of production equal to the rental rate of capital r_t^k.

Final goods output is then given by a familiar Cobb-Douglas production function:

$$Y_t = \ell^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} x_{it}^{\alpha} = \left(\frac{L}{M}\right)^{1-\alpha} \sum_{i=1}^{M} A_{it}^{1-\alpha} \left(A_{it}\ell\hat{k}_t\right)^{\alpha} = \hat{k}_t^{\alpha} L \cdot \frac{1}{M} \sum_{i=1}^{M} A_{it} = \hat{k}_t^{\alpha} L A_t = K_t^{\alpha} \left(A_t L\right)^{1-\alpha}$$

GDP per worker:

$$y_t = A_t \hat{k}_t^{\alpha}$$

Innovation and Growth

As before, a successful innovation will increase the productivity of an intermediate good by γ:

$$A_{i,t+1} = \gamma A_{it}$$

and the success probability depends on target productivity adjusted R&D resources:

$$z_{it} = \eta \frac{R_{it}}{A_{it}^*}$$

Technically speaking, the following expression for the value of the firm is now incorrect when the interest rate may change over time, but we'll focus on the BGP anyway, where r is constant:

$$V_{it} = \frac{\Pi_{it}}{r(\hat{k}) + z}$$

and where $\hat{k}$ denotes the level of capital per effective labor along the Balanced Growth Path. The interest rate and capital rental rate are related by:

$$r(\hat{k}) = r^k - \delta = \alpha^2 \hat{k}^{\alpha-1} - \delta$$

The expected net benefit of R&D activity is:

$$\eta \frac{R_{it}}{A_{it}^*} \cdot \frac{\pi(\hat{k}) A_{it}^* \ell}{r(\hat{k}) + z} - R_{it}$$

FOC:

$$R_{it} \quad : \quad \eta \frac{\pi(\hat{k})\ell}{r(\hat{k}) + z} - 1 = 0 \quad \longrightarrow \quad z = \eta\pi(\hat{k})\ell - r(\hat{k})$$

The growth rate along the BGP now depends positively on capital per effective labor, since higher capital per effective labor means higher profits and lower interest rates:

$$g = z\left(\gamma - 1\right) = \left(\gamma - 1\right)\left[\eta\pi(\hat{k})\ell - r(\hat{k})\right]$$

General Equilibrium

To keep things simple, we'll assume that the savings rate is a constant fraction of income, as in the Solow-Swan model. The capital accumulation equation is:

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)\,K_t \quad | \quad : A_t L \\ (1+g)\,\hat{k}_{t+1} &= s\hat{k}_t^{\alpha} + (1-\delta)\,\hat{k}_t \end{aligned}$$

and the BGP level of capital per effective labor is given by:

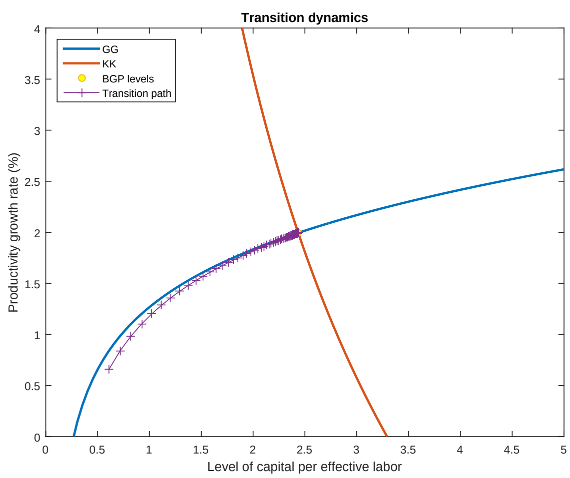
$$\hat{k} = \left(\frac{s}{\delta + g}\right)^{\frac{1}{1-\alpha}}$$

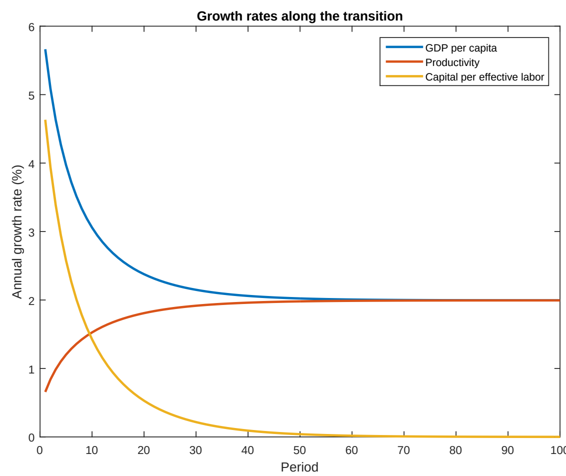
Note that the BGP level of capital per effective labor depends negatively on the growth rate.

The BGP level of capital per effective labor and the BGP productivity growth rate are jointly determined and influence each other:

$$\begin{cases} g = (\gamma - 1)\left[\eta\pi(\hat{k})\ell - r(\hat{k})\right] & \text{BGP productivity growth rate (GG curve)} \\ \hat{k} = \left[s/\left(\delta + g\right)\right]^{1/(1-\alpha)} & \text{BGP level of capital per effective labor (KK curve)} \end{cases}$$

Although this system does not have a closed-form solution, we can solve it numerically and produce a graphical illustration:





4 Technology Transfer and Cross-Country Convergence

We can also analyze the issue of international transfer of technology. Suppose that there exist two groups of countries: technology leaders and technology adopters[5]. The behavior of technology leaders is the same as described in the previous sections. Technology adopters enjoy an "advantage of backwardness" and can increase their productivity by adopting technologies developed in other countries. However, if a country does not innovate at all, then it will stagnate while the rest of the world continues to advance.

Productivity and Distance to Frontier

Assume that a successful innovator in any sector gets to implement a technology with a productivity parameter equal to a level $\bar{A}_{it}$, which represents the world technology frontier in this sector and which grows at a rate $\bar{g}$ determined outside the country. Each productivity parameter A_{it} will evolve according to:

$$A_{it} = \begin{cases} \bar{A}_{it} & \text{with probability } z \\ A_{i,t-1} & \text{with probability } 1-z \end{cases}$$

Then the country's average productivity parameter will evolve as follows:

$$A_t = \frac{1}{M}\sum_{i=1}^{M} A_{it} = \frac{1}{M}\sum_{i=1}^{M}\left[z\bar{A}_{it} + (1-z)\,A_{i,t-1}\right] = z\sum_{i=1}^{M}\bar{A}_{it} + (1-z)\,\frac{1}{M}\sum_{i=1}^{M} A_{i,t-1} = z\bar{A}_t + (1-z)\,A_{t-1}$$

That is, in the fraction z of sectors that innovate productivity is $\bar{A}_t$, whereas in the remaining fraction productivity is the same as in period $t-1$.

The country's "proximity" to the world technology frontier is the ratio of its average productivity parameter to the global frontier parameter:

$$a_t = A_t/\bar{A}_t$$

and evolves according to:

$$\begin{aligned} A_t &= z\bar{A}_t + (1-z)\,A_{t-1} \quad | \quad : \bar{A}_t \\ \frac{A_t}{\bar{A}_t} &= z + (1-z)\,\frac{A_{t-1}}{\bar{A}_t} = z + (1-z)\,\frac{A_{t-1}}{\bar{A}_{t-1}}\frac{\bar{A}_{t-1}}{\bar{A}_t} \\ a_t &= z + \frac{1-z}{1+\bar{g}}a_{t-1} \end{aligned}$$

There is a unique steady-state proximity a^*, which can be found by setting $a_t = a_{t-1} = a^*$:

$$\begin{aligned} a^* &= z + \frac{1-z}{1+\bar{g}}a^* \\ (1+\bar{g})\,a^* &= (1+\bar{g})\,z + (1-z)\,a^* \\ a^* &= \frac{(1+\bar{g})\,z}{\bar{g}+z} \end{aligned}$$

Once the steady-state proximity is reached, the country's productivity growth rate is given by:

$$g = \frac{A_t}{A_{t-1}} - 1 = \frac{z\bar{A}_t + (1-z)\,A_{t-1}}{A_{t-1}} - 1 = z\frac{\bar{A}_t}{\bar{A}_{t-1}}\frac{\bar{A}_{t-1}}{A_{t-1}} + (1-z) - 1 = z\,(1+\bar{g})\,\frac{1}{a^*} - z = \bar{g} + z - z = \bar{g}$$

Therefore, all technology adopters that innovate ($z > 0$) will converge to the same growth rate, although their steady-state proximity to the technology frontier may differ due to different z.

[5] This is a simplifying assumption. In reality this distinction is not strict, as even highly developed countries are technology adopters in some industries.

Convergence and Divergence

Recall the formula for the probability of innovating z (whenever the formula would yield negative values, a country does not innovate at all and sets $z = 0$):

$$z = \eta\pi\ell - r$$

Let us focus first on those technology adopters that innovate ($z > 0$). Another "advantage of backwardness" is that the growth rate of productivity is faster the further behind the technology frontier a country is. The average innovation size is given by:

$$\gamma_t = \frac{\bar{A}_t}{A_{t-1}} = \frac{\bar{A}_t}{\bar{A}_{t-1}} \frac{\bar{A}_{t-1}}{A_{t-1}} = \frac{(1+\bar{g})}{a_{t-1}}$$

And the country's productivity growth rate is:

$$g_t = z\,(\gamma_t - 1) = \frac{z\,(1+\bar{g})}{a_{t-1}} - z$$

Therefore, the further behind the frontier the country is, the higher its productivity growth rate will be. This fact limits how far behind the frontier a country can fall, because eventually it will get so far behind that its growth rate will be just as large as the growth rate of the frontier, at which point the gap will stop increasing.

However, if countries do not innovate at all ($z = 0$), maybe due to poor macroeconomic conditions, legal environment, education system, or credit markets, they will not benefit from technology transfer, but will instead stagnate. If this situation persists, their productivity level will remain constant and they will diverge from the club of innovating countries.

Together these two results help to explain the empirical fact that there is a group of countries that are converging to parallel growth paths (i.e., with identical long-run growth rates) and another group of countries that are falling further and further behind. Notice that even countries that are converging to parallel growth paths are not necessarily converging in levels. That is, one country's steady-state proximity to the frontier a^* can differ from another's if they have different values of the critical parameters governing the intensity of R&D.

This result helps us to account for the fact that there are systematic and persistent differences across countries in the level of productivity. That is, convergence in levels is not absolute but conditional. In our model, two countries will end up with the same productivity levels in the long run if they have the same parameter values, but not otherwise.